Acta Math. Univ. Comenianae
Vol. LXVII, 1(1998), pp. 181–195

THE FINITE VOLUME METHOD FOR AN ELLIPTIC–PARABOLIC EQUATION

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Abstract. In this note we prove the convergence of a finite volume scheme for the discretization of an elliptic-parabolic problem, namely the nonlinear diffusion equation $c(u)_t - \Delta u = 0$, together with Dirichlet boundary conditions and an initial condition. This is done by means of a priori estimates in L^2 and the use of Kolmogorov's theorem on relative compactness of subsets of L^2.

1. Introduction

In this note we prove the convergence of an implicit finite volume scheme for the numerical solution of the initial value problem for the elliptic-parabolic equation

$$c(u)_t - \Delta u = 0 \quad \text{in } Q_T = \Omega \times (0,T), \tag{1}$$

where Ω is a bounded connected open subset of $\mathbb{R}^N$ with smooth boundary and T a positive constant, together with the Dirichlet boundary condition

$$u = u^D \quad \text{on } \partial\Omega \times (0,T), \tag{2}$$

and the initial condition

$$c(u(x,0)) = c(u_0(x)) \quad \text{for all } x \in \Omega. \tag{3}$$

We denote by (P) the problem given by the equation (1), the boundary condition (2) and the initial condition (3). We assume that the function c satisfies the hypothesis

(H_c) c is a continuous nondecreasing function such that $c' \in L^1_{loc}(\mathbb{R})$;

and that the initial condition u_0 and the boundary data u^D satisfy the hypotheses

(H_0) $u_0 \in L^\infty(\Omega)$ and we define $U_0 := \|u_0\|_{L^\infty(\Omega)}$;

(H_D) u^D is Lipschitz continuous on $\overline{\Omega}$ with Lipschitz constant L_D.

Received November 17, 1997.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 35K55, 65M12, 65N12, 65N22; Secondary 76M25, 76S05.

Equation (1) changes type in $\Omega \times \mathbb{R}^+$: it is elliptic in regions where $c(u)$ is constant and parabolic elsewhere. Since we do not expect the solution to be smooth, we define a weak solution of Problem (P) as follows.

Definition 1.1. A function u is a weak solution of Problem (P) if

$$(4)\quad \begin{cases} \quad (i) \quad u - u_D \in L^2(0,T;H_0^1(\Omega)); \\ \quad (ii) \quad c(u) \in L^\infty(Q_T); \\ \quad (iii) \quad u \text{ satisfies the integral identity} \\ \displaystyle\int_0^T \int_\Omega \Big\{ \Big(c(u(x,t)) - c(u_0(x)) \Big) \psi_t(x,t) - \boldsymbol{\nabla} u(x,t)\, \boldsymbol{\nabla} \psi(x,t) \Big\}\, dx\, dt = 0, \\ \text{for all } \psi \in L^2(0,T;H_0^1(\Omega)) \text{ such that } \psi_t \in L^\infty(Q_T) \text{ and } \psi(.,T) = 0. \end{cases}$$

It follows from Otto [**Ott96**] that (P) has at most one weak solution.

Elliptic-parabolic equations have been studied a lot from the theoretical point of view. We refer in particular to the articles by van Duijn and Peletier [**VDP82**], Hulshof [**Hul86**], Hulshof and Wolanski [**HW88**], Alt and Luckhaus [**AL83**] and Otto [**Ott96**]. They prove the existence and the uniqueness of the solution of boundary value problems for class of equations including (1), as well as regularity properties of the interface between saturated and unsaturated regions.

For numerical studies we refer to Hornung [**Hor78**] for the discretization of the Richards equation by the finite difference method and to Knabner [**Kna87**] for its discretization by means of the finite element method. Kelanemer [**Kel94**] and Chounet, Hilhorst, Jouron, Kelanemer and Nicolas [**CHJKN97**] implement a mixed finite element method and Knabner et al [**Knaal97**] apply a slighty different finite volume scheme than the one presented here.

The purpose of this paper is to prove the convergence of a time implicit finite volume scheme for the discretization of Problem (P).

Finite volume schemes have first been developed by engineers in order to study complex coupled physical phenomena where the conservation of extensive quantities (such as masses, energy, impulsion, ...) must be carefully respected by the approximate solution. Another advantage of such schemes is that a large variety of meshes can be used. The basic idea is the following : one integrates the partial differential equation in each control volume and then approximates the fluxes across the volume boundaries.

Equation (1) is a simplified form of the Richards equation which is very basic in environmental sciences for computing the liquid pressure in aquifers and the velocity of groundwater flow. The finite volume method is one of the most popular method among the engineers performing computations in hydrology. Therefore it is of crucial importance to be able to present convergence proofs for precisely this method.

In Section 2, we introduce the finite volume scheme and define the approximate Problems $(P_{h,k})$. Then we prove the existence and uniqueness of the solution $u_{h,k}$ of Problem $(P_{h,k})$.

In Section 3, we derive a priori estimates. First we obtain an $L^\infty(Q_T)$-bound for $c(u_{h,k})$ and present an estimate on $u_{h,k}$ in a discrete norm corresponding to a norm in $L^2(0,T;\, H^1(\Omega))$. This yields an estimate on differences of space translates of $u_{h,k}$. Next we introduce the auxiliary function $W(s) = \displaystyle\int_0^s \min(c'(z),1)\, dz$ which we also rewrite as $W(s) = \mathcal{F}(c(s))$, where $\mathcal{F}$ is strictly increasing and continuous. We estimate differences of space and time translates of $W(u_{h,k})$, which imply that the sequence $\{W(u_{h,k})\}$ is relatively compact in $L^2(Q_T)$. A basic ingredient that we use to obtain these estimates is a discrete form of the Poincaré's inequality.

From these estimates, we deduce in Section 4 the existence of a subsequence of $\{u_{h,k}\}$ which converges to a function $u \in L^2(0,T;H^1(\Omega))$ weakly in $L^2(Q_T)$ and such that $\{c(u_{h,k})\}$ converges to a function χ strongly in $L^2(Q_T)$. Finally, we prove that $\chi = c(u)$ and that u is the unique weak solution of Problem (P).

For other articles about the convergence of the finite volume method for elliptic or parabolic equations, we refer to Baughman and Walkington [**BW93**], Herbin [**Her95**] and Eymard, Gallouët, Hilhorst and Naït Slimane [**EGHNS96**].

For the complete proofs of the results which we present here, we refer to [**Gut98**] and to [**EGHb**] where we consider the complete Richards equation which also involves a convection term. There we will suppose the Lipschitz continuity of the function c.

Acknowledgement. The authors are greatly indebted to Professor J. Kačur for his very constructive remarks about the regularity of the nonlinear function c.

2. The Finite Volume Scheme

In this section, we construct approximate solutions of Problem (P). To that purpose, we introduce a time implicit discretization and a finite volume scheme for the discretization in space. Let $\mathcal{T}$ be a mesh of Ω. The elements of $\mathcal{T}$ will be called control volumes in what follows. For any $(p,q) \in \mathcal{T}^2$ with $p \neq q$, we denote by $e_{pq} = \overline{p} \cap \overline{q}$ their common interface; it is included in a hyperplane of $\mathbb{R}^N$, which does not intersect p nor q. Then $m(e_{pq})$ denotes the measure of e_{pq} for the Lebesgue measure of the hyperplane, and $\vec{n}_{pq}$ denotes the unit vector normal to e_{pq}, oriented from p to q. The set of boundary control volumes is denoted by $\partial\mathcal{T} = \Big\{p \in \mathcal{T}, \mathrm{meas}(\partial p \cap \partial\Omega) \neq 0\Big\}$ and for all $p \in \partial\mathcal{T}$, we denote by e_p the intersection of the boundary of p and the boundary of Ω, i.e. $e_p = \partial p \cap \partial\Omega$.

We denote by $\mathcal{E}$ the set of pairs of adjacent control volumes together with the set of pairs (p, e_p) for all $p \in \partial\mathcal{T}$, that is $\mathcal{E} = \{(p,q) \in \mathcal{T}^2, p \neq q, m(e_{pq}) \neq 0\} \cup \{(p,e_p),\ p \in \partial\mathcal{T}\}$.

For all $p \in \mathcal{T} \setminus \partial\mathcal{T}$, $N(p) = \{q \in \mathcal{T}, (p,q) \in \mathcal{E}\}$ denotes the set of neighbors of p and for all $p \in \partial\mathcal{T}$, $N(p) = \{q \in \mathcal{T}, (p,q) \in \mathcal{E}\} \cup \{\partial p \cap \partial\Omega\}$ denotes the set of neighbors of p including the common boundary of p and Ω.

Furthermore, for all $p \in \mathcal{T}$, we denote by $m(p)$ the measure of p in $\mathbb{R}^N$.

We use the notation

$$h = \max_{p \in \mathcal{T}} \delta(p) \tag{5}$$

where $\delta(p)$ denotes the diameter of p, and suppose that there exists a family of points $x_p \in \Omega$ such that

$$\begin{aligned} &x_p \in p && \text{for all } p \in \mathcal{T}, \\ &\frac{x_q - x_p}{|x_q - x_p|} = \vec{n}_{pq} && \text{for all } (p,q) \in \mathcal{T}^2. \end{aligned} \tag{H$_\mathcal{T}$}$$

We denote by $d_{pq} = |x_q - x_q|$ and define the transmissivity by $T_{pq} = \dfrac{m(e_{pq})}{d_{pq}}$. If $p \in \partial\mathcal{T}$ and $q = e_p$, we define

$$T_{pq} = T_{p,e_p} = \frac{m(e_p)}{d_{p,e_p}}, \tag{6}$$

where $d_{p,e_p} = \left|x_{e_p} - x_p\right|$ and x_{e_p} is a point of e_p. We remark that Hypothesis (H$_\mathcal{T}$) means that e_{pq} and the segment $[x_p, x_q]$ are orthogonal.

The time implicit finite volume scheme is defined by the following equations in which $k > 0$ denotes the time step.

(i) The initial condition for the scheme is given by

$$u_p^0 = \frac{1}{m(p)} \int_p u_0(x)\; dx, \tag{7}$$

for all $p \in \mathcal{T}$;

(ii) The discrete equation

$$m(p)\,\frac{c(u_p^{n+1}) - c(u_p^n)}{k} - \sum_{q \in N(p)} T_{pq}\,(u_q^{n+1} - u_p^{n+1}) = 0, \tag{8}$$

for all $p \in \mathcal{T},\ n \in \{0, \dots, [T/k]\}$. The discrete Dirichlet condition is defined in the following way. For all $p \in \partial\mathcal{T}$ and for $q = e_p = \partial p \cap \partial\Omega$, we set

$$u_{e_p}^{n+1} = u_{e_p}^D = u^D(x_{e_p}), \tag{9}$$

where x_{e_p} is a point of e_p.

We remark that one cannot use an explicit finite volume scheme to solve the Richards equation since c can be constant on an interval of $\mathbb{R}^+$ so it has no inverse function.

This numerical scheme (7) and (8) allows to build an approximate solution, $u_{h,k}\colon \Omega \times \mathbb{R}^+ \mapsto \mathbb{R}$ given for all $p \in \mathcal{T}$ and all $n \in \{0, \dots, [T/k]\}$ by

$$u_{h,k}(x,t) = u_p^{n+1}, \qquad \text{for all } x \in p, \text{ for all } t \in (nk, (n+1)k]. \tag{10}$$

Since we have to deal with the inhomogeneous Dirichlet boundary condition $u = u^D$ on $\partial\Omega \times (0,T]$, we are led to consider the new unknown function

$$v_{h,k} = u_{h,k} - u_h^D, \tag{11}$$

where

$$u_h^D(x) = \begin{cases} u_p^D := u^D(x_p) & \text{if } x \in p, \\ u_{e_p}^D := u^D(x_{e_p}) & \text{if } x \in e_p. \end{cases} \tag{12}$$

Therefore

$$v_{h,k}(x) = \begin{cases} v_p^n = u_p^n - u_p^D & \text{if } x \in p, \\ v_{e_p}^n = 0 & \text{if } x \in e_p. \end{cases} \tag{13}$$

With these notations (8) can be rewritten as

$$m(p)\,\frac{c(u_p^{n+1}) - c(u_p^n)}{k} - \sum_{q \in N(p)} T_{pq}\,(v_q^{n+1} - v_p^{n+1}) - \sum_{q \in N(p)} T_{pq}\,(u_q^D - u_p^D) = 0, \tag{14}$$

for all $p \in \mathcal{T}$ and $n \in \{0, \dots, [T/k]\}$. Next we state some estimates which u_h^D satisfies.

Lemma 2.1. *The function u_h^D satisfies the L^2-estimate*

$$\left\| u_h^D \right\|_{L^2(\Omega)} = \sum_{p \in \mathcal{T}} m(p)\,(u_p^D)^2 \le m(\Omega)\, \left\| u^D \right\|_{C(\overline{\Omega})}^2, \tag{15}$$

as well as the "discrete H^1-estimate"

$$\sum_{(p,q) \in \mathcal{E}} T_{pq}\,(u_q^D - u_p^D)^2 \le C\, m(\Omega). \tag{16}$$

The discrete problem $(P_{h,k})$ is given by initial condition (7), boundary condition (9) or (13) and either the discrete equation (8) or the discrete equation (14).

Theorem 2.2. *Suppose that the hypotheses (H_c), (H_0), (H_D) and $(H_{\mathcal{T}})$ are satisfied. There exists a unique solution of the discrete problem $(P_{h,k})$.*

Proof. In order to prove the uniqueness of the solution of Problem $(P_{h,k})$, we write the equations for two solutions $\{u_{1p}^{n+1}\}$, $\{u_{2p}^{n+1}\}$, $p \in \mathcal{T}$, $n \in \{0, \ldots, [T/k]\}$, multiply the difference of the equations for u_{1p}^{n+1} and u_{2p}^{n+1} by $k\,(u_{1p}^{n+1} - u_{2p}^{n+1})$ and sum on $p \in \mathcal{T}$.

Next we prove the existence of the solution of Problem $(P_{h,k})$. To that end, we consider a sequence of smooth nondecreasing functions c_ε such that c_ε converges to c uniformly on $\mathbb{R}$ as $\varepsilon \downarrow 0$ and we denote by L_ε the Lipschitz constant of c_ε. To begin with we prove the existence of a unique solution of the problem

$$m(p)\,\frac{c_\varepsilon(u_p^\varepsilon) - c(u_p^n)}{k} - \sum_{q \in N(p)} T_{pq}\,(u_q^\varepsilon - u_p^\varepsilon) = 0. \tag{17}$$

We denote by $\mathcal{P}$ the number of elements of $\mathcal{T}$. A vector $U = (u_p)_{p\in\mathcal{T}}$ being given, we define $\tilde{U} = (\tilde{u}_p)_{p\in\mathcal{T}}$ as the solution of the linear system

$$m(p)\,\frac{c_\varepsilon(u_p) + L_\varepsilon(\tilde{u}_p - u_p) - c(u_p^n)}{k} - \sum_{q \in N(p)} T_{pq}\,(\tilde{u}_q - \tilde{u}_p) = 0. \tag{18}$$

Note that the matrix involved in the resolution of system (18) is strictly diagonal dominant so that it has a unique solution.

In order to prove the existence of u_p^ε for all $p \in \mathcal{T}$, we define the operator

$$\begin{aligned} T\colon \qquad \mathbb{R}^{\mathcal{P}} &\longrightarrow \mathbb{R}^{\mathcal{P}} \\ U = (u_p)_{p\in\mathcal{T}} &\longmapsto \tilde{U} = (\tilde{u}_p)_{p\in\mathcal{T}}, \end{aligned} \tag{19}$$

and the norm

$$\|U\|_{l^2} = \Big(\sum_{p\in\mathcal{T}} m(p)\,u_p^2\Big)^{1/2}, \tag{20}$$

on $\mathbb{R}^{\mathcal{P}}$. Next we show that the operator T is a strict contraction from $(\mathbb{R}^{\mathcal{P}}, \|\ \|_{l^2})$ into $(\mathbb{R}^{\mathcal{P}}, \|\ \|_{l^2})$. Let $U_1 = (u_{1p})_{p\in\mathcal{T}}$ and $U_2 = (u_{1p})_{p\in\mathcal{T}}$ be two vectors of $\mathbb{R}^{\mathcal{P}}$ and let $\tilde{U_1} = TU_1$ and $\tilde{U_2} = TU_2$. We set

$$\begin{aligned} \tilde{w}_p &= \tilde{u}_{1p} - \tilde{u}_{2p}, \\ w_p &= u_{1p} - u_{2p}, \end{aligned} \tag{21}$$

for all $p \in \mathcal{T}$. We subtract equation (18) for U_2 from equation (18) for U_1 to obtain

$$m(p)\,\frac{L_\varepsilon}{k}\left(\tilde{w}_p - w_p + \frac{c_\varepsilon(u_{1p}) - c_\varepsilon(u_{2p})}{L_\varepsilon}\right) - \sum_{q\in N(p)} T_{pq}\left(\tilde{w}_q - \tilde{w}_p\right) = 0. \tag{22}$$

for all $p \in \mathcal{T}$. Next we multiply (22) by $\tilde{w}_p$ and sum the result over $p \in \mathcal{T}$. This yields

$$\begin{aligned} \frac{L_\varepsilon}{k} \sum_{p\in\mathcal{T}} m(p)\,(\tilde{w}_p)^2 - \sum_{p\in\mathcal{T}} \sum_{q\in N(p)} T_{pq}\,(\tilde{w}_q - \tilde{w}_p)\,\tilde{w}_p \\ = \frac{L_\varepsilon}{k} \sum_{p\in\mathcal{T}} m(p)\,w_p\,\tilde{w}_p \left(1 - \frac{1}{L_\varepsilon}\,\mathcal{C}_p\right), \end{aligned} \tag{23}$$

where

$$\mathcal{C}_p = \frac{c_\varepsilon(u_{1p}) - c_\varepsilon(u_{2p})}{w_p} = \frac{c_\varepsilon(u_{1p}) - c_\varepsilon(u_{2p})}{u_{1p} - u_{2p}}. \tag{24}$$

for all $p \in \mathcal{T}$. By the choice of c_ε, we have that

$$0 \le \frac{1}{L_\varepsilon}\,\mathcal{C}_p \le 1. \tag{25}$$

for all $p \in \mathcal{T}$. Also using the discrete Poincaré inequality (cf. [**EGHa**]) we deduce from (23) that

$$\left(\sum_{p\in\mathcal{T}} m(p)\,(\tilde{w}_p)^2\right)^{1/2} \le C \left(\sum_{p\in\mathcal{T}} m(p)\,(w_p)\right)^{1/2}, \tag{26}$$

where

$$C = \frac{\frac{L_\varepsilon}{k}}{\frac{L_\varepsilon}{k} + \frac{1}{\delta^2(\Omega)}} < 1, \tag{27}$$

and $\delta(\Omega)$ is the diameter of domain Ω. We substitute (20) and (21) into (26) to obtain

$$\left\|\tilde{U}_1 - \tilde{U}_2\right\|_{l^2} \le C\, \|U_1 - U_2\|_{l^2}\,. \tag{28}$$

Therefore T is a strict contraction from $(\mathbb{R}^{\mathcal{P}}, \|\ \|_{l^2})$ into $(\mathbb{R}^{\mathcal{P}}, \|\ \|_{l^2})$ and has a unique fixed point U^ε which satisfies

$$m(p)\,\frac{c_\varepsilon(u_p^\varepsilon) - c(u_p^n)}{k} - \sum_{q\in N(p)} T_{pq}\left(u_q^\varepsilon - u_p^\varepsilon\right) = 0. \tag{29}$$

for all $p \in \mathcal{T}$. Finally we let $\varepsilon \downarrow 0$ and suppose that

$$-M \le c(u_p^n) \le M. \tag{30}$$

for all $p \in \mathcal{T}$. We show that

$$-M \leq c_\varepsilon(u_p^\varepsilon) \leq M, \tag{31}$$

and in turn that

$$\left|u_p^\varepsilon\right| \leq C = C(h), \tag{32}$$

for all $p \in \mathcal{T}$. Therefore there exists u_p^{n+1} and a subsequence $u_p^{\varepsilon_n}$ such that

$$u_p^{\varepsilon_n} \longrightarrow u_p^{n+1} \text{ as } \varepsilon \downarrow 0, \tag{33}$$

where u_p^{n+1} satisfies (8). □

The mathematical problem is to study the convergence of $\{u_{h,k}\}$ to the weak solution of Problem (P) as h and k tend to zero.

3. A Priori Estimates

In this section we show that $c(u_{h,k})$ satisfies a discrete maximum principle and present an estimate for $u_{h,k}$ in a discrete space analogous to $L^2(0,T;H^1(\Omega))$. For the proofs of these statements we refer to [**EGHb**] and to [**Gut98**].

Lemma 3.1. *Let $u_{h,k}$ be the solution of Problem $(P_{h,k})$, suppose that u_0 and u^D satisfy hypotheses (H_c), (H_0) and (H_D) and let $M = \max\Big(\|c(u_0)\|_{L^\infty(\Omega)}, \left\|c(u^D)\right\|_{L^\infty(\partial\Omega)}\Big)$. Then for all $p \in \mathcal{T}$ and $0 \leq n \leq [T/k]$, we have that*

$$-M \leq c(u_{h,k}(x,t)) \leq M \quad \text{for all} \quad x \in p,\ t \in (nk,(n+1)k]. \tag{34}$$

Lemma 3.2. *Suppose that the hypotheses (H_c), (H_0), (H_D) and $(H_{\mathcal{T}})$ are satisfied. There exists a positive constant C such that*

$$\sum_{n=0}^{[T/k]} k \sum_{(p,q)\in\mathcal{E}} T_{pq}\left(v_q^{n+1} - v_p^{n+1}\right)^2 \leq C. \tag{35}$$

One can then apply the discrete Poincaré inequality [**EGHa**] to deduce the following result.

Lemma 3.3. *Suppose that the hypotheses (H_c), (H_0), (H_D) and $(H_{\mathcal{T}})$ are satisfied. There exists a positive constant C such that*

$$\|v_{h,k}\|_{L^2(Q_T)} \leq C, \tag{36}$$

and

$$\|u_{h,k}\|_{L^2(Q_T)} \leq C. \tag{37}$$

Next we present an estimate on differences of space translates of the approximate solution. We introduce the function $W(s) = \displaystyle\int_0^s \min(c'(z),1)\,dz$ and derive estimates on differences of space and time translates of the function $W(u_{h,k})$ which imply that the sequence $\{W(u_{h,k})\}$ is relatively compact in $L^2(Q_T)$.

Lemma 3.4. *Suppose that the hypotheses (H_c), (H_0), (H_D) and ($H_{\mathcal{T}}$) are satisfied. There exists a positive constant C such that*

$$\int_{\Omega_\xi \times (0,T)} \Big(u_{h,k}(x+\xi,t) - u_{h,k}(x,t)\Big)^2 \, dx\, dt \le |\xi| \, (|\xi| + 2h)\, C, \tag{38}$$

and

$$\int_{\Omega_\xi \times (0,T)} \Big(v_{h,k}(x+\xi,t) - v_{h,k}(x,t)\Big)^2 \, dx\, dt \le |\xi| \, (|\xi| + 2h)\, C, \tag{39}$$

for all $\xi \in \mathbb{R}^N$, where $\Omega_\xi = \{x \in \Omega, [x, x+\xi] \subset \Omega\}$.

Next we define the function (cf. [**JK95**])

$$W(s) = \int_0^s \min(c'(z), 1)\, dz. \tag{40}$$

In view of Hypothesis (H_c), W is well defined on $\mathbb{R}$ and satisfies the following properties.

Lemma 3.5. *Suppose that Hypotheses (H_c) is satisfied. The function W is nondecreasing and satisfies the inequality*

$$|W(s_1) - W(s_2)| \le \min\Big(|s_1 - s_2| \, , |c(s_1) - c(s_2)| \Big). \tag{41}$$

Moreover, there exists a strictly increasing continuous function $\mathcal{F}$ such that

$$W(s) = \mathcal{F}(c(s)). \tag{42}$$

Proof. Let $s_1 \le s_2$. We have that

$$W(s_2) - W(s_1) = \int_{s_1}^{s_2} \min(c'(z), 1)\, dz. \tag{43}$$

In view of Hypotheses (H_c), $\min(c'(z), 1) \ge 0$. Hence we deduce that

$$W(s_2) - W(s_1) \ge 0, \tag{44}$$

so that W is nondecreasing. Next we consider $|W(s_2) - W(s_1)|$ for all $s_1, s_2 \in \mathbb{R}$ and suppose that $s_1 \le s_2$. Then

$$|W(s_2) - W(s_1)| = \int_{s_1}^{s_2} \min(c'(z), 1)\, dz. \tag{45}$$

On the one hand (45) implies that

$$|W(s_2) - W(s_1)| \leq \int_{s_1}^{s_2} c'(z)\,dz = c(s_2) - c(s_1), \tag{46}$$

and on the other hand it also implies that

$$|W(s_2) - W(s_1)| \leq \int_{s_1}^{s_2} \; dz = s_2 - s_1. \tag{47}$$

Following the same argument in the case $s_2 \leq s_1$, we deduce (41). Finally let $s_1, s_2 \in \mathbb{R}$ be such that

$$c(s_1) < c(s_2). \tag{48}$$

Since c is nondecreasing, we have that $s_1 < s_2$ so that $\text{meas}(s_1, s_2) > 0$. We define the set E_s by

$$E_s = \Big\{ z \in (s_1, s_2),\, c'(z) > 0 \Big\}. \tag{49}$$

Since $c \in L^1_{loc}$, we have that

$$c(s_2) - c(s_1) = \int_{s_1}^{s_2} c'(z)\,dz. \tag{50}$$

Then if $\text{meas}(E_s) = 0$, we deduce that

$$\int_{s_1}^{s_2} c'(z)\,dz = 0, \tag{51}$$

and hence $c(s_1) = c(s_2)$ which contradicts (48). Therefore $\text{meas}(E_s) > 0$ and

$$W(s_2) - W(s_1) = \int_{s_1}^{s_2} \min(c'(z), 1)\,dz > 0. \tag{52}$$

Hence we have proved that $c(s_1) < c(s_2)$ implies that $W(s_1) < W(s_2)$ so that there exists a strictly increasing function $\mathcal{F}$ satisfying (42). Moreover (41) yields

$$|\mathcal{F}(c(s_1)) - \mathcal{F}(c(s_2))| \leq |c(s_1) - c(s_2)|\,, \tag{53}$$

so that $\mathcal{F}$ is Lipschitz continuous. □

We deduce the next result from the Lemmas 3.4 and 3.5.

Corollary 3.6. *Suppose that the hypotheses (H_c), (H_0), (H_D) and ($H_{\mathcal{T}}$) are satisfied. We have with the same constant C as in Lemma 3.4 that*

$$\int_{\Omega_\xi \times (0,T)} \Big(W(u_{h,k})(x+\xi,t) - W(u_{h,k})(x,t) \Big)^2 \, dx\, dt \leq |\xi| \, (|\xi| + 2h) \, C, \tag{54}$$

for all $\xi \in \mathbb{R}^N$, where $\Omega_\xi = \{x \in \Omega, [x+\xi, x] \subset \Omega\}$.

We now consider differences of time translates of the function $W(u_{h,k})$.

Lemma 3.7. *Suppose that the hypotheses (H_c), (H_0), (H_D) and ($H_{\mathcal{T}}$) are satisfied. There exists a positive constant C such that*

$$\int_{\Omega \times (0,T-\tau)} \Big(W(u_{h,k})(x,t+\tau) - W(u_{h,k})(x,t) \Big)^2 \, dx\, dt \leq \tau \, C, \tag{55}$$

for all $\tau \in (0,T)$.

Proof. Let $\tau \in (0,T)$ and $t \in (0, T-\tau)$. We set

$$\mathcal{A}(t) = \int_\Omega \Big(W(u_{h,k})(x,t+\tau) - W(u_{h,k})(x,t) \Big)^2 \, dx\, dt. \tag{56}$$

Substituting (10) yields

$$\mathcal{A}(t) = \sum_{p \in \mathcal{T}} m(p) \Big(W(u_p^{[(t+\tau)/k]+1}) - W(u_p^{[t/k]+1}) \Big)^2. \tag{57}$$

In view of (41) and since c is nondecreasing we deduce that

$$\mathcal{A}(t) \leq \sum_{p \in \mathcal{T}} m(p) \Big(u_p^{[(t+\tau)/k]+1} - u_p^{[t/k]+1} \Big) \Big(c(u_p^{[(t+\tau)/k]+1}) - c(u_p^{[t/k]+1}) \Big), \tag{58}$$

which implies that

$$\mathcal{A}(t) \leq \sum_{p \in \mathcal{T}} \Big(u_p^{[(t+\tau)/k]+1} - u_p^{[t/k]+1} \Big) \sum_{\substack{n \in \mathbb{N}, \\ t < nk \leq t+\tau}} m(p) \Big(c(u_p^{n+1}) - c(u_p^n) \Big). \tag{59}$$

The proof of Lemma 3.7 then follows as in [**EGHb**] and [**Gut98**]. □

4. Convergence

In this section we prove the convergence of the approximate solution to the weak solution of Problem (P). To begin with we state a convergence result which will be useful in what follows.

Lemma 4.1. *Let $\{u_m\}$ be such that u_m converges to u weakly in $L^2(Q_T)$ and $\{W(u_m)\}$ converges to a limit χ strongly in $L^2(Q_T)$ and a.e. in Q_T. Then*

$$\chi = W(u) \quad \text{a.e. in } Q_T. \tag{60}$$

Proof. The proof is similar to that of Alt et Luckhaus [**AL83**]. □

We are now in a position to present our main result.

Theorem 4.2. *Let T be a fixed positive constant and suppose that the hypotheses (H_c), (H_0), (H_D) and (H_T) are satisfied. Then*

(i) *$u_{h,k}$ converges to u weakly in $L^2(Q_T)$;*
(ii) *$c(u_{h,k})$ converges to $c(u)$ strongly in $L^2(Q_T)$,*

as h and k tend to zero, where u is the unique weak solution of Problem (P).

Proof. Using the estimates (54), (55) and Kolmogorov's theorem (see Brezis [**Bre83**, Theorem IV.25, p. 72]), we deduce that $\{W(u_{h,k})\}$ is relatively compact in $L^2(Q_T)$. Then in view of the lemmas 3.3, 3.6, 3.7 and 4.1 we deduce the existence of a subsequence $\{u_{h_m,k_m}\}$ of $\{u_{h,k}\}$ and of a function $u \in L^2(Q_T)$ such that

$$\begin{cases} \text{(i)} \ \ u_{h_m,k_m} & \text{converges to } u \text{ weakly in } L^2(Q_T), \\ \text{(ii)} \ \ W(u_{h_m,k_m}) & \text{converges to } W(u) \text{ strongly in } L^2(Q_T) \text{ and a.e. in } Q_T, \end{cases} \tag{61}$$

as h_m and k_m tend to zero. In view of the definition of $\mathcal{F}$ in Lemma 3.5 we deduce from (61(ii)) that

$$\mathcal{F}(c(u_{h_m,k_m})) \text{ converges to } \mathcal{F}(c(u)) \text{ a.e. in } Q_T. \tag{62}$$

Since $\mathcal{F}$ is continuous and strictly increasing on $[-M, M]$, we deduce that $\mathcal{F}$ is inversible with a continuous inverse. Therefore

$$c(u_{h_m,k_m}) \text{ converges to } c(u) \text{ a.e. in } Q_T, \tag{63}$$

and hence

$$c(u_{h_m,k_m}) \text{ converges to } c(u) \text{ strongly in } L^2(Q_T). \tag{64}$$

We then show that u satisfies the integral equality

$$\begin{aligned} -\int_\Omega c(u_0(x))\,\psi(x,0)\,dx\,dt \ &- \int_0^T \int_\Omega c(u(x,t))\,\psi_t(x,t)\,dx\,dt \\ &- \int_0^T \int_\Omega u(x,t)\,\Delta\psi(x,t)\,dx\,dt = 0, \end{aligned} \tag{65}$$

for all $\psi \in \tilde{\mathcal{F}} := \Big\{ f \in C^{2,1}(\overline{\Omega} \times [0,T]),\ f = \dfrac{\partial f}{\partial n} = 0 \text{ on } \partial\Omega \times [0,T],\ f = 0$ on $\Omega \times \{T\} \Big\}$. By the definitions of $v_{h,k}$ and u_h^D and in view of the Lemmas 2.1 and we deduce that the sequence $\{v_{h_m,k_m}\}$ converges to $v = u - u^D$ weakly in $L^2(Q_T)$. Next we show that $v \in L^2(0,T;H_0^1(\Omega))$ and thus that $u \in L^2(0,T;H^1(\Omega))$. We define $\tilde{v}_{h,k}$ by

$$\tilde{v}_{h,k} = v_{h,k} \quad \text{a.e. in } \Omega \times [0,T], \qquad \tilde{v}_{h,k} = 0 \quad \text{a.e. in } (\mathbb{R}^N \setminus \Omega) \times [0,T]. \tag{66}$$

Therefore $\{\tilde{v}_{h_m,k_m}\}$ converges to $\tilde{v}$ with

$$\tilde{v} = v \quad \text{a.e. in } \Omega \times [0,T], \qquad \tilde{v} = 0 \quad \text{a.e. in } (\mathbb{R}^N \setminus \Omega) \times [0,T]. \tag{67}$$

Then for all $\xi \in \mathbb{R}^N$, $\xi \neq 0$ we have in view of Lemma 3.4 that

$$\int_0^T \int_{\mathbb{R}^N} \frac{|\tilde{v}_{h,k}(x+\xi,t) - \tilde{v}_{h,k}(x,t)|^2}{|\xi|^2}\, dx\, dt \leq \frac{|\xi| + 2h}{|\xi|}\, C, \tag{68}$$

which in turn implies a similar inequality for the function $\tilde{v}$. In particular

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \frac{\tilde{v}(x+\xi,t) - \tilde{v}(x,t)}{|\xi|}\, \varphi(x,t)\, dx\, dt \\ &\leq \left(\int_0^T \int_{\mathbb{R}^N} \frac{|\tilde{v}(x+\xi,t) - \tilde{v}(x,t)|^2}{|\xi|^2}\, dx\, dt \right)^{1/2} \left(\int_0^T \int_{\mathbb{R}^N} \varphi^2(x,t) \right)^{1/2} \\ &\leq C\, \|\varphi\|_{L^2(\mathbb{R}^N \times (0,T))}, \end{aligned} \tag{69}$$

which implies that

$$\frac{\partial \tilde{v}}{\partial x_i} \in L^2(\mathbb{R}^N \times (0,T)). \tag{70}$$

Therefore $\tilde{v} \in L^2(0,T;H^1(\mathbb{R}^N))$. Since also $\tilde{v} = 0$ a.e. in $\mathbb{R}^N \setminus \Omega$, $v \in L^2(0,T;H_0^1(\Omega))$ and thus $u \in L^2(0,T;H^1(\Omega))$ satisfies

$$u = u^D \text{ on } \partial\Omega \times (0,T). \tag{71}$$

Integrating by parts the integral equation (65), we deduce that u satisfies

$$\int_0^T \int_\Omega \Big\{ \Big(c(u(x,t)) - c(u_0(x)) \Big)\, \psi_t(x,t) - \boldsymbol{\nabla} u(x,t)\, \boldsymbol{\nabla} \psi(x,t) \Big\}\, dx\, dt = 0, \tag{72}$$

for all $\psi \in \tilde{\mathcal{F}}$. Also using the density of $\tilde{\mathcal{F}}$ in the set $\{\psi \in L^2(0,T;H_0^1(\Omega))$, $\psi_t \in L^\infty(Q_T)$, $\psi(\cdot,T) = 0\}$, we finally deduce that u coïncides with the unique weak solution of Problem (P); in particular the whole sequence $\{u_{h,k}\}$ converges to u. □

References

[**AL83**] Alt H. W. and Luckhaus S., *Quasilinear elliptic-parabolic differential equations*, Math.-Z. **183**(4) (1983), 311–341.

[**BW93**] Baughman L. A. and Walkington N. J., *Co-volume methods for degenerate parabolic problems*, Numer. Math. **64** (1993), 45–67.

[**Bre83**] Brezis H., *Analyse fonctionnelle, théorie et applications*, Masson, Paris, 1983.

[**CHJKN97**] Chounet L. M., Hilhorst D., Jouron C., Kelanemer Y. and Nicolas P., *Saturated-unsaturated simulation for coupled heat and mass transfer in the ground by means of a mixed finite element method*, Preprint, Université de Paris-Sud, 1997.

[**VDP82**] van Duijn C. J. and Peletier L. A., *Nonstationary filtration in partially saturated porous media,*, Arch. Rat. Mech. Anal. **78**(2) (1982), 173–198.

[**EGGH96**] Eymard R., Gallouët T., Ghilani M. and Herbin R., *Error estimates for the approximate solutions of a nonlinear hyperbolic equation given by some finite volume schemes,*, Preprint, ENPC, 1996.

[**EGHa**] Eymard R., Gallouët T. and Herbin R., *Finite volumes method* (Ph. Ciarlet and J. L. Lyons, eds.), in preparation for the Handbook of Numerical Analysis.

[**EGHb**] Eymard R., Gutnic M. and Hilhorst D., *The finite volume method for the Richards equation*, (to appear).

[**EGHS97**] Eymard R., Gallouët T., Hilhorst D. and Naït Slimane Y., *Finite volumes and nonlinear diffusion equations*, to appear in M2AN, 1998.

[**Gut98**] Gutnic M., *Sur des problèmes d'évolution non linéaires en milieu poreux*, Thèse de doctorat, Université Paris-Sud, Orsay, 1998.

[**Her95**] Herbin R., *An error estimate for a finite volume scheme for a diffusion convection problem on a triangular mesh*, Num. Meth. P.D.E. (1995), 165–173.

[**Hor78**] Hornung U., *Numerische Simulation von gesättigt-ungesättigt Wasserflüssen in porösen Medien,*, In Numerische Behandlung von Differentialgleichungen mit besonderer Berücksichtigung freier Randwertaufgaben, volume 39 of ISNM, Birkhaüser, Basel-Boston, 1978, pp. 214–232.

[**Hul86**] Hulshof J., *Elliptic-parabolic problems: the interface*, PhD thesis, RijksUniversiteit of Leiden, 1986.

[**HW88**] Hulshof J. and Wolanski N., *Monotone flows in N-dimensional partially saturated porous media : Lipschitz-continuity of the interface*, Arch. Rat. Mech. Anal. **102**(4) (1988), 287–305.

[**JK95**] Jäger W. and Kačur J., *Solution of doubly nonlinear and degenerate parabolic problems by relaxation scheme,*, RAIRO-Model. Math. Anal. Numer. **29**(5) (1995), 605–627.

[**Kel94**] Kelanemer Y., *Transferts couplés de masse et de chaleur dans les milieux poreux: modélisation et étude numérique*, Thèse de doctorat, Université Paris-Sud, Orsay, 1994.

[**Kna87**] Knabner P., *Finite element simulation of saturated-unsaturated flow through porous media,*, Large scale scientific computing, Prog. Sci. Comput. **7** (1987), 83–93.

[**Knaal97**] Frolkovič P., Knabner P., Tapp C. and Thiele K., *Adaptive finite volume discretization of density driven flows in porous media*, In Transport de contaminants en milieux poreux (support de cours), CEA-EDF-INRIA, INRIA, June 1997, pp. 322–355.

[**Ott96**] Otto F., *L^1-contraction and uniqueness for quasilinear elliptic-parabolic equations*, Journal of differential equations **131** (1996), 20–38.

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