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CLONES IN TOPOLOGY AND ALGEBRA

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Abstract. Clones of continuous maps of topological spaces and clones of homomorphisms of universal algebras are investigated and their initial segments compared. We show, for instance, that for every triple $2 \leq n_1 \leq n_2 \leq n_3$ of integers there exist algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ with two unary operations such that the initial k-segments of their clones of homomorphisms are equal exactly when $k \leq n_1$, isomorphic exactly when $k \leq n_2$ and elementarily equivalent exactly when $k \leq n_3$.

1. Concepts and Results

1.1. According to [**5**], a **clone** $\mathrm{Clo}(X)$ on a nonvoid set X is a collection of finitary operations $X^n \to X$ with $n \in \omega$ (where ω denotes the set of all finite ordinals), containing all Cartesian projections $p_i^{(n)}\colon X^n \to X$ with $i \in n = \{0, \dots, n-1\}$, that is, maps $p_i^{(n)}(x_0, \dots, x_{n-1}) = x_i$, and closed under all operations C_m^n with $m, n \in \omega$ (called compositions in [**5**]), defined for every m-tuple $f_0, \dots, f_{m-1}\colon X^n \to X$ of elements of $\mathrm{Clo}(X)$ and any $g\colon X^m \to X$ in $\mathrm{Clo}(X)$ with the result $f\colon X^n \to X$ given by

$$f(x_0, \dots, x_{n-1}) = g(f_0(x_0, \dots, x_{n-1}), \dots, f_{m-1}(x_0, \dots, x_{n-1})).$$

It is often convenient to regard $\mathrm{Clo}(X)$ also as a category k whose objects are all finite powers

$$X^0, X, X^2, \dots$$

of X, the set of k-morphisms $X^n \to X$ consists of all n-ary operations of $\mathrm{Clo}(X)$ including all projections $p_i^{(n)}$, the set of k-morphisms $X^n \to X^m$ consists of all maps $f = f_0 \dot{\times} \cdots \dot{\times} f_{m-1}$ where $(f_0, \dots, f_{m-1})$ is an m-tuple of n-ary operations in $\mathrm{Clo}(X)$ and f is defined by

$$f(x_0, \dots, x_{n-1}) = (f_0(x_0, \dots, x_{n-1}), \dots, f_{m-1}(x_0, \dots, x_{n-1})),$$

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and the composition in k is the usual composition of mappings (denoted $\circ$ here). Then k is, indeed, a category in which $p_0^{(n)} \dot{\times} \cdots \dot{\times} p_{n-1}^{(n)}$ is the identity morphism on X^n, and the object X^n together with the projections $p_0^{(n)}, \dots, p_{n-1}^{(n)}$ is a categorial product of n copies of X. The operations C_m^n of [**5**] then take on the form

$$C_m^n(g; f_0, \dots, f_{m-1}) = g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{m-1}).$$

We shall regard $\mathrm{Clo}(X)$ both as the category k just described, and also in the sense of [**5**], that is, as the set of finitary operations including all $p_i^{(n)}$ with $i \in n \in \omega$ and closed under the composition operations C_m^n.

1.2. The notion of an **abstract clone** (see [**5**]) arises naturally when properties of clones invariant under an isomorphism of categories are investigated. This notion coincides with that of a (monosorted) finitary algebraic theory in the sense of [**2**], [**3**]. Thus, an **algebraic theory** $\mathcal{T}$ with a **base object** a is a category whose set of objects $\{a^n \mid n \in \omega\}$ consists of mutually distinct objects, and one in which morphisms

$$\pi_i^{(n)} : a^n \to a \text{ with } i \in n$$

are specified so that $(a^n, \{\pi_i^{(n)} \mid i \in n\})$ is a categorial product of n copies of the base object a. Let $\circ$ denote the composition also in $\mathcal{T}$. Then, for any choice of $f_0, \dots, f_{m-1} \in \mathcal{T}(a^n, a)$, we let $f_0 \dot{\times} \cdots \dot{\times} f_{m-1}$ denote the unique morphism $f \in \mathcal{T}(a^n, a^m)$ satisfying $f_j = \pi_j^{(m)} \circ f$ for all $j \in m$.

A **homomorphism** H of an algebraic theory $\mathcal{T}$ with a base object a into an algebraic theory $\mathcal{T}'$ with a base object a' is any functor $H : \mathcal{T} \to \mathcal{T}'$ satisfying

$$H(a^n) = (a')^n \text{ and } H(\pi_i^{(n)}) = (\pi_i^{(n)})' \text{ for } i \in n \in \omega.$$

If H is one-to-one and surjective, we say that $\mathcal{T}$ and $\mathcal{T}'$ are **isomorphic** algebraic theories.

1.3. Algebraic theories (or, equivalently, abstract clones) can also be described by means of the operations C_m^n from 1.1. Any algebraic theory $\mathcal{T}$ with a base object a is, indeed, an ω-sorted universal algebra whose carrier Y_n of the n-th sort is the set $\mathcal{T}(a^n, a)$, and the (heterogeneous) structure on $\{Y_n \mid n \in \omega\}$ has the following operations:

(c) for every $n \in \omega$, there are n nullary operations in Y_n, namely $\pi_i^{(n)} \in Y_n$ with $i \in n$,

(s) for every $n, m \in \omega$, there is a heterogeneous operation

$$C_m^n : Y_m \times \overbrace{Y_n \times \cdots \times Y_n}^{m \text{ times}} \to Y_n$$

defined by $C_m^n(g; f_0, \dots, f_{m-1}) = g \circ (f_0 \dot{\times} \cdots \dot{\times} f_{m-1})$.

Any ω-sorted algebra arising from an algebraic theory $\mathcal{T}$ also satisfies equations reflecting the associativity of composition and the fact that the nullary operations from (c) are product projections. This description is particularly appropriate if we wish to speak about the first order language of clone theory (see [**8**], for instance), for it is exactly the first order language of the ω-sorted universal algebras just described. In particular, two algebraic theories are elementarily equivalent if and only if they satisfy exactly the same sentences in this first order language. Since any clone on a nonvoid set is also an algebraic theory, it is possible to speak about their elementary equivalence as well. In this context, the actual form of the maps from $\mathrm{Clo}(X)$ becomes irrelevant, of course.

1.4. Every nonvoid topological space X determines an algebraic theory $\mathrm{Clo}(X)$. This is simply the full subcategory of the category Top (= the category of all topological spaces and all continuous maps) determined by all finite powers X^0, X, $X^2, \ldots$ of the space X. If $X = (P, \tau)$, that is, if τ is a topology on the underlying set P of X, then X determines a clone $\mathrm{Clo}(P)$ on the nonvoid set P in the sense of 1.1: a mapping $f\colon P^n \to P$ belongs to $\mathrm{Clo}(P)$ if and only if $f\colon X^n \to X$ is continuous.

More generally, if $(\mathcal{K}, U)$ is a concrete category with finite **concrete products** (that is, if the category $\mathcal{K}$ has finite products and the faithful functor $U\colon \mathcal{K} \to \mathrm{Set}$ into the category Set of all sets and maps preserves them), then any $\mathcal{K}$-object a with nonvoid $U(a)$ determines an algebraic theory $\mathrm{Clo}(a)$, and also a clone $\mathrm{Clo}(U(a))$ on the nonvoid set $U(a)$. We restrict ourselves, however, to the case of Top and of categories $\mathrm{Alg}(\Delta)$ of all (monosorted) universal algebras of a finitary type Δ and all their homomorphisms in the sense of [**1**]. When endowed with their natural forgetful functors, the categories Top and $\mathrm{Alg}(\Delta)$ have concrete products.

In our main results (Theorems 1 and 2 in 1.7 and 1.8 below), the symbols $\mathrm{Alg}(1,1)$ and $\mathrm{Alg}(1,1,1)$ denote the respective categories of all universal algebras with two or three unary operations and all their homomorphisms.

1.5. Let $\mathcal{T}$ be an algebraic theory with a base object a. Its n-**segment** $\mathcal{T}_n$ is the full subcategory of $\mathcal{T}$ determined by the objects $a^0, a, \ldots, a^{n-1}$. The definition of a homomorphism and an isomorphism of algebraic theories in 1.2 can be restated for their n-segments in an obvious manner. A description of $\mathcal{T}_n$ as an n-sorted algebra with nullary operations $\pi_i^{(m)}$ with $i \in m$ and heterogeneous operations C_m^l from 1.3 restricted to $l, m < n$ is also quite straightforward. It is then clear that the first order language of n-segments of algebraic theories is just the appropriate fragment of the first order language of algebraic theories, and we can speak about isomorphism and elementary equivalence of n-segments of algebraic theories in the evident way. Since clones are algebraic theories as well, their n-segments, and isomorphism and elementary equivalence of their n-segments are also included in these definitions.

Since clones carry additional concrete information, we can also speak about their equality and about the equality of clone segments. Just as in [**11**], we regard clones as pairs $(\mathcal{T}, F)$ where $\mathcal{T}$ is an algebraic theory with a base object a, and

$$F\colon \mathcal{T} \to \mathrm{Set}$$

is a one-to-one functor such that

$$F(a^n) = (F(a))^n, \quad F(\pi_i^{(n)}) = p_i^{(n)} \text{ for all } i \in n \in \omega,$$

where $p_i^{(n)}$ is the i-th Cartesian projection $p_i^{(n)}(x_0, \dots, x_{n-1}) = x_i$ from 1.1. An n-**segment of** $(\mathcal{T}, F)$ is then just a pair $(\mathcal{T}_n, F_n)$ where $\mathcal{T}_n$ is an n-segment of $\mathcal{T}$ and $F_n\colon \mathcal{T}_n \to \mathrm{Set}$ is the domain restriction of F.

Let $(\mathcal{T}_n, F_n)$ and $(\mathcal{T}'_n, F'_n)$ be clone n-segments. As we have already noted, $(\mathcal{T}_n, F_n)$ is elementarily equivalent to $(\mathcal{T}'_n, F'_n)$ if $\mathcal{T}_n$ is elementarily equivalent to $\mathcal{T}'_n$. And $(\mathcal{T}_n, F_n)$ is isomorphic to $(\mathcal{T}'_n, F'_n)$ if there exists an isofunctor Φ of $\mathcal{T}_n$ onto $\mathcal{T}'_n$. If the isofunctor Φ also satisfies

$$F'_n \circ \Phi = F_n,$$

we say that $(\mathcal{T}_n, F_n)$ **equals** $(\mathcal{T}'_n, F'_n)$ and write $(\mathcal{T}_n, F_n) = (\mathcal{T}'_n, F'_n)$.

The equality of clones is defined analogously.

To be able to discuss these three relations simultaneously, we write

$$(\mathcal{T}_n, F_n) \overset{j}{\simeq} (\mathcal{T}'_n, F'_n),$$

where $\overset{1}{\simeq}$ stands for the equality $=$, while $\overset{2}{\simeq}$ means the existence of an isomorphism $\simeq$, and $\overset{3}{\simeq}$ means the elementary equivalence. For the sake of brevity, we shall write

$$(\mathcal{T}_n, F_n) \subseteq (\mathcal{T}'_n, F'_n)$$

if there is a one-to-one homomorphism

$$H\colon \mathcal{T}_n \to \mathcal{T}'_n$$

of n-segments of algebraic theories such that $F'_n \circ H = F_n$. Similarly, we write $(\mathcal{T}, F) \subseteq (\mathcal{T}', F')$ if there exists a one-to-one homomorphism $H\colon \mathcal{T} \to \mathcal{T}'$ with $F \circ H = F'$.

1.6. Problem 1 in Taylor's monograph [**8**] asked about the existence of topological spaces X and Y (as nice as possible) whose monoids of continuous selfmaps are isomorphic, while their algebraic theories $\mathrm{Clo}(X)$ and $\mathrm{Clo}(Y)$ are **not** elementarily equivalent. This problem was solved in [**9**] where, for every $n \geq 2$, the second author constructed metrizable spaces X and Y such that $\mathrm{Clo}_n(X)$ is isomorphic to $\mathrm{Clo}_n(Y)$, but $\mathrm{Clo}_{n+1}(X)$ is not elementarily equivalent to $\mathrm{Clo}_{n+1}(Y)$. This result spawned further investigations of equality, isomorphism and elementary equivalence of clone segments. As we proceed, we describe certain results of these investigations, and particularly those needed to prove our two results.

1.7. For a nonvoid topological space $X = (P, \tau)$, let $\mathrm{Clo}(X)$ denote not only its algebraic theory (as in 1.4), but also the clone $(\mathrm{Clo}(X), F)$ where $F\colon \mathrm{Clo}(X) \to \mathrm{Set}$ is the domain restriction of the forgetful functor $\mathrm{Top} \to \mathrm{Set}$, that is, the functor which sends the algebraic theory $\mathrm{Clo}(X)$ onto the clone $\mathrm{Clo}(P)$ on the nonvoid set P, and let $\mathrm{Clo}_n(X)$ denote its n-segment. With this convention applied, the initial result of [**9**] was extended in [**6**] as follows.

[**6**]: For every triple n_1, n_2, n_3 of elements of the set $\{2, 3, 4, \ldots, \infty\}$ satisfying

$$n_1 \le n_2 \le n_3,$$

there exist metrizable spaces $X_1 = (P, \tau_1)$ and $X_2 = (P, \tau_2)$ such that

$$n_j = \sup\{m \mid \mathrm{Clo}_m(X_1) \overset{j}{\simeq} \mathrm{Clo}_m(X_2)\} \text{ for } j = 1, 2, 3.$$

Below is a similar extension of the result of [**9**], this time to the category $\mathrm{Alg}(\Delta)$ whose infinite type Δ is that of one binary and countably many unary operations.

[**11**]: For every triple n_1, n_2, n_3 of elements of the set $\{2, 3, 4, \ldots, \infty\}$ satisfying

$$n_1 \le n_2 \le n_3,$$

there exist algebras $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{Alg}(\Delta)$ with the same underlying set, and such that

$$n_j = \sup\{m \mid \mathrm{Clo}_m(\mathcal{A}_1) \overset{j}{\simeq} \mathrm{Clo}_m(\mathcal{A}_2)\} \text{ for } j = 1, 2, 3.$$

We reiterate the obvious fact that, for any algebra $\mathcal{A}$ in $\mathrm{Alg}(\Delta)$, the clone $\mathrm{Clo}(\mathcal{A})$ is formed by all **homomorphisms** between finite powers of $\mathcal{A}$. As such, the clone $\mathrm{Clo}(\mathcal{A})$ is the centralizer of an often-investigated clone of all term operations of $\mathcal{A}$, and this is why $\mathrm{Clo}(\mathcal{A})$ is usually called the **centralizer clone of** $\mathcal{A}$, see [**5**], for instance.

Our first result, Theorem 1 below, strengthens the algebraic result of [**11**] cited above by showing that the algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ satisfying its claim exist already in $\mathrm{Alg}(1, 1)$.

Theorem 1. *For any choice of* $n_1, n_2, n_3 \in \{2, 3, \ldots, \infty\}$ *satisfying*

$$n_1 \le n_2 \le n_3,$$

there exist algebras $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *in* $\mathrm{Alg}(1,1)$ *with the same underlying set, and such that*

$$n_j = \sup\{k \mid \mathrm{Clo}_k(\mathcal{A}_1) \overset{j}{\simeq} \mathrm{Clo}_k(\mathcal{A}_2)\} \text{ for } j = 1,2,3.$$

1.8. Additional concepts are needed to introduce our second result, an algebraic version of a clone representation theorem from [**7**].

Let $(\mathcal{K}_1, U_1)$ and $(\mathcal{K}_2, U_2)$ be concrete categories with concrete finite products, and let $\Phi\colon \mathcal{K}_1 \to \mathcal{K}_2$ be a functor satisfying $U_2 \circ \Phi = U_1$.

We recall that a leading theme of [**7**] is the comparison of clone segments of a quadruple of objects

$$a_1, a_2, \Phi(a_1), \Phi(a_2) \text{ with } a_1, a_2 \in \mathrm{obj}\,\mathcal{K}_1$$

as to their equality and isomorphism. To be specific, first we denote

$$\begin{aligned} t_i &= \sup\{k \mid \mathrm{Clo}_k(a_i) = \mathrm{Clo}_k(\Phi(a_i))\} \quad \text{for } i = 1,2,\\ r &= \sup\{k \mid \mathrm{Clo}_k(a_1) = \mathrm{Clo}_k(a_2)\},\\ s &= \sup\{k \mid \mathrm{Clo}_k(\Phi(a_1)) = \mathrm{Clo}_k(\Phi(a_2))\}, \end{aligned}$$

and then, having replaced the equality $\mathrm{Clo}_k(\dots) = \mathrm{Clo}_k(\dots)$ in these four expressions by $\mathrm{Clo}_k(\dots) \simeq \mathrm{Clo}_k(\dots)$, we obtain four more parameters $\tilde{t}_1$, $\tilde{t}_2$, $\tilde{r}$, $\tilde{s}$ reflecting isomorphism properties of the respective clone segments. (Elementary equivalence is not investigated in [**7**].) Since the equality and the isomorphism are transitive relations, it is not difficult to see that in the quadruples $R = (r, s, t_1, t_2)$ and $\widetilde{R} = (\tilde{r}, \tilde{s}, \tilde{t}_1, \tilde{t}_2)$

(g) no entry is strictly smaller than the remaining three.

Just as in [**7**], quadruples satisfying (g) are called **grounded quadruples**. Since equal segments are always isomorphic, we have $r \le \tilde{r}$, $s \le \tilde{s}$ and $t_i \le \tilde{t}_i$ for $i = 1,2$. The latter four relations will be abbreviated by writing $R \le \widetilde{R}$.

The paper [**7**] compares clone segments of topological spaces X_1 and X_2 and of their suitable topological modifications ΦX_1 and ΦX_2 (where Φ could be, amongst others, the compactly generated modification, the sequential modification, or the completely regular modification). It is shown there that any two grounded quadruples $R \le \widetilde{R}$ whose entries belong to the set $\{2, 3, 4, \dots, \infty\}$ can be realized, in the sense described above, by clone segments of two topological spaces and their suitable modifications. Our Theorem 2 below is an algebraic instance of such realization. We let

$$\Phi\colon\ \mathrm{Alg}(1,1,1) \to \mathrm{Alg}(1,1)$$

denote the functor eliminating the last operation of any algebra in its domain, that is, the functor sending every algebra $(A, \{\alpha, \beta, \gamma\})$ in its domain to its reduct $(A, \{\alpha, \beta\})$.

Theorem 2. *For every pair $R \leq \widetilde{R}$ of grounded quadruples of elements of $\{2,3,4,\ldots,\infty\}$, there exist algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in* $\mathrm{Alg}(1,1,1)$ *on the same underlying set, and such that*

$$(*)\qquad \begin{cases} t_i = \sup\{k \mid \mathrm{Clo}_k(\mathcal{A}_i) = \mathrm{Clo}_k(\Phi(\mathcal{A}_i))\} \quad \textit{for } i=1,2,\\ r = \sup\{k \mid \mathrm{Clo}_k(\mathcal{A}_1) = \mathrm{Clo}_k(\mathcal{A}_2)\},\\ s = \sup\{k \mid \mathrm{Clo}_k(\Phi(\mathcal{A}_1)) = \mathrm{Clo}_k(\Phi(\mathcal{A}_2))\},\\ \quad \textit{and}\\ \tilde{t}_i = \sup\{k \mid \mathrm{Clo}_k(\mathcal{A}_i) \simeq \mathrm{Clo}_k(\Phi(\mathcal{A}_i))\} \quad \textit{for } i=1,2,\\ \tilde{r} = \sup\{k \mid \mathrm{Clo}_k(\mathcal{A}_1) \simeq \mathrm{Clo}_k(\mathcal{A}_2)\},\\ \tilde{s} = \sup\{k \mid \mathrm{Clo}_k(\Phi(\mathcal{A}_1)) \simeq \mathrm{Clo}_k(\Phi(\mathcal{A}_2))\}. \end{cases}$$

The remainder of the paper proves Theorem 1 and Theorem 2. Throughout the paper, we recall all relevant and needed parts [**6**], [**7**], [**11**] and comment upon their nature, in the hope that the reader may find these comments useful. Our general theme is twofold: while algebraic tools clarify certain topological results, these topological results do, in turn, have algebraic consequences. In conclusion, we propose a problem that naturally arises from these considerations.

2. Algebra Helps Topology

2.1. As noted already in the first section, in [**9**] it was shown that for every integer $n \geq 2$ there exist metrizable spaces $X_1 = (P,\tau_1)$ and $X_2 = (P,\tau_2)$ such that $\mathrm{Clo}_n(X_1) = \mathrm{Clo}_n(X_2)$ but $\mathrm{Clo}_{n+1}(X_1)$ is not elementarily equivalent to $\mathrm{Clo}_{n+1}(X_2)$. Subsequently, variants and improvements of this result were presented and proved in a number of papers, such as [**6**], [**7**], [**10**], [**11**]. Surprisingly few purely topological arguments were needed, and algebraic and combinatorial reasoning dominated in the latter papers. In this section we briefly explain this phenomenon, and illustrate how an algebraic approach assists in the proof of Proposition 2.7, which we need to prove Theorem 2, our second main result.

2.2. All results applied here are based on representations of suitable algebraic theories by continuous maps. We denote these algebraic theories $\mathcal{T}(\Sigma,\Omega)$ because they depend on a given (monosorted finitary) type $\Sigma = \bigcup_{n=0}^{\infty}\Sigma_n$ with an infinite nullary part Σ_0 and a given subset $\Omega \subseteq \bigcup_{n=2}^{\infty}\Sigma_n$. The actual structure of $\mathcal{T}(\Sigma,\Omega)$ plays a central role in all papers mentioned in 2.1, and this is why it is also essential here. We recall it briefly below. For a more detailed description, we refer the reader to [**10**] or [**11**].

If $\Omega = \emptyset$, then $\mathcal{T}(\Sigma,\emptyset) = \mathcal{T}(\Sigma)$ is the free algebraic theory of the finitary type Σ. This is the algebraic theory with a base object a such that, for every integer $n \geq 0$, the set of all morphisms $a^n \to a$ is the Σ-algebra freely generated by the set $\{\pi_i^{(n)} \mid i \in n\}$. Explicitly, this algebra is the set $\bigcup_{k=0}^{\infty} M_k^{(n)}$ of all Σ-terms, in which the 'initial layer'

$$M_0^{(n)} = \{\pi_i^{(n)} \mid i \in n\} \cup \{\sigma \circ \tau^{(n)} \mid \sigma \in \Sigma_0\}$$

consists of Σ-terms of depth 0, and these are of two kinds: the terms $\pi_i^{(n)}: a^n \to a$ with $i \in n$ are the product projections, while the terms $\sigma \circ \tau^{(n)}$ are constants, that is, expressions in which $\sigma \in \Sigma_0$ and $\tau^{(n)}: a^n \to a^0$ is the unique morphism into the terminal object a^0 of $\mathcal{T}(\Sigma)$; next, for $k \geq 0$,

$$M_{k+1}^{(n)} = M_k^{(n)} \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \{\sigma(t_0, \ldots, t_{\operatorname{ar}\sigma - 1}) \mid t_i \in M_k^{(n)}\}$$

where $\sigma(t_0, \ldots, t_{\operatorname{ar}\sigma-1})$ is a Σ-term of depth $\leq k+1$, built from the Σ-terms $t_0, \ldots, t_{\operatorname{ar}\sigma-1}$ of depth $\leq k$ (and $\operatorname{ar}\sigma = m$ means that σ is an m-ary operation symbol, i.e. $\sigma \in \Sigma_m$). The $\mathcal{T}(\Sigma)$-morphisms $a^n \to a^m$ with $m > 1$ are just the m-tuples of morphisms $a^n \to a$. The composition in $\mathcal{T}(\Sigma)$, defined recursively in [**10**], is, in fact, nothing but a substitution for which $\pi_i^{(n)}$ with $i \in n$ play the role of variables.

For a given $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, we define a subtheory $\mathcal{T}(\Sigma, \Omega)$ of $\mathcal{T}(\Sigma)$ as follows. For any $\sigma \in \Omega$, we delete from $\mathcal{T}(\Sigma)$ all Σ-terms $a^n \to a$ of the form

$$\sigma(\pi_{\varphi(0)}^{(n)}, \ldots, \pi_{\varphi(\operatorname{ar}\sigma-1)}^{(n)})$$

where $\varphi\colon \operatorname{ar}\sigma \to n$ is a one-to-one map, and also all Σ-terms that contain at least one subterm of this form. For any $n \geq 0$, here is the precise recursive definition of the set $T^{(n)}$ of all morphisms $a^n \to a$ in $\mathcal{T}(\Sigma, \Omega)$:
$T^{(0)} = M^{(0)}$ and $T^{(n)} = \bigcup_{k=0}^{\infty} T_k^{(n)}$ for $n \geq 1$, where

$$\begin{aligned}
T_0^{(n)} &= M_0^{(n)} = \{\pi_0^{(n)}, \ldots, \pi_{m-1}^{(n)}\} \cup \{\sigma_0 \circ \tau^{(n)} \mid \sigma_0 \in \Sigma_0\},\\
T_1^{(n)} &= T_0^{(n)} \cup \bigcup_{\sigma \in \Sigma \setminus (\Sigma_0 \cup \Omega)} \big\{\sigma(t_0, \ldots, t_{\operatorname{ar}\sigma-1}) \mid t_i \in T_0^{(n)} \text{ for all } i \in \operatorname{ar}\sigma\big\}\\
&\quad \cup \bigcup_{\sigma \in \Omega} \big\{\sigma(t_0, \ldots, t_{\operatorname{ar}\sigma-1}) \mid t_i \in T_0^{(n)} \text{ for all } i \in \operatorname{ar}\sigma, \text{ and}\\
&\qquad \text{either } t_i = t_j \text{ for some } i, j \in \operatorname{ar}\sigma \text{ with } i \neq j\\
&\qquad \text{or } t_i = \sigma_0 \circ \tau^{(n)} \text{ for some } i \in \operatorname{ar}\sigma \text{ and } \sigma_0 \in \Sigma_0\big\},\\
T_{k+1}^{(n)} &= T_k^{(n)} \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \big\{\sigma(t_0, \ldots, t_{\operatorname{ar}\sigma-1}) \mid t_i \in T_k^{(n)} \text{ for all } i \in \operatorname{ar}\sigma\\
&\qquad \text{and } t_j \notin T_{k-1}^{(n)} \text{ for some } j \in \operatorname{ar}\sigma\big\}, \text{ for any } k \geq 1.
\end{aligned}$$

2.3. If $\mathcal{K}$ is a category with finite products and $\mathcal{T}$ is an algebraic theory with a base object a, then a functor $F\colon \mathcal{T} \to \mathcal{K}$ is called a **representation of $\mathcal{T}$ in $\mathcal{K}$**

whenever it is full, one-to-one and preserves finite products. It is then clear that $\mathcal{T}$ and the full subcategory of $\mathcal{K}$ determined by $\{F(a^n) \mid n \in \omega\}$ are isomorphic algebraic theories. Since F preserves finite products, $(F(a^n), \{F(\pi_i^{(n)}) \mid i \in n\})$ is the product of n copies of $F(a)$. Hence the object $F(a)$ determines a representation F uniquely up to natural equivalence.

All results on clone segments in Top are based on the following theorem.

Theorem ([**10**], [**11**]). *Let Σ be a type such that*

$$\operatorname{card} \Sigma_0 \geq 2^{\aleph_0} + \operatorname{card}(\Sigma \setminus \Sigma_0),$$

and let $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$. Then the algebraic theory $\mathcal{T}(\Sigma, \Omega)$ has a representation in Top. *Moreover, the space representing $\mathcal{T}(\Sigma, \Omega)$ may be chosen to be metrizable.*

While formulated explicitly in [**10**], this theorem was proved implicitly already in [**11**]. (The purpose of [**10**], however, was to represent $\mathcal{T}(\Sigma, \Omega)$ in the category Metr of all metric spaces and their non-expanding maps.)

We shall need certain additional facts and notions concerning the spaces used in the proof of this Theorem.

2.4. Suppose that a type Σ with $\operatorname{card} \Sigma_0 \geq 2^{\aleph_0} + \operatorname{card}(\Sigma \setminus \Sigma_0)$ is given. Let

$$\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$$

denote the initial Σ-algebra, that is, the Σ-algebra freely generated by the empty set. Thus P is the set $P = \bigcup_{k=0}^{\infty} P_k$ of Σ-terms

$$\begin{aligned} P_0 &= \Sigma_0 \\ P_{k+1} &= P_k \cup \bigcup_{\sigma \in \Sigma \setminus \Sigma_0} \{\sigma(t_0, \dots, t_{\operatorname{ar}\sigma - 1}) \mid t_i \in P_k\} \end{aligned}$$

and each operation $p_\sigma \colon P^{\operatorname{ar}\sigma} \to P$ of $\mathbb{P}$ is given by

$$p_\sigma(t_0, \dots, t_{\operatorname{ar}\sigma-1}) = \sigma(t_0, \dots, t_{\operatorname{ar}\sigma-1}).$$

Then, for every $n \in \omega$, there is a natural one-to-one map $\Lambda^{(n)}$ of the set $\bigcup_{k=0}^{\infty} M_k^{(n)}$ of all $\mathcal{T}(\Sigma)$-morphisms $a^n \to a$ (see 2.2) into the set of all maps $P^n \to P$ defined inductively as follows:

$$\begin{aligned} \Lambda^{(n)}(\pi_i^{(n)}) &= p_i^{(n)} && \text{where } p_i^{(n)} \colon P^n \to P \text{ is the } i\text{-th Cartesian projection;} \\ \Lambda^{(n)}(\sigma \circ \tau^{(n)}) &= c_\sigma^{(n)} && \text{where } c_\sigma^{(n)} \colon P^n \to P \text{ is the constant map with the value } \sigma \in \Sigma_0 \subseteq P; \\ \Lambda^{(n)}(\sigma(t_0, \dots, t_{\operatorname{ar}\sigma-1})) &= p_\sigma \circ (\Lambda^{(n)}(t_0) \dot{\times} \cdots \dot{\times} \Lambda^{(n)}(t_{\operatorname{ar}\sigma-1})). \end{aligned}$$

Hence, for example, for any $\sigma \in \Sigma_n$ with $n > 0$,

$$p_\sigma = \Lambda^{(n)}(\sigma(\pi_0^{(n)}, \dots, \pi_{n-1}^{(n)})).$$

Next, we recall that for any given

$$\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1),$$

the proof of the Theorem in 2.3 constructs a **metric** ϱ_Ω **on the set** P so that the space $X_\Omega = (P, \varrho_\Omega)$ is such that

(r) for any $n \in \omega$, a map $f\colon X_\Omega^n \to X_\Omega$ is continuous if and only if $f = \Lambda^{(n)}(t)$ for some $t\colon a^n \to a$ in $\mathcal{T}(\Sigma, \Omega)$.

Hence the space X_Ω determines a representation $F_\Lambda : \mathcal{T}(\Sigma, \Omega) \to \mathrm{Top}$ of $\mathcal{T}(\Sigma, \Omega)$.

Furthermore, we recall that the space X_Ω has these two properties:

(α) if $\sigma \in \Sigma \setminus (\Sigma_0 \cup \Omega)$, then p_σ is a homeomorphism of $X_\Omega^{\operatorname{ar}\sigma}$ onto a closed subspace of X_Ω and

(β) if $\sigma \in \Omega$, then p_σ is not continuous, but p_σ^{-1} is a continuous map of the closed subset $p_\sigma(P^{\operatorname{ar}\sigma})$ of X_Ω onto $X_\Omega^{\operatorname{ar}\sigma}$.

We remark that the construction of the metric ϱ_Ω is an application of the topological Main Theorem of [**9**], enriched again by certain algebraic arguments. Statements (α) and (β) were explicitly formulated and proved in [**11**].

2.5. Hence, given a type Σ with

$$\operatorname{card}\Sigma \geq 2^{\aleph_0} + \operatorname{card}(\Sigma \setminus \Sigma_0),$$

and any $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, all spaces $X_\Omega = (P, \varrho_\Omega)$ on the underlying set P of the initial Σ-algebra $\mathbb{P} = (P, \{p_\sigma \mid \sigma \in \Sigma\})$ that determine representations $F_\Omega\colon \mathcal{T}(\Sigma, \Omega) \to \mathrm{Top}$ are constructed 'in a uniform way': for any choice of Ω whatsoever, the same collection $\Lambda = \{\Lambda^{(n)} \mid n \in \omega\}$ determines which maps in $\{P^n \to P \mid n \in \omega\}$ are continuous maps $X_\Omega^n \to X_\Omega$. Thus if $U\colon \mathrm{Top} \to \mathrm{Set}$ denotes the forgetful functor, then

$$\Lambda^{(n)}(t) = U \circ F_\Omega(t)$$

for every $\Omega \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, $n \in \omega$ and $t\colon a^n \to a$ in $\mathcal{T}(\Sigma, \Omega)$. The collection $\Lambda = \{\Lambda^{(n)} \mid n \in \omega\}$ has an obvious extension to a unique faithful functor, again denoted

$$\Lambda\colon \mathcal{T}(\Sigma) \to \mathrm{Set},$$

and the above formula $\Lambda^{(n)}(t) = U \circ F_\Omega(t)$ then says that the clones $\mathrm{Clo}(X_\Omega)$ and $(\mathcal{T}(\Sigma, \Omega), \Lambda_\Omega)$ are equal in the sense of 1.5 — where $\Lambda_\Omega\colon \mathcal{T}(\Sigma, \Omega) \to \mathrm{Set}$ denotes the domain-restriction of Λ, of course.

2.6. Let $\mathrm{Clo}(\Sigma,\Omega)$ denote the clone $(\mathcal{T}(\Sigma,\Omega),\Lambda_\Omega)$ just defined. By 2.4, given any choice $n_1 \le n_2 \le n_3$ of elements from $\{2,3,\ldots,\infty\}$, to construct spaces X_1 and X_2 with

$$n_j = \sup\{k \mid \mathrm{Clo}_k(X_1) \overset{j}{\simeq} \mathrm{Clo}_k(X_2)\} \quad \text{for } j = 1,2,3,$$

we need only find Σ with $\mathrm{card}\,\Sigma_0 \ge 2^{\aleph_0} + \mathrm{card}(\Sigma \setminus \Sigma_0)$ and $\Omega_1,\Omega_2 \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ such that

$$n_j = \sup\{k \mid \mathrm{Clo}_k(\Sigma,\Omega_1) \overset{j}{\simeq} \mathrm{Clo}_k(\Sigma,\Omega_2)\} \text{ for } j = 1,2,3.$$

Then the spaces $X_1 = X_{\Omega_1}$ and $X_2 = X_{\Omega_2}$ will have the required properties. This, indeed, is how such spaces were constructed in [**6**].

2.7. Proposition. *Let $R = (r,s,t_1,t_2)$ and $\widetilde{R} = (\tilde r,\tilde s,\tilde t_1,\tilde t_2)$ be grounded quadruples of elements of $\{2,3,4,\ldots,\infty\}$, and let $R \le \widetilde{R}$. Then there exist metrizable spaces X_1, X_2, X_3 and X_4 such that*

(i) $\mathrm{Clo}(X_1) \subseteq \mathrm{Clo}(X_3)$ *and* $\mathrm{Clo}(X_2) \subseteq \mathrm{Clo}(X_4)$.
(ii) *The following statements hold:*

$$\begin{aligned} t_i &= \sup\{k \mid \mathrm{Clo}_k(X_i) = \mathrm{Clo}_k(X_{i+2})\} \quad \textit{for } i = 1,2,\\ r &= \sup\{k \mid \mathrm{Clo}_k(X_1) = \mathrm{Clo}_k(X_2)\},\\ s &= \sup\{k \mid \mathrm{Clo}_k(X_3) = \mathrm{Clo}_k(X_4)\},\end{aligned}$$

and

$$\begin{aligned} \tilde t_i &= \sup\{k \mid \mathrm{Clo}_k(X_i) \simeq \mathrm{Clo}_k(X_{i+2})\} \quad \textit{for } i = 1,2,\\ \tilde r &= \sup\{k \mid \mathrm{Clo}_k(X_1) \simeq \mathrm{Clo}_k(X_2)\},\\ \tilde s &= \sup\{k \mid \mathrm{Clo}_k(X_3) \simeq \mathrm{Clo}_k(X_4)\}.\end{aligned}$$

Proof. By 2.6, it suffices to find a type Σ with

$$\mathrm{card}\,\Sigma \ge 2^{\aleph_0} + \mathrm{card}(\Sigma \setminus \Sigma_0),$$

and select $\Omega_1,\Omega_2,\Omega_3,\Omega_4 \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$ so that the statements corresponding to those claimed by our Proposition are satisfied for $\mathrm{Clo}_k(\Sigma,\Omega_j)$ with $j = 1,2,3,4$. In general, a proper selection of these four sets depends on the mutual relations of **all** entries in the two quadruples, and requires a lengthy discussion of numerous separate cases. Fortunately, the existence of these sets Ω_j was established already in [**7**], where such four algebraic theories $\mathcal{T}(\Sigma,\Omega_j)$ were also needed.

(In passing, we note that the topological part of [**7**] was made more complex by the requirement that the spaces representing $\mathcal{T}(\Sigma,\Omega_3)$ and $\mathcal{T}(\Sigma,\Omega_4)$ be certain specific modifications of the spaces respectively representing $\mathcal{T}(\Sigma,\Omega_1)$ and $\mathcal{T}(\Sigma,\Omega_2)$).

We thus illustrate the proof only in its simplest non-trivial case, that of all eight entries of the two quadruples equal to some integer $n \geq 2$. We choose $\Sigma = \Sigma_0 \cup \Sigma_n$ with $\operatorname{card} \Sigma_0 = 2^{\aleph_0}$ and $\Sigma_n = \{\alpha, \beta\}$, so that the type Σ has precisely two n-ary operation symbols and $2^{\aleph_0}$ nullary operation symbols. If we set $\Omega_1 = \{\alpha, \beta\}$, $\Omega_2 = \{\beta\}$, $\Omega_3 = \{\alpha\}$, and $\Omega_4 = \emptyset$, then the clones $\mathrm{Clo}_k(\Sigma, \Omega_j)$, $j = 1, \ldots, 4$ satisfy (1) and (2), but with just one exception: it is not clear why, say, the segment $\mathrm{Clo}_{n+1}(\Sigma, \Omega_1)$ is not isomorphic to $\mathrm{Clo}_{n+1}(\Sigma, \Omega_2)$. In [**6**], however, the composites $\gamma = \sigma \circ (\pi^{(n)}_{\psi(0)} \dot{\times} \cdots \dot{\times} \pi^{(n)}_{\psi(n-1)})$ with $\sigma \in \Sigma_n$ and a permutation $\psi\colon n \to n$, called n-**cells** there, were characterized in terms of the first order language of clone segments. Consequently, the number of n-cells is invariant under any isomorphism of clone segments. Since the number of operation symbols in $\Sigma_n \setminus \Omega$ and the number of n-cells in $\mathrm{Clo}_{n+1}(\Sigma, \Omega)$ determine each other uniquely, for $\Omega_1 = \{\alpha, \beta\}$ and $\Omega_2 = \{\beta\}$ it then immediately follows that $\mathrm{Clo}_{n+1}(\Sigma, \Omega_1) \not\simeq \mathrm{Clo}_{n+1}(\Sigma, \Omega_2)$.$\square$

3. Topology Helps Algebra

3.1. Let Δ be the type of one binary operation symbol denoted as $\circ$ and $\aleph_0$ unary operation symbols denoted as γ and π_k for $k \in \omega$.

In [**11**], the topological result of [**6**] cited in 1.7 was translated to a similar result for the category $\mathrm{Alg}(\Delta)$:

[**11**]: for any choice $n_1 \leq n_2 \leq n_3$ of members of the set $\{2, 3, \ldots, \infty\}$, there exist algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathrm{Alg}(\Delta)$ such that

$$n_j = \sup\{k \mid \mathrm{Clo}_k(\mathcal{A}_1) \overset{j}{\simeq} \mathrm{Clo}_k(\mathcal{A}_2)\} \text{ for } j = 1, 2, 3.$$

Since this translation is not entirely straightforward, we outline it next. Then, in 3.5 below, we shall use it to translate Proposition 2.7 to an analogous result for the category $\mathrm{Alg}(\Delta)$.

3.2. In [**11**], to any metrizable topological space $X = (P, \tau)$ we assigned an algebra $\mathbb{A}(X) = (Q, \{\circ^X, \gamma^X\} \cup \{\pi^X_k \mid k \in \omega\})$ in $\mathrm{Alg}(\Delta)$, and called it its **algebraic trace**. (In what follows, we shall omit the superscript X, and use $\circ$, γ, π_k also to denote the operations of the trace $\mathbb{A}(X)$). The underlying set Q of $\mathbb{A}(X)$ is the disjoint union

$$Q = P \cup S \cup \{\lambda\},$$

in which the set S consists of all one-to-one sequences $\{p_k \mid k \in \omega\}$ of members of P. The operations of $\mathbb{A}(X)$ are defined as follows:

$$q \circ q' = \begin{cases} q, & \text{if } q = q' \in P \\ \lambda, & \text{otherwise;} \end{cases}$$

for every $k \in \omega$,

$$\pi_k(q) = \begin{cases} p_k, & \text{if } q = \{p_k \mid k \in \omega\} \in S \\ q, & \text{for every } q \in P \cup \{\lambda\}; \end{cases}$$

and finally, the operation γ that traces the topology τ of X is given by

$$\gamma(q) = \begin{cases} p, & \text{if } q \in S \text{ and } q \text{ converges to } p \text{ in } X \\ \lambda, & \text{if } q \in S \text{ does not converge in } X \\ q, & \text{for every } q \in P \cup \{\lambda\}. \end{cases}$$

It is clear that $\mathbb{A}(X)$ is isomorphic to $\mathbb{A}(X')$ if and only if X is homeomorphic to X'.

3.3. Let $k \in \omega$. It is clear that a successful translation of Proposition 2.7 to its analogon in $\mathrm{Alg}(\Delta)$ by means of algebraic trace $\mathbb{A}(X)$ of a metrizable space X calls for a close relationship between continuous maps

$$X^k \to X$$

and homomorphisms

$$\mathbb{A}(X)^k \to \mathbb{A}(X).$$

Statements (a) and (b) below were proved in [**11**] by purely algebraic arguments depending only on the definition of the operations $\circ$, γ, π_k of $\mathbb{A}(X)$:

(a) Any homomorphism $h\colon \mathbb{A}(X)^k \to \mathbb{A}(X)$ is either the constant map $\lambda^{(k)}$ with the value λ, or else $h(P^k) \subseteq P$ and the domain-range restriction of h maps X^k continuously into X.

(b) If $h_1, h_2\colon \mathbb{A}(X)^k \to \mathbb{A}(X)$ are homomorphisms such that $h_1(p) = h_2(p)$ for all $p \in P^k$, then $h_1 = h_2$.

Since nothing more can be proved about algebraic traces of metrizable spaces in general, algebra needs the help from topology now. This is why [**11**] brings in the following notion:

> a topological space X is **disciplined** if every continuous map $f\colon X^k \to X$ with $k \in \omega$ decomposes as
>
> $$f = g \circ \pi^{(M)} \text{ for some } M \subseteq n,$$
>
> where $\pi^{(M)}\colon X^n \to X^M$ is the surjective projection given by
>
> $$[\pi^{(M)}(x_0, \dots, x_{n-1})](m) = x_m \text{ for all } m \in M,$$
>
> and $g\colon X^M \to X$ is a homeomorphism onto a closed subset of X.

The help from topology comes in the form of a following statement, proved in [**11**].

(e) If $X = (P, \tau)$ is a metrizable space that is disciplined, then every continuous map $f\colon X^k \to X$ extends to a homomorphism $\mathbb{A}(f)\colon \mathbb{A}(X)^k \to \mathbb{A}(X)$. The extension $\mathbb{A}(f)$ is constant if and only if f is constant.

3.4. From the three statements in 3.3 it follows that, for any disciplined metrizable space $X = (P, \tau)$, the clone $\mathrm{Clo}(\mathbb{A}(X))$ of its trace $\mathbb{A}(X)$ is the least clone containing $\mathrm{Clo}(X)$ and all 'external' constant maps $\lambda^{(k)}$ with $k \in \omega$. This close similarity of the two clones implies the claim below, proved in [**11**]. We recall that a point x of a space X is its **absolute fixpoint** if $f(x) = x$ for every nonconstant continuous map $f \colon X \to X$, and note that the absence of absolute fixpoints is needed only in arguments involving elementary equivalence.

[**11**] If $X_1 = (P, \tau_1)$ and $X_2 = (P, \tau_2)$ are disciplined metrizable spaces with no absolute fixpoints, then for every $k \in \omega$ and for $j = 1, 2, 3$,

$$\mathrm{Clo}_k(X_1) \overset{j}{\simeq} \mathrm{Clo}_k(X_2) \text{ if and only if } \mathrm{Clo}_k(\mathbb{A}(X_1)) \overset{j}{\simeq} \mathrm{Clo}_k(\mathbb{A}(X_2)).$$

To translate the topological result of [**6**] to the algebraic result of [**11**] described in 3.1, we still need to know that all spaces $X_\Omega = (P, \varrho_\Omega)$ determining representations of the algebraic theories $\mathcal{T}(\Sigma, \Omega)$, see 2.4, have no absolute fixpoints (which is easy) and that they are disciplined. The (partially) topological proof of the second claim, presented in [**11**], requires not only that the continuous maps $X_\Omega^k \to X_\Omega$ are precisely the maps $\Lambda^{(k)}(t)$ with $t \colon a^k \to a$ in $\mathcal{T}(\Sigma, \Omega)$, but also the properties of X_Ω formulated as (α), (β) in 2.4.

This concludes the needed review of the proof of the result from [**11**] quoted in 3.1. Now we proceed to prove a weaker version of Theorem 2.

3.5. Proposition. *Let Δ be the countable finitary type from* 3.1*, and let $\Delta' = \Delta \cup \{\delta\}$ be the type extending Δ by a new unary operation symbol. Let*

$$\Phi \colon \mathrm{Alg}(\Delta') \to \mathrm{Alg}(\Delta)$$

denote the forgetful functor which sends any Δ'-algebra to its Δ-reduct.

Let $R = (r, s, t_1, t_2)$ and $\widetilde{R} = (\tilde{r}, \tilde{s}, \tilde{t}_1, \tilde{t}_2)$ be grounded quadruples of elements of $\{2, 3, 4, \dots, \infty\}$ such that $R \leq \widetilde{R}$. Then there exist algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathrm{Alg}(\Delta')$ *that satisfy* (*) *of Theorem* 2.

Proof. For given grounded quadruples R and $\widetilde{R}$, from Proposition 2.7 we obtain metrizable spaces $X_j = X_{\Omega_j}$ for $j = 1, \dots, 4$, associated with suitable sets $\Omega_j \subseteq \Sigma \setminus (\Sigma_0 \cup \Sigma_1)$, and such that (1), (2) of 2.7 are satisfied. These four spaces have the same underlying set P, that is, $X_j = (P, \tau_j)$ for $j = 1, \dots, 4$. Since they satisfy also (α) and (β) of 2.3, these spaces are disciplined, and have no absolute fixpoint, by [**11**]. Consequently, their algebraic traces $\mathbb{A}(X_j) \in \mathrm{Alg}(\Delta)$ satisfy

$$\begin{aligned} t_i &= \sup\{k \mid \mathrm{Clo}_k(\mathbb{A}(X_i)) = \mathrm{Clo}_k(\mathbb{A}(X_{i+2}))\} \text{ for } i = 1, 2, \\ r &= \sup\{k \mid \mathrm{Clo}_k(\mathbb{A}(X_1)) = \mathrm{Clo}_k(\mathbb{A}(X_2))\}, \\ s &= \sup\{k \mid \mathrm{Clo}_k(\mathbb{A}(X_3)) = \mathrm{Clo}_k(\mathbb{A}(X_4))\}, \end{aligned}$$

and also analogous equalities for $\tilde{t}_i$, $\tilde{r}$, $\tilde{s}$ and isomorphisms $\mathrm{Clo}_k(\dots) \simeq \mathrm{Clo}_k(\dots)$.

For $j = 1, \dots, 4$, let $\circ^{(j)}$, $\gamma^{(j)}$, $\pi_k^{(j)}$ denote the operations of $\mathbb{A}(X_j)$ corresponding to the operation symbols $\circ$, γ, π_k in Δ, and let Q denote the underlying set of these four algebras. Since $\mathrm{Clo}(X_1) \subseteq \mathrm{Clo}(X_3)$, every continuous map $f\colon X_1^m \to X_1$ is continuous also as a map $X_3^m \to X_3$. Hence its extension $\mathbb{A}(f)\colon \mathbb{A}(X_1)^m \to \mathbb{A}(X_1)$ (as defined in 3.3) is also a homomorphism of $\mathbb{A}(X_3)^m \to \mathbb{A}(X_3)$. This means that $\gamma^{(3)} \circ \mathbb{A}(f) = [\mathbb{A}(f)] \circ (\gamma^{(3)})^m$, so that adding the operation $\gamma^{(3)}$ to the original operations $\circ^{(1)}$, $\gamma^{(1)}$, $\pi_k^{(1)}$ of $\mathbb{A}(X_1)$ produces an algebra $\mathcal{A}_1$ in $\mathrm{Alg}(\Delta')$ such that $\mathrm{Clo}(\mathcal{A}_1) = \mathrm{Clo}\,\mathbb{A}(X_1)$ in the sense of 1.5. Set $\delta^{(1)} = \gamma^{(1)}$. Since the forgetful functor Φ eliminates δ, we have $\Phi(\mathcal{A}_1) = \mathbb{A}(X_3)$. Similarly, since $\mathrm{Clo}(X_2) \subseteq \mathrm{Clo}(X_4)$, adding the operation $\gamma^{(4)}$ to the operations $\circ^{(2)}$, $\gamma^{(2)}$, $\pi_k^{(2)}$ of $\mathbb{A}(X_2)$ produces an algebra $\mathcal{A}_2$ in $\mathrm{Alg}(\Delta')$ with $\mathrm{Clo}\,\mathcal{A}_2 = \mathrm{Clo}\,\mathbb{A}(X_2)$ and, setting $\delta^{(2)} = \gamma^{(2)}$ yields $\Phi(\mathcal{A}_2) = \mathbb{A}(X_4)$. The algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ thus have all the required properties. $\square$

4. Type Reduction

4.1. Proposition. *For any countable finitary type Δ, there exists a one-to-one full functor $\Psi\colon \mathrm{Alg}(\Delta) \to \mathrm{Alg}(1,1)$ that preserves finite products.*

Proof. The functor Ψ will be a composite of three functors having the properties we desire for Ψ. All three functors, and hence also Ψ itself will be carried by the functor $\hom(\omega, -)\colon \mathrm{Set} \to \mathrm{Set}$. In particular, all these functors will assign algebras with the underlying set Z^ω to algebras whose underlying set is Z. We shall use the following observation in all three steps of the proof.

(0) For each $k \in \omega$, define $p_k\colon Z^\omega \to Z^\omega$ by

$$[p_k(\varphi)](i) = \varphi(k) \text{ for all } i \in \omega.$$

Let Δ_1 be the type of ω unary operations. It is well-known and easy to verify that the functor $\Psi_0\colon \mathrm{Alg}(\emptyset) \to \mathrm{Alg}(\Delta_1)$ given by $\Psi_0(Z) = (Z^\omega, \{p_k \mid k \in \omega\})$ is full. Since Ψ_0 is carried by the functor $\hom(\omega, -)$, it is one-to-one and preserves finite products.

(1) The first step is well-known. To any algebra $\mathcal{A} = (Z, \{\alpha_i \mid i \in \omega\})$ of a given countable finitary type $\Delta = \{n_i \mid i \in \omega\}$, we assign an algebra $\Psi_1(\mathcal{A}) = (Z^\omega, \{\alpha\} \cup \{p_k \mid k \in \omega\})$ of countable unary type Δ_1 whose operations p_k with $k \in \omega$ are those from (0), and the operation α is given by

$$[\alpha(\varphi)](i) = \alpha_i(\varphi \restriction n_i) \text{ for all } \varphi \in Z^\omega \text{ and } i \in \omega.$$

In view of (0), we only need to verify that Ψ_1 defines a full functor, but this is easy. First, from (0) it follows that for every homomorphism $g\colon \Psi_1(\mathcal{A}) \to \Psi_1(\mathcal{A}')$ there is a unique map $f\colon Z \to Z'$ such that $g(\varphi) = f \circ \varphi$ for all $\varphi \in Z^\omega$. Secondly, $f \circ \alpha(\varphi) = g(\alpha(\varphi)) = \alpha(g(\varphi)) = \alpha(f \circ \varphi)$, and hence $f(\alpha_i(\varphi \restriction n_i)) = \alpha_i'(f \circ (\varphi \restriction n_i))$ for every operation α_i of $\mathcal{A}$. Therefore $\Psi_1\colon \mathrm{Alg}(\Delta) \to \mathrm{Alg}(\Delta_1)$ is full.

(2) The second step modifies an idea from [**4**]. To any given (now unary) algebra $\mathcal{A} = (Z, \{\beta_i \mid i \in \omega\})$ from $\mathrm{Alg}(\Delta_1)$ we assign the algebra $\Psi_2(\mathcal{A}) = (Z^\omega, \{p_0, s, \beta\})$ in $\mathrm{Alg}(1,1,1)$, where p_0 is as in (0) and the operations $s, \beta \colon Z^\omega \to Z^\omega$ are defined, for any $\varphi \in Z^\omega$ and $i \in \omega$, by

$$[s(\varphi)](i) = \varphi(i+1) \quad \text{and} \quad [\beta(\varphi)](i) = \beta_i(\varphi(0)).$$

Again, we need only show that Ψ_2 is full. First we note that $(s^k(\varphi))(0) = \varphi(k)$ for any $k \geq 1$ in ω. Whence $p_0 \circ s^k = p_k$ is a term operation of $\Phi_2(\mathcal{A})$ for every $k \in \omega$. As in (1) above, from (0) it follows that any homomorphism $g \colon \Psi_2(\mathcal{A}) \to \Psi_2(\mathcal{A}')$ has the form $g(\varphi) = f \circ \varphi$, and using the operation β we easily find that $f \colon \mathcal{A} \to \mathcal{A}'$ is a homomorphism in $\mathrm{Alg}(\Delta_1)$.

(3) Finally, to any algebra $\mathcal{A} = (Z, \{\gamma_0, \gamma_1, \gamma_2\})$ from $\mathrm{Alg}(1,1,1)$ we assign the algebra $\Psi_3(\mathcal{A}) = (Z^\omega, \{s, \gamma\})$ in $\mathrm{Alg}(1,1)$, where s is defined as in (2) above and, for any $\varphi \in Z^\omega$,

$$[\gamma(\varphi)](i) = \begin{cases} \gamma_i(\varphi(0)), & \text{for } i = 0,1,2, \\ \varphi(0), & \text{for } i \in \omega \setminus 3. \end{cases}$$

Analogously to (2), it is easy to see that $s^3 \circ \gamma \circ s^k = p_k$ for all $k \in \omega$. Thus $g(\varphi) = f \circ \varphi$ once again, and $f \colon \mathcal{A} \to \mathcal{A}'$ is a homomorphism because of the first clause in the definition of γ. This shows that Ψ_3 is also full.

It is clear that the composite $\Psi = \Psi_3 \circ \Psi_2 \circ \Psi_1$ has the properties claimed. $\square$

4.2. Let $\Psi \colon \mathrm{Alg}(\Delta) \to \mathrm{Alg}(1,1)$ be the functor from Proposition 4.1. Let $\mathcal{A}$ be an algebra in $\mathrm{Alg}(\Delta)$ and let Z be its underlying set. Then, for each $m \in \omega$, the mapping $e_Z^{(m)} \colon (Z^\omega)^m \to (Z^m)^\omega$ given, for any $\varphi_0, \dots, \varphi_{m-1} \in Z^\omega$ and all $i \in \omega$ by

$$[e_Z^{(m)}(\varphi_0, \dots, \varphi_{m-1})](i) = (\varphi_0(i), \dots, \varphi_{m-1}(i))$$

carries an isomorphism $\epsilon_{\mathcal{A}}^{(m)} \colon \Psi(\mathcal{A})^m \to \Psi(\mathcal{A}^m)$ in $\mathrm{Alg}(1,1)$ with the inverse

$$\psi_{\mathcal{A}}^{(m)} = \Psi(\pi_0^{(m)}) \dot{\times} \cdots \dot{\times} \Psi(\pi_{m-1}^{(m)}).$$

Since Ψ is full and one-to-one, all $\mathrm{Alg}(1,1)$-homomorphisms $\Psi(\mathcal{A})^m \to \Psi(\mathcal{A})$ are of the form $\Psi(f) \circ \epsilon_{\mathcal{A}}^{(m)}$ with a uniquely determined $\mathrm{Alg}(\Delta)$-homomorphism $f \colon \mathcal{A}^m \to \mathcal{A}$. For any other algebra $\mathcal{A}'$ in $\mathrm{Alg}(\Delta)$ with the same underlying set Z, and for any $k \in \{2, 3, \dots, \infty\}$, we thus have

$$\mathrm{Clo}_k(\Psi(\mathcal{A})) \overset{j}{\simeq} \mathrm{Clo}_k(\Psi(\mathcal{A}')) \text{ if and only if } \mathrm{Clo}_k(\mathcal{A}) \overset{j}{\simeq} \mathrm{Clo}_k(\mathcal{A}').$$

for $j = 1, 2$. To see that this is true also for $j = 3$, that is, for the relation of elementary equivalence, we need only use the fact that $\mathrm{Clo}(\Psi(\mathcal{A}))$ is always isomorphic to $\mathrm{Clo}(\mathcal{A})$.

4.3. Our Theorem 1 now follows from 3.4, 4.2 and the result of [**11**] quoted in 3.1.

4.4. Now we turn to the proof of Theorem 2. For any algebra $\mathcal{A} = (Q, \{\circ, \gamma\} \cup \{\pi_k \mid k \in \omega\})$ in $\mathrm{Alg}(\Delta)$ as in 3.2, denote $\Psi(\mathcal{A}) = (Q^\omega, \{\alpha, \beta\})$. If $\Delta' = \Delta \cup \{\delta\}$ is the extended type introduced in 3.5, and $\tilde{\mathcal{A}} = (Q, \{\circ, \gamma\} \cup \{\pi_k \mid k \in \omega\} \cup \{\delta\})$ is an algebra in $\mathrm{Alg}(\Delta')$, then $\mathcal{A}$ is the reduct of $\tilde{\mathcal{A}}$ in $\mathrm{Alg}(\Delta)$. We define

$$\widetilde{\Psi}(\tilde{\mathcal{A}}) = (Q^\omega, \{\alpha, \beta, \mu\}),$$

where the unary operation $\mu\colon Q^\omega \to Q^\omega$ is given by

$$\mu(\varphi) = \delta \circ \varphi \text{ for all } \varphi \in Q^\omega.$$

The functor $\widetilde{\Psi}\colon \mathrm{Alg}(\Delta') \to \mathrm{Alg}(1,1,1)$ is clearly one-to-one and full. It also preserves finite products and completes the commutative square

$$\begin{tikzcd}
\mathrm{Alg}(\Delta') \arrow[r] \arrow[d, "\widetilde{\Psi}"'] & \mathrm{Alg}(\Delta) \arrow[d, "\Psi"] \\
\mathrm{Alg}(1,1,1) \arrow[r] & \mathrm{Alg}(1,1)
\end{tikzcd}$$

in which the horizontal arrows represent forgetful functors. Then, in view of 4.2, an application of Proposition 3.5 completes the proof of Theorem 2.

4.5. In both theorems, the representing algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are very large. An inspection of the respective proofs shows that the spaces X_1 and X_2 from which the algebras were constructed have cardinality $2^{\aleph_0}$ in Theorem 2, and $\max(2^{\aleph_0}, \aleph_2)$ in Theorem 1 (where the elementary (non)equivalence calls for theories $\mathcal{T}(\Sigma, \Omega_i)$ with a rather large set $\Sigma \setminus \Sigma_0$, see [**6**]). The 'translation' into $\mathrm{Alg}(1,1)$ does not enlarge these cardinalities, however.

4.6. Problem. *Could Theorem* 1 *and Theorem* 2 *be proved by purely algebraic methods? And would the new representing algebras* $\mathcal{A}_1$ *and* $\mathcal{A}_2$ *then be smaller?*

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