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SUBASSOCIATIVE ALGEBRAS

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ABSTRACT. An algebra is subassociative if the associator $[x, y, z]$ of any three elements x, y, z is their linear combination. In this paper we prove that any such algebra is Lie-admissible and that almost any such algebra is proper in the sense that there exists an invariant bilinear form A for which there holds the following identity: $[x, y, z] = A(y, z)x - A(x, y)z$, which enables a close connection with associative algebras. We discuss also the improper subassociative algebras.

1. INTRODUCTION

In certain algebras, it happens that some (sometimes all) associators $[x, y, z] := (xy)z - x(yz)$ are linear combinations of their arguments. Simple, but important examples can be found in [**2**, Proposition 2.1(v), (vi), (viii)]. In the same article there is also a discussion of anticommutative algebras which meet the requirement

$$[x, y, y] = -[y, y, x] = F(x, y)y - F(y, y)x$$

with F a bilinear form. This identity holds, for example, in every central simple non-Lie Maltsev algebra over a field of characteristic $\neq 2, 3$ (which has finite dimension (Filippov), more precisely, this dimension is 7 (Kuzmin)). An algebra whith this identity should be in the style of Definition 1 called **subalternative** and we expect that it has very special properties even without supposition of anticommutativity. We believe these properties worthy of special research and the first step should be the present article, where we discus **subassociative** algebras (in which **all** associators are linear combinations of their arguments).

Throughout $\mathbb{F}$ will be a commutative field of characteristic chr$\mathbb{F}$. We shall use two special symbols: a subset S of a linear space spans the subspace lin S, and a subset T of an algebra generates the subalgebra alg T.

Definition 1. Let H be an algebra over $\mathbb{F}$ with multiplication $(x, y) \mapsto xy$. H is a **subassociative algebra**, if the associator of any three elements from H is their linear combination: $\forall(x, y, z) \in H^3\ \exists(\alpha, \beta, \gamma) \in \mathbb{F}^3$:

$$[x, y, z] = \alpha x + \beta x + \gamma z\,. \tag{1}$$

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Examples 2.

(i) Any associative algebra is subassociative.

(ii) $\mathbb{R}^3$ with the usual vector product $(x,y) \mapsto x \wedge y$ and inner product $(x,y) \mapsto \langle x,y \rangle$ is a subassociative algebra since

$$(x \wedge y) \wedge z - x \wedge (y \wedge z) = \langle x,y \rangle z - \langle y,z \rangle x \,. \tag{2}$$

(iii) Let H be any linear space over $\mathbb{F}$ and $U, V \colon H \to \mathbb{F}$ linear functionals. If we define for every $(x,y) \in H^2$

$$xy := U(y)x + V(x)y \,, \tag{3}$$

we find out that H with this multiplication is a subassociative algebra:

$$[x,y,z] = U(y)V(x)z - U(z)V(y)x \,. \tag{4}$$

(iv) We can generalize the previous case introducing two properties, noting P, Q: if H is an algebra with the property

$$\forall (x,y) \in H^2 \;\; \exists (\alpha, \beta) \in \mathbb{F}^2 : xy = \alpha x + \beta y \,, \tag{P}$$

then H is a subassociative algebra of the following kind:

$$\forall (x,y,z) \in H^3 \;\; \exists (\pi, \rho) \in \mathbb{F}^2 : [x,y,z] = \pi x + \rho z \,. \tag{Q}$$

Definition 3. Let H be an algebra over $\mathbb{F}$ with multiplication $(x,y) \mapsto xy$ and suppose that there exists such a bilinear form $A \colon H^2 \to \mathbb{F}$ that

$$[x,y,z] = A(y,z)x - A(x,y)z \tag{5}$$

for any $(x,y,z) \in H^3$. Then we call H a **proper** subassociative algebra.

Using (9) in the fourth section it is easy to prove the following identity concerning the form A from (5):

$$A(xy,z) = A(x,yz) \,. \tag{6}$$

Examples 4. The first three cases in Examples 2 are proper subassociative algebras.

In (i): $A = 0$.

In (ii): $A(x,y) = -\langle x,y \rangle$.

In (iii): $A(x,y) = -U(y)V(x)$.

The case (iv) will be discussed later.

The next proposition shows how we can make a proper subassociative algebra from any associative algebra with unit.

Proposition 5. *Let G be an associative algebra with multiplication $(a,b) \mapsto a*b$ and with unit e. Further let $P\colon G \to \mathbb{F}$ be a linear functional and $P(e)=1$. Define in $H := \operatorname{Ker} P$ a new multiplication*

$$(x,y) \mapsto xy := x*y - A(x,y)e\,,$$

where $A(x,y) := P(x*y)$. Then H is a proper subassociative algebra and A the bilinear form from Definition 3.

The proof is straightforward, as is the proof of the opposite proposition:

Proposition 6. *Let H be a proper subassociative algebra from Definition* 3, *and $G := \mathbb{F}e \oplus H$, where $e \notin H$. Introduce in G a new multiplication*

$$(\alpha e + x, \beta e + y) \mapsto (\alpha e + x) * (\beta e + y) := (\alpha\beta + A(x,y))e + \alpha y + \beta x + xy\,.$$

G with this multiplication is an associative algebra with unit e.

Since we shall prove that almost all subassociative algebras are proper, Propositions 5 and 6 justify the name "subassociative".

2. Two Lemmas

In order to prove that subassociative algebras are usually proper we shall need two lemmas.

Suppose that $n \in \mathbb{Z}^+$ and L is a linear space over $\mathbb{F}$, $\dim L \geq n$, and let $R\colon L^n \to L$ be an n-linear map with the property:

$$\forall (x_1, \dots, x_n) \in L^n \ \exists (\alpha_1, \dots, \alpha_n) \in \mathbb{F}^n : \ R(x_1, \dots, x_n) = \sum_{j=1}^{n} \alpha_j x_j\,.$$

If $n=1$, it is easy to see that there exists one and only one constant $\rho \in \mathbb{F}$ such that

$$\forall x \in L : R(x) = \rho x\ .$$

Therefore we shall suppose $n \geq 2$.

Lemma 7. *Let L, n, R be as before and also $\dim L > n$ or $\mathbb{F} \neq \mathbb{Z}_2 = \{0,1\}$. Then for each $j = 1, \dots, n$ there exists a unique multilinear map $A_j\colon L^{n-1} \to \mathbb{F}$ such that the equation*

$$R(x_1, \dots, x_n) = \sum_{j=1}^{n} A_j(x_1, \dots, \widehat{x}_j, \dots, x_n)x_j$$

holds identically. Here $\widehat{x}_j$ designates the absence of j-th argument.

Proof. Let $x_1, \dots, x_{n-1}$ be any $n-1$ elements from L and $M := \operatorname{lin}\{x_1, \dots, x_{n-1}\}$. Take $x_n, x_n' \notin M$.

$$R(x_1, \dots, x_{n-1}, x_n) = \sum_{j=1}^{n-1} \alpha_j x_j + \alpha_n x_n \,, \tag{$*$}$$

$$R(x_1, \dots, x_{n-1}, x_n') = \sum_{j=1}^{n-1} \beta_j x_j + \beta_n x_n' \;; \tag{$**$}$$

$$\sum_{j=1}^{n-1} (\alpha_j - \beta_j) x_j + \alpha_n x_n - \beta_n x_n' = R(x_1, \dots, x_{n-1}, x_n - x_n') \,.$$

In short, there are such $\delta \in \mathbb{F}$ and $y \in M$ that

$$\begin{aligned} \alpha_n x_n - \beta_n x_n' &= y + \delta(x_n - x_n') \,, \\ (\beta_n - \delta)\, x_n' &= (\alpha_n - \delta)\, x_n - y \,. \end{aligned}$$

If $\alpha_n - \delta = 0$ or $\beta_n - \delta = 0$, it follows immediately: $\alpha_n = \beta_n$. Suppose the opposite, i.e. $x_n' = \varphi x_n + \psi y$ ($\varphi \neq 0$) and use ($**$):

$$\varphi R(x_1, \dots, x_{n-1}, x_n) + \psi R(x_1, \dots, x_{n-1}, y) = \sum_{j=1}^{n-1} \beta_j x_j + \beta_n \varphi x_n + \beta_n \psi y \,.$$

Considering ($*$) we get: $\varphi \alpha_n = \beta_n \varphi$ and $\alpha_n = \beta_n$.
(**Remark.** This part of the proof will be needed once again later for the first argument of R).

But what if $x_n' = \sum_{j=1}^{n-1} \gamma_j x_j \in M$? Then we can always write:

$$R(x_1, \dots, x_{n-1}, x_n') = \sum_{j=1}^{n-1} (\beta_j + \beta_n \gamma_j - \alpha_n \gamma_j) x_j + \alpha_n x_n' \,.$$

Therefore, α_n is independent of x_n and uniquely determined by $x_1, \dots, x_{n-1}$:

$$R(x_1, \dots, x_{n-1}, x_n) = \sum_{j=1}^{n-1} \alpha_j x_j + A_n(x_1, \dots, x_{n-1}) x_n \,.$$

Again we take $x_n \notin M$. From the expression of $R(\lambda x_1, x_2, \dots)$ there follows the homogeneity of A_n in the first argument. Similarly we prove that A_n is homogeneous in the other arguments.

To prove the additivity of the form A_n in the first argument we will use the equation

$$R(x_1 + x_1', x_2, \dots, x_{n-1}, x_n) = R(x_1, \dots) + R(x_1', \dots) \,.$$

If x_n can be chosen to be linearly independent of $x_1, x_1', x_2, \dots, x_{n-1}$, the additivity is obvious. Suppose then that $x_1, x_1', x_2, \dots, x_{n-1}$ are linearly independent and that $\dim L = n$. $x_n = x_1 + \lambda x_1'$ for some λ different from 0 or 1. Then the elements x_1, x_1' and $x_1 + x_1'$ are not in $\operatorname{lin}\{x_2, \dots, x_n\}$ and we can use the consideration above the remark in the first part of the proof:

$$\begin{aligned} R(x_1, x_2, \dots, x_n) &= \alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_{n-1} x_{n-1} + A_n(x_1, x_2, \dots, x_{n-1}) x_n\,, \\ R(x_1', x_2, \dots) &= \alpha_1 x_1' + \beta_2 x_2 + \dots + \beta_{n-1} x_{n-1} + A_n(x_1', x_2, \dots) x_n\,, \\ R(x_1 + x_1', x_2, \dots) &= \alpha_1 (x_1 + x_1') + \gamma_2 x_2 + \dots + \gamma_{n-1} x_{n-1} \\ &\quad + A_n(x_1 + x_1', x_2, \dots) x_n\,. \end{aligned}$$

Hence, the additivity of the form A_n in the first argument (and in the same way in all other arguments) is proven and A_n is a multilinear form.

Suppose that $n = 2$. Then for a fixed x_2

$$Q_{x_2}(x_1) := R(x_1, x_2) - A_2(x_1) x_2$$

is a linear map and there exists such an $A_1(x_2) \in \mathbb{F}$, uniquely determined by x_2, that $Q_{x_2}(x_1) = A_1(x_2) x_1$, and then

$$R(x_1, x_2) = A_1(x_2) x_1 + A_2(x_1) x_2\,.$$

But since $Q_{x_2}(x_1)$ is linear in both variables, A_1 is necessarily a linear functional. So the lemma is valid for $n = 2$.

Let now $n > 2$ and suppose that the lemma is valid for all $k < n$. Fix again x_n and define

$$Q_{x_n}(x_1, \dots, x_{n-1}) := R(x_1, \dots, x_{n-1}, x_n) - A_n(x_1, \dots, x_{n-1}) x_n\,.$$

Q_{x_n} is an $(n-1)$-linear map, so

$$Q_{x_n}(x_1, \dots, x_{n-1}) = \sum_{j=1}^{n-1} A_j(x_1, \dots, \widehat{x}_j, \dots, x_{n-1}, x_n) x_j\,,$$

where A_j are $(n-2)$-linear forms, additionally dependent on parameter x_n. To see that A_j is also linear in this last argument it is sufficient to choose $x_j \notin \operatorname{lin}\{x_1, \dots, \widehat{x}_j, \dots, x_{n-1}\}$ and to put instead of x_n first λx_n ($\lambda \in \mathbb{F}$) and then $x_n + x_n'$. $\square$

Lemma 8. *Let $n \geq 2$ and $0 < m \leq n$. Further let L be a linear space over $\mathbb{F}$, $\dim L \geq m$, and $R\colon L^n \to L$ an n-linear map with the property:*

$$\forall (x_1, \dots, x_n) \in L^n \ \ \exists (\alpha_1, \dots, \alpha_m) \in \mathbb{F}^m : R(x_1, \dots, x_n) = \sum_{i=1}^{m} \alpha_i x_i.$$

Excluding the case where $\dim L = m$ *and* $\mathbb{F} = \mathbb{Z}_2$ *at the same time, there exist such unique* $(n-1)$*-linear forms* $A_i\colon L^{n-1} \to \mathbb{F}$ $(i = 1, \dots, m)$, *that*

$$(***) \qquad R(x_1, \dots, x_n) = \sum_{i=1}^{m} A_i(x_1, \dots, \widehat{x}_i, \dots, x_n) x_i$$

holds identically.

Proof. For $m = n$ we get Lemma 7. So, suppose that $m < n$. Fix $x_{m+1}, \dots, x_n$; then there exist, according to Lemma 7, such unique maps $A_i(x_1, \dots, \widehat{x}_i, \dots, x_m, x_{m+1}, \dots, x_n)$, linear in the first $m-1$ arguments, that $(***)$ holds. The linearity of these maps in the last $n-m$ arguments is proved by choosing $x_i \notin \operatorname{lin}\{x_1, \dots, \widehat{x}_i, \dots, x_m\}$. □

Example 9. In any algebra H with the property (P) from Example 2(iv) (with possible exception of $\dim H = 2$ and $\mathbb{F} = \mathbb{Z}_2$), there exist by Lemma 7 such linear functionals U, V that (3) and (4) hold. If $\dim H \geq 2$, U and V are unique. In the exceptional case when $\dim H = 2$, $\mathbb{F} = \mathbb{Z}_2$ and linear functionals U, V do not exists, we can find a base $\{p, q\}$, having Table 1 as its multiplication table. The parameters α, β, γ are arbitrary. In this case, U and V do not exist even as nonlinear forms.

$\cdot$	p	q
p	$(1+\alpha)p$	$\beta p + (\alpha+\gamma)q$
q	$(\alpha+\beta)p + \gamma q$	$(1+\alpha)q$

Table 1.

$\cdot$	p	q
p	$\alpha p + \beta q$	$\gamma p + \delta q$
q	$\gamma p + \delta q$	$\varepsilon p + \zeta q$

Table 2.

3. Two-dimensional Subassociative Algebras

Definition 10. A subassociative algebra H is **non-strange** if there exist three bilinear forms $A, B, C\colon H^2 \to \mathbb{F}$ such that

$$(7) \qquad [x, y, z] = A(y, z)x + B(x, z)y + C(x, y)z$$

holds for each $(x, y, z) \in H^3$. Otherwise H is **strange**.

Of course, any proper subassociative algebra is non-strange.

Since any algebra of dimension < 2 is associative, we find out, using Lemma 7, that if H is a strange subassociative algebra then either

(i) $\dim H = 2$, or
(ii) $\dim H = 3$, $\mathbb{F} = \mathbb{Z}_2$.

In this section we will look through all two-dimensional subassociative algebras. A sufficient condition for them to be subassociative is:

$$\forall x \in H\ \exists \alpha \in \mathbb{F} : [x,x,x] = \alpha x\,.$$

A	p	q
p	$\beta\gamma + \delta^2 - \alpha\delta - \beta\zeta$	$\beta\varepsilon - \gamma\delta - \omega$
q	$\beta\varepsilon - \gamma\delta + \omega$	$\gamma^2 + \delta\varepsilon - \alpha\varepsilon - \gamma\zeta$

Table 3.

B	p	q
p	0	ω
q	$-\omega$	0

Table 4.

All commutative algebras are of this kind; they are all non–strange and even proper (if one takes $\omega = 0$): Tables 2, 3, 4. The form C from (7) can be computed from A and B by the equation

$$C = -2B - A \tag{8}$$

and the parameter ω in Table 4 is arbitrary.

The noncommutative non-strange case is given in Table 5, its form A in Table 6, the form B in Table 4 (with arbitrary ω) and the form C is given by (8). Since we may choose $\omega = 0$, the algebra is proper. The algebra has the property: $\forall x \in H : x^2 = \lambda x$.

$\cdot$	p	q
p	$(\beta+\delta)p$	$\alpha p + \beta q$
q	$\gamma p + \delta q$	$(\alpha+\gamma)q$

Table 5. $pq \neq qp$

A	p	q
p	$-\beta\delta$	$-\omega - \alpha\beta$
q	$\omega - \gamma\delta$	$-\alpha\gamma$

Table 6.

There exist also 7 nonisomorphic strange algebras over the field $\mathbb{F} = \mathbb{Z}_2$ with Table 7 as the multiplication table, and 6 nonisomorphic strange algebras over the field $\mathbb{F} = \mathbb{Z}_3 = \{0, \pm 1\}$ with Table 8 as a multiplication table.

$\cdot$	p	q
p	γq	$\alpha p + \beta q$
q	$(\alpha+1)p + (\beta+\gamma+1)q$	0

Table 7. $\mathbb{F} = \mathbb{Z}_2$, $(1+\alpha)(1+\beta)(1+\gamma) = 0$

$\cdot$	p	q
p	q	αp
q	βp	$(\alpha+\beta-1)q$

Table 8. $\mathbb{F} = \mathbb{Z}_3, \alpha \neq \beta$

4. Teichmüller Equation

The well known Teichmüller equation is valid in any algebra:

$$w[x,y,z]+[w,x,y]z=[wx,y,z]-[w,xy,z]+[w,x,yz]\,. \tag{9}$$

With this equation we will discuss the non-strange algebras. Of course we may suppose that the dimension of algebra is at least 3. Using (7) we transform (9) into

$$\begin{aligned} & B(x,z)wy+B(w,y)xz+B(w,z)xy+[A(x,y)+C(x,y)]wz \\ & \quad =[A(x,yz)-A(xy,z)]w+B(w,yz)x \\ & \quad +B(wx,z)y+[C(wx,y)-C(w,xy)]z \end{aligned} \tag{$E(w,x,y,z)$}$$

Proposition 11. *Suppose that an algebra H (of any dimension) has property* (Q) *from Example* 2*(iv). Then H is a proper subassociative algebra; the only exceptions are given by Table* 7 *for $\gamma=0$.*

Proof. It is easy to see that Proposition 11 is valid for $\dim H<3$. Hence, suppose that $\dim H\geq 3$, and, using Lemma 8, that

$$[x,y,z]=A(y,z)x+C(x,y)z$$

for all x,y,z, which is a special case of (7) with $B=0$. If there exist such x and y that $A(x,y)+C(x,y)\neq 0$, we find out from $E(w,x,y,z)$ that $wz=\lambda w+\mu z$ for all w and z. Therefore, $\dim\operatorname{alg}\{x,y\}=2$ and (8) gives us a contradiction. □

Now we shall prove the following identities:

$$[x,x,x]=0\,, \tag{10}$$

$$A(x,x)+B(x,x)+C(x,x)=0\,. \tag{11}$$

(10) and (11) are of course equivalent identities. They surely hold for such x that $x^2=\lambda x$. Therefore assume that $x^2\neq\lambda x$ and that (11) does not hold. From $E(x^2,x,x,x)$ then there follows:

$$\begin{aligned} x^2x &= \alpha x+\beta x^2\,, \\ xx^2 &= x^2x-[x,x,x]=\gamma x+\beta x^2\,, \\ x^2x^2 &= x(xx^2)+[x,x,x^2]=\delta x+\varepsilon x^2\,. \end{aligned}$$

Hence, $\dim\operatorname{alg}\{x\}=2$ and $\operatorname{alg}\{x\}$ is an algebra from Table 2. But such algebra is commutative and $[x,x,x]=0$, which is a contradiction.

If for any x we choose a y such that $y\notin\operatorname{lin}\{x,x^2\}$, then there follows from $E(y,x,x,x)$:

$$A(x^2,x)=A(x,x^2)\,. \tag{12}$$

From $E(x,x,x,y)$ we also get

$$C(x^2,x) = C(x,x^2)\,. \tag{13}$$

Suppose that x and x^2 are linearly independent. Then from $E(x,x,x,x)$ there follows

$$2B(x,x) = 0\,. \tag{14}$$

Now say that $x^2 = \tau x$ and $2B(x,x) \neq 0$ and y is not colinear with x. From $E(x,y,x,x)$ we get: $yx = \alpha x + \beta y$; from $E(x,x,y,x)$ it follows: $xy = \gamma x + \delta y$; finally from $E(x,y,x,y)$ we find out: $y^2 = \varepsilon x + \zeta y$. Hence $\dim \operatorname{alg}\{x,y\} = 2$. But we already know that in non-strange two-dimensional algebras there is always $B(x,x) = 0$, which is a contradiction. Therefore (14) holds always.

Now it is time for the main theorem.

Theorem 12. *Let H be a non-strange subassociative algebra (of any dimension) over a field $\mathbb{F}$ of characteristic $\operatorname{chr}\mathbb{F} \neq 2$. Then H is a proper subassociative algebra.*

Proof. In view of Proposition 11 it is enough to prove that $B = 0$ when $\dim H \geq 3$. (14) tells us that $B(x,x) = 0$ for any $x \in H$. From $E(x,x,x,y)$ and $E(y,x,x,x)$ there follows that if $x^2 \neq \lambda x$ then $B(x,y) = B(y,x) = 0$. So, if $B(x,y) \neq 0$ for some x,y, then $x^2 = \lambda x$ and $y^2 = \mu y$.

Hence, let us suppose that $B(x,y) \neq 0$ for x,y linearly independent. $0 = B(x+y,x+y) = B(x,y)+B(y,x)$, therefore $B(y,x) = -B(x,y) \neq 0$. $B(x,x+y) = B(x,y) \neq 0 \Rightarrow (x+y)^2 = \nu(x+y)$, which gives us:

$$xy + yx = \alpha x + \beta y\,. \tag{$*$}$$

$E(x,y,y,x)$ implies

$$xy - yx = \gamma x + \delta y\,. \tag{$**$}$$

$(*)$ and $(**)$ are enough for the conclusion: $\dim \operatorname{alg}\{x,y\} = 2$. $\operatorname{alg}\{x,y\}$ is therefore an algebra from Table 5, of course without the condition $pq \neq qp$. We can take $x = p$, $y = q$ and $B(p,q) = \omega \neq 0$.

Furthermore, let r be another element from H, $r \notin \operatorname{alg}\{p,q\}$. Then $E(p,p,q,r)$ gives us: $[A(p,q) + B(p,q) + C(p,q)]pr = -\omega pr \in \operatorname{lin}\{p,q,r\}$. $E(q,q,p,r)$ gives: $qr \in \operatorname{lin}\{p,q,r\}$, $E(r,q,p,p)$ gives: $rp \in \operatorname{lin}\{p,q,r\}$, $E(r,p,q,q)$ proves: $rq \in \operatorname{lin}\{p,q,r\}$, and finally $E(r,q,r,p)$ gives $r^2 \in \operatorname{lin}\{p,q,r\}$.

Hence, $\operatorname{lin}\{p,q,r\} = \operatorname{alg}\{p,q,r\}$ is a 3-dimensional subalgebra. Say, $pr = \kappa_1 p + \lambda_1 q + \mu_1 r$, $qr = \kappa_2 p + \lambda_2 q + \mu_2 r$.

$$\begin{aligned}[p,p,r] &= C(p,p)r + \cdots = \beta\delta r + \cdots \\ &= p^2 r - p(pr) = \mu_1(\beta + \delta - \mu_1)r + \cdots\end{aligned}$$

and: $\mu_1(\beta+\delta-\mu_1)=\beta\delta$, or

$$(***) \qquad (\mu_1-\beta)(\delta-\mu_1)=0\,.$$

$$\begin{aligned}[p,q,r] &= C(p,q)r+\cdots=(\alpha\beta-\omega)r+\cdots\\ &= (pq)r-p(qr)=(\mu_1\alpha+\mu_2\beta-\mu_1\mu_2)r+\cdots\end{aligned}$$

and: $\alpha\beta-\omega=\mu_1\alpha+\mu_2\beta-\mu_1\mu_2$, or

$$\omega=(\mu_1-\beta)(\mu_2-\alpha)\,,$$

which gives together with $(***)$: $\mu_1=\delta$.

$$\begin{aligned}[q,p,r] &= C(q,p)r+\cdots=(\gamma\delta+\omega)r+\cdots\\ &= (qp)r-q(pr)=\gamma\delta r+\cdots\,,\end{aligned}$$

which is a contradiction. □

So, we found out that the subassociative algebras are the following:

(i) proper algebras, which are in a direct connection with the associative algebras, by Propositions 5 and 6;
(ii) strange algebras of dimension 2 (over $\mathbb{F}=\mathbb{Z}_2$ and $\mathbb{Z}_3$);
(iii) strange algebras of dimension 3 (over $\mathbb{F}=\mathbb{Z}_2$), and
(iv) improper non-strange algebras (of dimension >2 and of characteristic 2).

5. Strange 3-dimensional Algebras

Perhaps the best way to determine all strange 3-dimensional subassociative algebras (over $\mathbb{Z}_2$) is to use a computer. There are 2^{27} different multiplication tables. It is easy to find all (that is 5) algebras with the property (P). In any other algebra we can find such a base $\{a_1,a_2,a_3\}$ that $a_1a_2=a_3$. There are "only" 2^{24} multiplication tables of this kind. If $a_ia_j=\lambda_{ij}^1a_1+\lambda_{ij}^2a_2+\lambda_{ij}^3a_3$ $(i,j=1,2,3)$, the multiplication table is determined with these binary digits λ_{ij}^k, which constitute a number in the binary system:

$$\lambda_{11}^1\lambda_{11}^2\lambda_{11}^3\lambda_{13}^1\lambda_{13}^2\lambda_{13}^3\lambda_{21}^1\lambda_{21}^2\lambda_{21}^3\lambda_{22}^1\lambda_{22}^2\lambda_{22}^3\lambda_{23}^1\lambda_{23}^2\lambda_{23}^3\lambda_{31}^1\lambda_{31}^2\lambda_{31}^3\lambda_{32}^1\lambda_{32}^2\lambda_{32}^3\lambda_{33}^1\lambda_{33}^2\lambda_{33}^3$$

Its decadic form is a code of an algebra. A capable computer can eliminate isomorphic algebras; the result is $5+801.163=801.168$ nonisomorphic three-dimensional algebras over $\mathbb{Z}_2$. With a computer we select 156 subassociative algebras, from those we select 60 improper algebras and among them there are 22 strange algebras; their codes are in Table 9.

438	2454	65904	67920	79449
81529	107352	107480	108121	109432
110201	524598	590320	592336	632537
658224	756720	1115232	8692568	8693337
9219833	9741896			

Table 9.

It is easy to check that in any of these algebras the identity $[x,x,x]=0$ holds which means that all these algebras are Lie-admissible. Also, none of these algebras is anticommutative (in the sense $\exists x : x^2 \neq 0$).

6. Improper Non-strange Algebras

In this section we will discuss improper nonstrange algebras. Therefore we will suppose that such an algebra H has $\dim H > 2$ and that $\operatorname{chr} \mathbb{F} = 2$.

Suppose that for certain x, y we have

$$A(x,y) + B(x,y) + C(x,y) \neq 0\,.$$

Of course, because of (11) x and y are linearly independent.
From $E(x,x,y,x)$ there follows $:\ x^2 = \lambda_1 x + \mu_1 y$,
from $E(y,x,y,y) :\ y^2 = \lambda_2 x + \mu_2 y$,
from $E(x,x,y,y) :\ xy = \lambda_3 x + \mu_3 y$,
from $E(x,y,x,y) :\ B(x,y)yx = \lambda_4 x + \mu_4 y$ and
from $E(y,x,y,x) :\ [A(x,y)+C(x,y)]yx = \lambda_5 x + \mu_5 y$.
Hence: $\operatorname{alg}\{x,y\} = \operatorname{lin}\{x,y\}$ and $\operatorname{alg}\{x,y\}$ is an algebra from Table 2 or Table 5. Then $A = C$ and $B(x,y) \neq 0$. We can take $x = p$, $y = q$, $B(p,q) = \omega \neq 0$.

Let $r \notin \operatorname{alg}\{p,q\}$ be another element.

$$E(p,p,q,r) \Rightarrow pr \in \operatorname{lin}\{p,q,r\}\,;$$
$$E(p,q,q,r) \Rightarrow qr \in \operatorname{lin}\{p,q,r\}\,;$$
$$E(r,p,p,q) \Rightarrow rp \in \operatorname{lin}\{p,q,r\}\,;$$
$$E(r,p,q,q) \Rightarrow rq \in \operatorname{lin}\{p,q,r\}\,.$$

Put $pr = \kappa_1 p + \lambda_1 q + \mu_1 r$, $rq = \kappa_2 p + \lambda_2 q + \mu_2 r$. With the direct computation we get:

$$[p,r,q] = (\ldots)p + (\ldots)q\,.$$

But since

$$[p,r,q] = A(r,q)p + \omega r + C(p,r)q\,,$$

we find the wrong conclusion $\omega = 0$.

Therefore we have proved the following identity:

$$A(x,y) + B(x,y) + C(x,y) = 0 \,. \tag{15}$$

Noting that $B = 0$ in the case where H is a proper subassociative algebra of dimension > 2 and that we may choose $B = 0$ for nonstrange algebras at dimension ≤ 2 we can say that (15) is the general equation for any non-strange subassociative algebra over a field $\mathbb{F}$ of any characteristic.

We have now the tool for the last theorem.

Theorem 13. *Any subassociative algebra is Lie-admissible:* $[x,y,z]+[z,x,y]+[y,z,x] = [y,x,z]+[z,y,x]+[x,z,y]$ *for all* x,y,z. *Any anticommutative* ($x^2 = 0$ *for all* x) *subassociative algebra is proper.*

Proof. The non-strange algebras are Lie-admissible because of (15). A straightforward computation shows that the strange algebras from Table 8 are Lie-admissible. The strange algebras from Table 7 and these of dimension 3 fulfill the identity $[x,x,x] = 0$ which in characteristic 2 suffices for Lie-admissibility.

It is easy to check that strange anticommutative algebras do not exist. Suppose now that H is an anticommutative and improper non-strange subassociative algebra. Since the characteristic is 2, H is also commutative.

$$0 = [x,y,x] = (A(y,x) + C(x,y))x + B(x,x)y$$

Then: $B(x,x) = 0$ and $C(x,y) = A(y,x)$ for all $x,y \in H$.

Suppose that $B(x,y) = A(x,y) + A(y,x) \neq 0$ for certain x,y, which means that x and y are linearly independent.

$$\begin{aligned} x(xy) &= [x,x,y] = A(y,x)x + A(x,x)y \,, \\ y(xy) &= A(y,y)x + A(x,y)y \,. \end{aligned}$$

Therefore x and y generate an algebra of dimension ≤ 3. The case $\dim \operatorname{alg}\{x,y\} = 2$ is not possible, by (8). Therefore, this subalgebra has Table 10 as its multiplication table.

$\cdot$	x	y	z
x	0	z	$\alpha x + \beta y$
y	z	0	$\gamma x + \delta y$
z	$\alpha x + \beta y$	$\gamma x + \delta y$	0

Table 10.

$A(y,x) = \alpha$ and $A(x,y) = \delta$, hence $\alpha \neq \delta$. But writing down $[z,z,x]$ and $[z,z,y]$ we find a contradiction. $\square$

For chr $\mathbb{F} \neq 2$, it is a trivial fact that anticommutative Lie-admissible algebra is Lie. We do not know if the anticommutative subassociative algebra is Lie also in the case chr $\mathbb{F} = 2$; however, it is at least Maltsev (i.e.: $x^2 = 0$, $[xz, x, y] = [x, y, z]x$, for all x, y, z).

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