Acta Math. Univ. Comenianae
Vol. LXVI, 2(1997), pp. 329–336

RADEMACHER VARIABLES IN CONNECTION WITH COMPLEX SCALARS

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Abstract. We shall see that the Sidon constant of the Rademacher system equals $\pi/2$. This is also the best constant for the contraction principle if complex scalars are involved.

1. The Rademacher System and Its Sidon Constant

Rademacher variables are generally understood as an i.i.d sequence of random variables taking the values -1 and $+1$ each with probability $1/2$. We model them as follows. Let $\mathbb{E}$ denote the multiplicative group of the two elements -1 and $+1$ in $\mathbb{C}$. Let us consider the cartesian power $\mathbb{E}^\infty = \prod_{j\in\mathbb{N}} \mathbb{E}$ and the natural maps

$$r_j : \mathbb{E}^\infty \to \mathbb{T}, \qquad (j \in \mathbb{N}),$$

which assign to any sequence $\varepsilon = (\varepsilon_j)_{j=1}^\infty$ their jth coordinate ε_j. Here $\mathbb{T}$ denotes the group of complex numbers of modulus 1.

If we equip $\mathbb{E}^\infty$ with the coarsest topology that will make all r_j continuous we find by Tychonoff's theorem that $\mathbb{E}^\infty$ is compact. Moreover, if we define multiplication in $\mathbb{E}^\infty$ coordinate wise the r_j become homorphisms.

As we are usually concerned with only finitely many Rademacher variables at a time, and since we are interested in their distributional properties, barely, we may equally think of $r_1, \dots, r_n$ (n fixed) as to be defined on $\mathbb{E}^n$ rather than $\mathbb{E}^\infty$. This should cause no troubles.

In either case, it turns out that we move in a convenient setting.

Given a compact abelian group G a continuous homeomorphism $\chi\colon G \to \mathbb{T}$ is called **character**. We say a sequence of characters $\mathcal{X} = (\chi_1, \chi_2, \dots)$ is a **Sidon set**, provided we can find a constant C such that however we choose a natural n and complex numbers $a_1, \dots, a_n$ we have

$$\sum_{j=1}^n |a_j| \le C \left\| \sum_{j=1}^n a_j \chi_j \right\|_\infty . \tag{1}$$

Received March 14, 1997.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 46B20, 46E40; Secondary 43A46.
The author is supported by the DFG.

Of course, $\|\cdot\|_\infty$ is shorthand for the norm in $C(G)$, the space of continuous functions on G. If $\mathcal{X}$ is a Sidon set, we label the smallest of all admissible constants C as

$$S(\mathcal{X}).$$

This is the **Sidon constant** of $\mathcal{X}$ (cf. [**4**]).

The system of Rademacher variables $\mathcal{R} = (r_1, r_2, \dots)$ is an obvious example for a Sidon set, for if we split the term $\sum_{j=1}^n a_j r_j$ into its real and imaginary parts we certainly get away with $C = 2$ in (1). We will see, that we can do slightly better, albeit the precise value of $S(\mathcal{R})$ is rather of aesthetic interest. The proof rests upon the following fact, which is almost a blueprint of [**1**, Lemma 3.6, p. 21].

Lemma 1. *For $j = 1, \dots, n$ let K_j be compact topological spaces and let f_j be continuous complex functions on K_j. Suppose there exist points $a_j, b_j \in K_j$ such that*

$$\|f_j\|_\infty = |f_j(a_j)| = |f_j(b_j)| \quad \textit{and} \quad f_j(a_j) = -f_j(b_j).$$

If we define $f \in C\left(\prod_{j=1}^n K_j\right)$ by

$$f(t_1, \dots, t_n) = \sum_{j=1}^n f_j(t_j), \qquad (t_1, \dots, t_n) \in \prod_{j=1}^n K_j,$$

then

$$\sum_{j=1}^n \|f_j\|_\infty \le \frac{\pi}{2} \|f\|_\infty.$$

Moreover, the constant $\pi/2$ is best possible.

Proof. Let us fix some $\vartheta \in [0, 2\pi)$ for an instant. Define

$$t_j = \begin{cases} a_j, & \text{if } \operatorname{Re}(e^{i\vartheta} f_j(a_j)) > 0 \\ b_j, & \text{if } \operatorname{Re}(e^{i\vartheta} f_j(b_j)) \ge 0 \end{cases} \qquad j = 1, \dots, n$$

and choose $\sigma_j \in [0, 2\pi)$ such that

$$\|f_j\|_\infty = e^{-i\sigma_j} f_j(b_j) = e^{i(\pi - \sigma_j)} f_j(a_j).$$

Then we get

$$\begin{aligned} \operatorname{Re}(e^{i\vartheta} f_j(t_j)) &= \max\Big\{ \operatorname{Re}(e^{i(\vartheta+\sigma_j)}) \|f_j\|_\infty \,,\ \operatorname{Re}(e^{i(\vartheta+\sigma_j+\pi)}) \|f_j\|_\infty \Big\} \\ &= \|f_j\|_\infty |\cos(\vartheta + \sigma_j)|. \end{aligned}$$

Hence,

$$\|f\|_\infty \ge \left| e^{i\vartheta} \sum_{j=1}^n f_j(t_j) \right| \ge \sum_{j=1}^n \operatorname{Re}\left(e^{i\vartheta} f_j(t_j)\right) = \sum_{j=1}^n \|f_j\|_\infty |\cos(\vartheta + \sigma_j)|.$$

Integration with respect to ϑ will settle our issue, since

$$2\pi\|f\|_\infty \geq \sum_{j=1}^{n} \|f_j\|_\infty \int_0^{2\pi} |\cos(\vartheta+\sigma_j)|\, d\vartheta = 4\sum_{j=1}^{n}\|f_j\|_\infty.$$

The fact that $\pi/2$ is best possible will be clear by the example included in the proof of the following theorem. □

Theorem 2. *The Sidon constant of the Rademacher system equals $\pi/2$.*

Proof. Given $a_1,\dots,a_n \in \mathbb{C}$ we define $f_j\colon \mathbb{E} \to \mathbb{C}$ by $f_j(-1) = -a_j$, $f_j(+1) = a_j$. Let $f\colon \mathbb{E}^n \to \mathbb{C}$ be given by $f(\varepsilon_1,\dots,\varepsilon_n) = \sum_{j=1}^n f_j(\varepsilon_j)$, then we may just as well write

$$f = \sum_{j=1}^{n} a_j r_j,$$

where $r_1,\dots,r_n$ are to be understood as defined on $\mathbb{E}^n$ rather than $\mathbb{E}^\infty$. Now, the preceeding lemma applies and we get

$$\sum_{j=1}^{n}|a_j| \leq \frac{\pi}{2}\Big\|\sum_{j=1}^{n} a_j r_j\Big\|_\infty.$$

As for the optimality of $\pi/2$ fix $n \in \mathbb{N}$ for an instant. Let $\beta = e^{i2\pi/n}$ be an n-the root of unity. We are going to consider the function $g_n = \sum_{j=1}^n \beta^{j-1} r_j$ on $\mathbb{E}^n$. If $\operatorname{sign}(a) \in \mathbb{T}$ is defined by $|a|/a$ $(a \neq 0)$ and $\operatorname{sign}(0) = 1$ then

$$|g_n(\varepsilon)| = \operatorname{sign}\Big(g_n(\varepsilon)\Big) g_n(\varepsilon) = \sum_{j=1}^{n} \operatorname{Re}\Big(\beta^{j-1}\operatorname{sign}(g_n(\varepsilon))\Big)\varepsilon_j \qquad (\varepsilon = (\varepsilon_1,\dots,\varepsilon_n)).$$

Since, obviously $\sum_{j=1}^n \operatorname{Re}(\beta^{j-1}e^{i\vartheta})\varepsilon_j \leq |g_n(\varepsilon)|$ it follows that

$$\|g_n\|_\infty = \max_{\varepsilon_j=\pm 1}\max_{0\leq\vartheta<2\pi}\sum_{j=1}^{n}\operatorname{Re}(\beta^{j-1}e^{i\vartheta})\varepsilon_j.$$

Note that $\vartheta \mapsto \max_{\varepsilon_j=\pm1}\sum_{j=1}^n \operatorname{Re}(\beta^{j-1}e^{i\vartheta})$ is $2\pi/n$-periodic and determine $\vartheta_n \in [0,\frac{2\pi}{n})$ and $\varepsilon_1^*,\dots,\varepsilon_n^*$ such that

$$\|g_n\|_\infty = \sum_{j=1}^{n}\operatorname{Re}(\beta^{j-1}e^{i\vartheta_n})\varepsilon_j^* = \sum_{j=1}^{n}\cos\Big(\vartheta_n + \frac{2\pi(j-1)}{n}\Big)\varepsilon_j^*.$$

By maximality every summand $\cos\Big(\vartheta_n + \frac{2\pi(j-1)}{n}\Big)\varepsilon_j^*$ is bound to be non-negative. Thus, in actual fact we have

$$\|g_n\|_\infty = \sum_{j=1}^{n}\Big|\cos\Big(\vartheta_n + \frac{2\pi(j-1)}{n}\Big)\Big|.$$

Since $\sum_{j=1}^{n} |\beta^{j-1}| = n$ we may conclude $n \le S(\mathcal{R}_n)\|g_n\|_\infty$ or, equivalently,

$$S(\mathcal{R}_n)^{-1} \le \frac{\|g_n\|_\infty}{n} = \frac{1}{n}\sum_{j=1}^{n}\Big|\cos\Big(\vartheta_n + \frac{2\pi(j-1)}{n}\Big)\Big| .$$

By continuity of the cosine we find that the right hand side tends to $\int_0^1 |\cos(2\pi t)|dt = \frac{2}{\pi}$ as $n \to \infty$. We conclude $S(\mathcal{R}) = \sup_n S(\mathcal{R}_n) \ge \frac{\pi}{2}$. □

2. The Contraction Principle Using Complex Scalars

The result on the Sidon constant of the Radmacher system can be applied to the complex version of the **contraction principle**. It is well known, and easily seen, that given reals $a_1, \dots, a_n$ and vectors $x_1, \dots, x_n$ in some (real or complex) Banach space X we always have

$$\Big\|\sum_{j=1}^{n} a_j x_j r_j\Big\|_{L_p^X(\mathbb{E}^n)} \le \max_{j=1,\dots,n} |a_j| \,\Big\|\sum_{j=1}^{n} x_j r_j\Big\|_{L_p^X(\mathbb{E}^n)}$$

for $1 \le p \le \infty$ (cf. [**3**, p. 91]).

If we want to extend this result to complex scalars and complex Banach spaces the basic tool is Pełczyński's celebrated result on commensurate sequences [**5**] which we state in a disguised form.

Lemma 3 (Pełczyński [**5**], Pisier [**6**]). *Let $\chi_1, \dots, \chi_n$ and $\psi_1, \dots, \psi_n$ be characters on compact abelian groups G and H, respectively. If $C \ge 1$ is such that*

$$\Big\|\sum_{j=1}^{n} a_j\psi_j\Big\|_\infty \le C\Big\|\sum_{j=1}^{n} a_j\chi_j\Big\|_\infty \quad \textit{for} \quad a_1, \dots, a_n \in \mathbb{C}$$

then we find for all $s \in H$ and $1 \le p \le \infty$

$$\Big\|\sum_{j=1}^{n} x_j\psi_j(s)\chi_j\Big\|_{L_p^X(G)} \le C\Big\|\sum_{j=1}^{n} x_j\chi_j\Big\|_{L_p^X(G)},$$

regardless of the choice of the Banach space X and vectors $x_1, \dots, x_n$ in X.

As for the proof there is no point in going into details. Everything can be found in Pełczyński's paper ([**5**, Theorem 1], compare [**6**, Théorème 2.1]). Three notes may be helpful.

- Here, $L_p^X(G)$ is the space of Bochner-p-integrable X-valued functions on G (with respect to the Haar measure). Of course, $\sum_{j=1}^{n} x_j\varphi_j \colon G \longrightarrow X$ is continuous, so we need not bother about integrability.

- The claim remains true also for Orlicz spaces $L_\phi^X(G)$ with literally the same proof as in [**5**], since all that is employed is Young's inequality.
- We suggest to call the best constant C in the inequality above **relative Sidon constant of** $\Psi_n = (\psi_1, \dots, \psi_n)$ **vs.** $\mathcal{X}_n = (\chi_1, \dots, \chi_n)$ for the following reason. If $\psi_1, \dots, \psi_n$ are Steinhaus variables, i.e. $\psi_j : \mathbb{T}^n \to \mathbb{T}$ is the projection on the jth coordinate, then
$$\Big\| \sum_{j=1}^n a_j \psi_j \Big\|_\infty = \sum_{j=1}^n |a_j|$$
and we find that the best constant C equals $S(\mathcal{X}_n)$.

Corollary 4. *The best constant in the principle of contraction for complex scalars is* $\pi/2$.

Proof. We have to proof the inequality
$$\Big\| \sum_{j=1}^n a_j x_j r_j \Big\|_{L_p^X(\mathbb{E}^n)} \le \frac{\pi}{2} \Big\| \sum_{j=1}^n x_j r_j \Big\|_{L_p^X(\mathbb{E}^n)}$$
whenever $a_1, \dots, a_n$ are complex scalars of modulus ≤ 1. Just as in the real case (cf. [**3**, pp. 95]), we may argue by convexity to see that the function
$$(a_1, \dots, a_n) \mapsto \Big\| \sum_{j=1}^n a_j x_j r_j \Big\|_{L_p^X(\mathbb{E}^n)}$$
takes its maximum on $\{a : \|a\|_\infty \le 1\} \subset \ell_\infty^n$ in an extreme point, say in $s = (s_1, \dots, s_n)$ where $|s_1| = \dots = |s_n| = 1$. If $\psi_1, \dots, \psi_n$ are Steinhaus variables the lemma implies
$$\Big\| \sum_{j=1}^n x_j \psi_j(s) r_j \Big\|_{L_p^X(\mathbb{E}^n)} \le C \Big\| \sum_{j=1}^n x_j r_j \Big\|_{L_p^X(\mathbb{E}^n)}$$
with $C = S(\mathcal{R}_n) \nearrow \pi/2$ $(n \to \infty)$.

As for the question of optimality it is useful to note that if we restrict our attention to scalars a_j of modulus 1 the inequalities
$$\Big\| \sum_{j=1}^n a_j r_j x_j \Big\|_{L_p^X(\mathbb{E}^n)} \le C_n \Big\| \sum_{j=1}^n r_j x_j \Big\|_{L_p^X(\mathbb{E}^n)} \qquad \text{for } |a_1| = \dots = |a_n| = 1 \tag{2}$$
and
$$\Big\| \sum_{j=1}^n r_j x_j \Big\|_{L_p^X(\mathbb{E}^n)} \le C_n \Big\| \sum_{j=1}^n a_j r_j x_j \Big\|_{L_p^X(\mathbb{E}^n)} \qquad \text{for } |a_1| = \dots = |a_n| = 1 \tag{3}$$
are equivalent, leading to the same constant C_n.

Let us consider a second set of n Rademacher variables which we would like to interpret as vectors x_j in $X = C(\mathbb{E}^n)$ given by $x_j(\widetilde{\varepsilon}_1, \dots, \widetilde{\varepsilon}_n) = \widetilde{\varepsilon}_j$.

An inspection on the proof of Theorem 2 reveals that

$$\frac{1}{n}\Big\|\sum_{j=1}^n e^{2\pi ij/n} x_j\Big\|_\infty$$

tends to $2/\pi$ as $n \to \infty$. The key observation is that for arbitrary complex numbers a_j

$$\Big\|\sum_{j=1}^n a_j x_j\Big\|_\infty = \max_{\widetilde{\varepsilon}_j = \pm 1}\Big|\sum_{j=1}^n a_j \widetilde{\varepsilon}_j\Big| = \max_{\widetilde{\varepsilon}_j = \pm 1}\Big|\sum_{j=1}^n a_j \varepsilon_j \widetilde{\varepsilon}_j\Big|$$

however we choose $\varepsilon_1, \dots, \varepsilon_n \in \{-1, +1\}$. Accordingly,

$$\Big\|\sum_{j=1}^n e^{2\pi ij/n} r_j x_j\Big\|_{L_p^X(\mathbb{E}^n)} = \Big\|\sum_{j=1}^n e^{2\pi ij/n} x_j\Big\|_\infty.$$

On the other hand

$$\frac{1}{n}\Big\|\sum_{j=1}^n r_j x_j\Big\|_{L_p^X(\mathbb{E}^n)} = \frac{1}{n}\Big\|\sum_{j=1}^n x_j\Big\|_\infty = 1.$$

Combining these, the best constants C_n in

$$\Big\|\sum_{j=1}^n r_j x_j\Big\|_{L_p^X(\mathbb{E}^n)} \le C_n \Big\|\sum_{j=1}^n a_j r_j x_j\Big\|_{L_p^X(\mathbb{E}^n)}$$

for $|a_1| = \dots = |a_n| = 1$ satisfy $\liminf\limits_{n\to\infty} C_n \ge \pi/2$. By the equivalence of (2) and (3) our proof is complete. $\square$

3. Concluding Remark

Rademacher variables also are involved when ℓ_1^n is to be embedded into ℓ_∞ or a suitable ℓ_∞^N $(N \ge n)$.

Let us recall the real situation. If we take $N = 2^n$ we may identify ℓ_∞^N with $\ell_\infty(\mathbb{E}^n)$. If the unit vectors in $\ell_\infty(\mathbb{E}^n)$ are labelled e_ε $(\varepsilon \in \mathbb{E}^n)$, we can define vectors u_j in $\ell_\infty(\mathbb{E}^n)$ via

$$u_j = \sum_{\varepsilon \in \mathbb{E}^n} r_j(\varepsilon) e_\varepsilon.$$

Then, for reals $a_1, \dots, a_n$ we certainly have

$$\sum_{j=1}^n |a_j| = \max_{\varepsilon_j = \pm 1}\Big|\sum_{j=1}^n a_j \varepsilon_j\Big| = \Big\|\sum_{j=1}^n a_j u_j\Big\|_\infty,$$

which amounts to saying that

$$\ell_1^n \to \ell_\infty(\mathbb{E}^n), \qquad e_j \mapsto u_j \quad (j = 1, \ldots, n)$$

is an isometric embedding.

If we look at our previous discussion, the same mapping reinterpreted as acting between the corresponding complex spaces will have operator norm arbitrarily close to $\pi/2$ as $n \to \infty$ — and not any better.

Nevertheless, the universal character of ℓ_∞ still allows us to embed $\ell_1^n \xrightarrow{\imath} W_n \hookrightarrow \ell_\infty$ with $dim(W_n) = n$ and $\|\imath\| \, \|\imath^{-1}\| \le 1 + \delta$ for arbitrarily small $\delta > 0$.

Again, isometry is possible if we invoke **Kronecker's theorem** on diophantine approximation (for a proof and related discussions consult e.g. [**2**, Chapt. XXIII, pp. 371–391]).

Theorem 5 (Kronecker 1884). *Let $\beta_1, \ldots, \beta_n$ be in $\mathbb{R}$ such that the set*

$$1, \beta_1, \ldots, \beta_n$$

is linearly independent over the field $\mathbb{Q}$. Given arbitrary $\xi_1, \ldots, \xi_n$ in $\mathbb{R}$ and $\delta > 0$ there exist a natural number m and integers $k_1, \ldots, k_n$ such that

$$\Big| \xi_j - m\beta_j - k_j \Big| < \delta \qquad (j = 1, \ldots, n).$$

We employ this result for our purposes.

Since

$$\left| e^{ia} - e^{ib} \right| \le |a - b| \qquad (a, b \in \mathbb{R})$$

we see that

$$\Big| \exp\left(2\pi i \alpha_j\right) - \exp\left(2\pi i m \beta_j\right) \Big| = \Big| \exp\big(2\pi i \alpha_j\big) - \exp\big(2\pi i (m\beta_j + k_j)\big) \Big| \le 2\pi\delta.$$

We conclude that the sequence of vectors $\{x_m\}_{m=1}^\infty$ in $\mathbb{C}^n$ defined by

$$x_m = \sum_{j=1}^n \exp(2\pi i m \beta_j) e_j \in \mathbb{C}^n \qquad (m \in \mathbb{N})$$

is dense in $\mathbb{T}^n$.

Put

$$w_j = \left(e^{2\pi i m \beta_j} \right)_{m=1}^\infty \in \ell_\infty. \qquad (j = 1, \ldots, n)$$

Given complex numbers $a_1, \ldots, a_n$, by density we get

$$\Big\| \sum_{j=1}^n a_j w_j \Big\|_\infty = \sup_{m \in \mathbb{N}} \Big| \sum_{j=1}^n a_j e^{2\pi i m \beta_j} \Big| = \sup_{\|(\xi_j)_1^n\|_\infty = 1} \Big| \sum_{j=1}^n a_j \xi_j \Big| = \sum_{j=1}^n |a_j|.$$

Finally, if $W_n = span\{w_1, \ldots, w_m\}$ we get the desired isometry $\imath$

$$\ell_1^n \xrightarrow{\imath} W_n \subset \ell_\infty$$
$$e_j \mapsto w_j \quad (j = 1, \ldots, n).$$

Acknowledgement. I would like to thank my teacher Prof. Dr. Albrecht Pietsch for his support and advice.

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