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ON THE LONG TIME BEHAVIOUR OF SOLUTIONS OF A CLASS OF AUTONOMOUS REACTION–DIFFUSION SYSTEMS

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1. Introduction

Let λ_1, λ_2 be positive real numbers, $\lambda_1 \neq \lambda_2$, and set

$$\Lambda := \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}.$$

We examine the reaction-diffusion system

$$(1)\quad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \Lambda \frac{d^2}{dx^2}\begin{pmatrix} u \\ v \end{pmatrix} + g\left(\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\|_{L^2}^2\right)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -v \\ u \end{pmatrix},\ t > 0,\ x \in \Omega := (0,1)$$

with Dirichlet boundary conditions $u, v|_{\partial\Omega} = 0|_{\partial\Omega}$, where $g\colon [0,\infty) \to \mathbb{R}$ is continuously differentiable and satisfies the following conditions

(i) g is decreasing, i.e. $g(y_1) > g(y_2)$ for all $0 \leq y_1 < y_2$,
(ii) $g(0) > 0$,
(iii) $g^- := \inf\{g(y) : y \geq 0\} \in (-\infty, 0]$,
(iv) $yg'(y)$ is bounded where g' denotes the derivative of g.

We are interested in the long-time behaviour of solutions of (1).

There are results [**1**] for the reaction-diffusion system

$$(2)\quad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda \frac{d^2}{dx^2}\begin{pmatrix} u \\ v \end{pmatrix} + \left(1 - \sqrt{u^2 + v^2}\right)\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} -v \\ u \end{pmatrix},\ t > 0,\ x \in \Omega := (0,1)$$

which show that

- the trivial solution is the only stationary solution of (2),
- system (2) has periodic solutions if λ is sufficiently small,
- there are no periodic solutions if λ is sufficiently large,
- the solutions of (2) form a (global) semiflow on the Sobolev space $H_0^1((0,1)) \times H_0^1((0,1))$, and all solutions tend either to the trivial solution or to a periodic solution if t tends to $+\infty$.

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If we had $\lambda_1 = \lambda_2$, we could proceed as in [**1**], and we would get similar results for system (1).

If we deal with the case $\lambda_1 \neq \lambda_2$, the situation becomes more complicated. In particular, the methods used in [**1**] do not work. The reason is, loosely speaking, that there is some interaction between the rotation term

$$\begin{pmatrix} -v \\ u \end{pmatrix}$$

and the different diffusion rates. It is our aim to understand this interaction.

The nonlinearity has the same influence on u and v. It is chosen in a way that a full description of the long-time behaviour of all solutions is possible. Of course, one can study more general nonlinearities or look at nonlinearities which depend only on local terms (i.e. they depend on the values of u and v, but not on the functions $u(t,\cdot)$, $v(t,\cdot)$). But if we do so, the influence of the nonlinearity makes it (much) more complicated to understand the dynamics and, of course, the interaction between rotation and different diffusion rates.

The paper is structured as follows.

In the first section we show that there is a (global) semiflow of solutions of (1) on the (large) space $L^2((0,1)) \times L^2((0,1)) =: L^2 \times L^2$.

In other applications where the nonlinearity does only depend on local terms, it is only possible to construct a semiflow on a subspace of $L^2 \times L^2$ such as for example the Sobolev space $H_0^1((0,1)) \times H_0^1((0,1))$ (see for example [**3**], [**4**]). Here, we get solutions $(u,v)\colon [0,\infty) \to L^2 \times L^2$ of (1) for every initial value in $L^2 \times L^2$. Furthermore, $(u,v)(t)$, $t > 0$, has a smooth representative $(u(t,\cdot), v(t,\cdot)) \in C^\infty([0,1]) \times C^\infty([0,1])$. A similar situation has been examined in [**2**].

The proof of the existence of the semiflow of solutions of (1) on $L^2 \times L^2$ is done by reducing the given PDE to an ODE. This reduction, which is motivated by the reduction made in [**2**], turns out to be useful for all questions concerning the long-time behaviour of the solutions of (1). The reduction works as follows: We introduce a positive number $p = p(\lambda_1, \lambda_2) := \frac{1}{\pi}\sqrt{\frac{2}{|\lambda_1 - \lambda_2|}}$ and denote the smallest integer which is larger than p by p^+, the largest integer which is smaller than p by p^-. If $p = p(\lambda_1, \lambda_2)$ is an integer, then we call (λ_1, λ_2) critical, otherwise non-critical. Thus, we get $p^+ - p^- = 2$ in the critical and $p^+ - p^- = 1$ in the non-critical case. For all positive integers n, we set

$$A_n := -\Lambda\pi^2 n^2 + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .$$

The matrix A_n has complex conjugated eigenvalues with real part $-\overline{\lambda} n^2\pi^2$ for $n \le p^-$, and real eigenvalues

$$\mu_n^{(1)} := -\frac{\lambda_1+\lambda_2}{2}\pi^2 n^2 + \sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2 \pi^4 n^4 - 1}$$
$$\mu_n^{(2)} := -\frac{\lambda_1+\lambda_2}{2}\pi^2 n^2 - \sqrt{\left(\frac{\lambda_1-\lambda_2}{2}\right)^2 \pi^4 n^4 - 1}$$

for all $n \ge p^+$. In the critical case, A_p has the eigenvalue $\mu_p := -(\lambda_1+\lambda_2)\pi^2 p^2/2$ with multiplicity 2. We construct vectors $e_n^{(j)} \in \mathbb{R}^2$, $n \in \mathbb{N}$, $j = 1,2$, such that

- $e_n^{(1)}$ and $e_n^{(2)}$ are linear independent for all n,
- $A_n e_n^{(j)} = \mu_n^{(j)} e_n^{(j)}$ for all $n \ge p$, $j = 1,2$,
- $||e_n^{(j)}|| = 1$ for all $n \ge p^+$, $j = 1,2$,
- $A_n(\alpha e_n^{(1)} + \beta e_n^{(2)}) = \left[-\overline{\lambda} n^2\pi^2 \left(\alpha e_n^{(1)} + \beta e_n^{(2)}\right) + \sqrt{1-\Delta\lambda_n^2}\left(\beta e_n^{(1)} - \alpha e_n^{(2)}\right)\right]$ for all $n \le p^-$, $\alpha, \beta \in \mathbb{R}$,
- $A_p(\alpha e_p^{(1)} + \beta e_p^{(2)}) = (2\beta + \mu_p\alpha)e_p^{(1)} + \mu_p\beta e_p^{(2)}$ in the critical case (i.e. if p is an integer).

We write the initial data (u_0, v_0) in the form

$$(u_0, v_0) = \sqrt{2} \sum_{n\in\mathbb{N},\ j=1,2} b_n^{(j)} e_n^{(j)} \sin(n\pi\cdot)$$

with real coefficients $b_n^{(j)}$, which we call the coefficients corresponding to (u_0, v_0). Then we set

$$c_{(u_0,v_0)}(t) := \begin{cases} e_p^{(1)}\left(b_p^{(1)} + 2tb_p^{(2)}\right)\exp(\mu_p t) + e_p^{(2)} b_p^{(2)} \exp(\mu_p t) & \text{if } p \text{ is an integer,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\begin{aligned} R_{(u_0,v_0)} : \mathbb{R}^2 \ni (t,y) \mapsto y^2 \sum_{k=1}^{p^-} \exp(-2\overline{\lambda} k^2\pi^2 t) \Big[& e_k^{(1)}\left(b_k^{(1)}\cos(p_k t) - b_k^{(2)}\sin(p_k t)\right) \\ & + e_k^{(2)}\left(b_k^{(1)}\sin(p_k t) + b_k^{(2)}\cos(p_k t)\right)\Big]^2 \\ & + y^2 \sum_{k=p^+}^{\infty} \left[e_k^{(1)} b_k^{(1)} \exp(\mu_k^{(1)} t) + e_k^{(2)} b_k^{(2)} \exp(\mu_k^{(2)} t)\right]^2 \\ & + y^2 c_{(u_0,v_0)}^2(t) \in \mathbb{R} \end{aligned}$$

and define the **basic solution** $b_{(u_0,v_0)} \in C^1(\mathbb{R},\mathbb{R})$ as the solution of the IVP

$$\frac{d}{dt} b_{(u_0,v_0)} = g\Big(R(t, b_{(u_0,v_0)})\Big) b_{(u_0,v_0)}\,, \qquad b_{(u_0,v_0)}(0) = 1\,,$$

where we define p_n for $n \leq p^-$ by

$$p_n := \sqrt{1 - \left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 n^2\pi^2}\,.$$

Using the basic solution, we define the **coefficient functions** $a_n^{(j)} : \mathbb{R} \to \mathbb{R}$

$$\begin{aligned}
\begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \end{pmatrix}(t) &:= b_{(u_0,v_0)}(t) \exp(-\overline{\lambda} n^2\pi^2 t) \begin{pmatrix} \cos(p_n t) & -\sin(p_n t) \\ \sin(p_n t) & \cos(p_n t) \end{pmatrix} \begin{pmatrix} b_n^{(1)} \\ b_n^{(2)} \end{pmatrix}, \\
&\qquad \text{for } n \leq p^-, \\
a_n^{(j)}(t) &:= b_{(u_0,v_0)}(t) \exp(\mu_n^{(j)} t) b_n^{(j)} \quad \text{for } n \geq p^+,\ j = 1,2,
\end{aligned}$$

$$\left.\begin{aligned}
a_p^{(1)}(t) &:= b_{(u_0,v_0)}(t) \exp(\mu_p t) \left(b_p^{(1)} + 2t b_p^{(2)}\right) \\
a_p^{(2)}(t) &:= b_{(u_0,v_0)}(t) \exp(\mu_p t) b_p^{(2)}
\end{aligned}\right\} \quad \text{in the critical case.}$$

We show that the functions $u(\cdot,\cdot), v(\cdot,\cdot) \colon (0,\infty) \times [0,1] \to \mathbb{R}$,

$$\begin{pmatrix} u(t,x) \\ v(t,x) \end{pmatrix} := \sqrt{2} \sum_{n \in \mathbb{N}} \left(a_n^{(1)}(t) e_n^{(1)} + a_n^{(2)}(t) e_n^{(2)}\right)$$

are well defined, they satisfy $u(t,\cdot), v(t,\cdot) \in C^\infty([0,1])$, $u(\cdot,x), v(\cdot,x) \in C^1((0,\infty))$ and they solve the PDE (1) in the classical sense. Furthermore, we show that this solution is uniquely determined.

In the next section we examine all stationary solutions of (1). We introduce the set I by $I := \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ in the non-critical case and by $I := \{p^+, p^+ + 1, \dots\} \times \{1,2\} \cup \{(p,1)\}$ in the critical case.

Let (u_0, v_0) be a stationary solution with coefficients $b_n^{(j)}$, $(n,j) \in \mathbb{N} \times \{1,2\}$. We show that we have $b_n^{(j)} = 0$ for all $(n,j) \notin I$ and that there are at most three $(n,j) \in I$ such that $b_n^{(j)} \neq 0$.

If we take $(n,j) \in I$ such that $-\mu_n^{(j)} < g(0)$ (with $\mu_p^{(1)} := \mu_p$ in the critical case), then

$$\pm\sqrt{g^{-1}(-\mu_n^{(j)})}\, e_n^{(j)} \sqrt{2} \sin(n\pi\cdot)$$

are stationary solutions of (1). These stationary solutions have exactly one coefficient which is non-zero. If there is a real number μ such that $-\mu < g(0)$ and $\mu_{n_\ell}^{(j_\ell)} = \mu$ for k, pairs $(n_\ell, j_\ell) \in I$, then

$$\sqrt{g^{-1}(-\mu)} \sum_{\ell=1}^{k} c_\ell e_{n_\ell}^{(j_\ell)} \sqrt{2} \sin(n_\ell \pi\cdot)\,, \qquad \sum_{\ell=1}^{k} c_\ell^2 = 1\,, \tag{3}$$

are also stationary solutions of (1). We show that every stationary solution can be written in the form (3) with $k \in \{0,1,2,3\}$. If there is a stationary solution with $k \geq 2$, then there is an infinite number of stationary solutions. Furthermore, the set of all stationary solutions of (1) is a finite union of compact and connected subsets of $L^2 \times L^2$.

We will actually construct all stationary solutions.

In the following section we examine periodic solutions of (1). We show that periodic solutions may occur. We show that for each periodic solution there is exactly one integer $n \in \{1, \dots, p^-\}$ with the property that the coefficient functions $a_n^{(j)}$, $j = 1, 2$, do not vanish identically. We call this integer the period index. Every periodic solution with period index n has period $2\pi/p_n$. This implies that p^- is the maximal number of different periods.

We set $\overline{\lambda} := (\lambda_1 + \lambda_2)/2$. We show that there are periodic solutions with period index n if and only if

$$n^2 < \frac{g(0)}{\overline{\lambda}\pi^2}. \tag{4}$$

If we have $\mu_m^{(j)} \neq -\overline{\lambda}\pi^2 n^2$ for all $(m,j) \in I$, then there is exactly one periodic solution with period index n. If n satisfies condition (4) and the set

$$E_n := \left\{(m,j) \in I : \mu_m^{(j)} = -\overline{\lambda}\pi^2 n^2\right\}$$

is not empty, then we get an infinite number of periodic orbits with period index n. For each of these periodic solutions, the quotient

$$q_m^{(j)}(t) := \frac{a_m^{(j)}(t)}{\sqrt{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2}}$$

turns out to be independent of t for all $(m,j) \in E_n$. On the other hand, if we take real numbers $\epsilon_{(m,j)}$, $(m,j) \in E_n$, we show that there is exactly one periodic orbit defined by periodic solutions with period index n which satisfy

$$\frac{a_m^{(j)}(t)}{\sqrt{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2}} = \epsilon_{(m,j)} \qquad \text{for all } t \in \mathbb{R},\ (m,j) \in E_n.$$

We show that every periodic orbit can be described by its period index n, the set E_n and the values $\epsilon_{(m,j)} \in \mathbb{R}$, $(m,j) \in E_n$. Furthermore, we show that the set E_n is either empty or it has one, two or three elements (where the number of elements of E_n depends on n and the diffusion constants λ_1, λ_2). If $|E_n|$ denotes the number of elements of E_n, then the periodic solutions with period index n form a smooth manifold of dimension $|E_n|+1$ (i.e. the dimension is 1, 2, 3 or 4).

In Section 5 we determine the long-time behaviour of every solution by looking at the coefficients $b_n^{(j)}$ corresponding to the initial value $(u_0,v_0)\in L^2\times L^2$.

First, we introduce the dominant multiplier (with $\mu_p^{(j)}:=\mu_p$ in the critical case)

$$\mu_{(u_0,v_0)} := \max\Big(\big\{\mu_n^{(j)} : n\ge p,\ b_n^{(j)}\neq 0\big\} \\ \cup \big\{-\overline{\lambda} n^2\pi^2 : n\le p^-, \Big(b_n^{(1)}\Big)^2+\Big(b_n^{(2)}\Big)^2\neq 0\big\}\Big).$$

We say that we have **critical dominant behaviour** if (λ_1,λ_2) is critical, $\mu_{(u_0,v_0)}=\mu_p$ and $b_p^{(2)}\neq 0$; otherwise, we have **non-critical dominant behaviour**. If we have non-critical dominant behaviour, we set

$$\begin{aligned}
N^p_{(u_0,v_0)} &:= \Big\{(n,1),(n,2) : n\le p, \Big(b_n^{(1)}\Big)^2+\Big(b_n^{(2)}\Big)^2\neq 0, -\overline{\lambda} n^2\pi^2=\mu_{(u_0,v_0)}, \\
&\qquad \overline{\lambda} n^2\pi^2 < g(0)\Big\} \\
N^s_{(u_0,v_0)} &:= \Big\{(n,j) : n>p, j=1,2, b_n^{(j)}\neq 0, \mu_n^{(j)}=\mu_{(u_0,v_0)}, -\mu_n^{(j)}<g(0)\Big\} \\
N^0_{(u_0,v_0)} &:= \Big\{(n,j) : n>p, j=1,2, b_n^{(j)}\neq 0, \mu_n^{(j)}=\mu_{(u_0,v_0)}, -\mu_n^{(j)}\ge g(0)\Big\} \\
&\quad \cup \Big\{(n,1),(n,2) : n\le p, \Big(b_n^{(1)}\Big)^2+\Big(b_n^{(2)}\Big)^2\neq 0, -\overline{\lambda} n^2\pi^2=\mu_{(u_0,v_0)}, \\
&\qquad \overline{\lambda} n^2\pi^2 \ge g(0)\Big\} \\
N_{(u_0,v_0)} &:= N^p_{(u_0,v_0)} \cup N^s_{(u_0,v_0)} \cup N^0_{(u_0,v_0)}
\end{aligned}$$

and call $N_{(u_0,v_0)}$ the set of dominant indices. Let $(u,v)\colon [0,\infty)\to L^2\times L^2$ be a solution of (1) with initial value (u_0,v_0). Then we get the following result:

- We have either $N_{(u_0,v_0)}=N^0_{(u_0,v_0)}$, $N_{(u_0,v_0)}=N^s_{(u_0,v_0)}$ or $N_{(u_0,v_0)}=N^s_{(u_0,v_0)}\cup N^p_{(u_0,v_0)}$.
- If $N_{(u_0,v_0)}=N^0_{(u_0,v_0)}$, then (u,v) tends to the zero-solution.
- If $N_{(u_0,v_0)}=N^s_{(u_0,v_0)}$, then (u,v) tends to a stationary solution which is not the trivial one. Furthermore, the limit can be described using the coefficients $\{b_n^{(j)} : (n,j)\in N^s_{(u_0,v_0)}\}$.
- If $N_{(u_0,v_0)}=N^s_{(u_0,v_0)}\cup N^p_{(u_0,v_0)}$ and $N^p_{(u_0,v_0)}\neq\emptyset$, then (u,v) tends to a periodic solution, which can be described using only the coefficients $b_n^{(j)}$ for $(n,j)\in N^s_{(u_0,v_0)}\cup N^p_{(u_0,v_0)}$.

This means that we only have to consider the dominant indices in order to determine the long-time behaviour of the solution (u,v). This result can be looked at as the main result of the paper. If we have critical dominant behaviour, we get the same result if we set $N^s_{(u_0,v_0)} = \{(p,1)\}$ and $N^0_{(u_0,v_0)} = N^p_{(u_0,v_0)} = \emptyset$.

Using this result, we can determine all stationary and periodic solutions which are stable and all stationary and periodic solutions which are attractive. This is done in Section 6.

Since we get similar results for the critical and the non-critical case, but the critical case becomes more technical, we restrict ourself to the non-critical case in Section 6. We note that nearly all $(\lambda_1, \lambda_2) \in (0,\infty) \times (0,\infty)$ are non-critical.

We show that — in the non-critical case — a stationary solution $(u_0, v_0) \neq (0,0)$ is stable if and only if the dominant multiplier $\mu_{(u_0,v_0)}$ coincides with the maximal multiplier

$$\mu := \max\left(\{\mu_n^{(j)} : n \geq p^+\} \cup \{-\overline{\lambda}\pi^2\}\right) .$$

Furthermore, a periodic orbit is stable if and only if the corresponding period index is 1 and $\mu = -\overline{\lambda}\pi^2$.

We show that stationary solutions as well as periodic orbits may occur which are stable but not attractive. On the other hand there are no stationary solutions and no periodic orbits which are attractive but not stable.

We show that a stationary solution is attractive if and only if $\mu = \mu_{(u_0,v_0)}$, $-\overline{\lambda}\pi^2 < \mu_{(u_0,v_0)}$ and $\mu_n^{(j)} = \mu_{(u_0,v_0)}$ for exactly one pair $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$. A periodic orbit is attractive if and only if the period index is 1 and $\mu_n^{(j)} < -\overline{\lambda}\pi^2$ for all $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$.

In the last section we determine the stable manifolds associated with stationary solutions or periodic orbits, provided that (λ_1, λ_2) is non-critical. We recall that the stable manifold of a stationary solution (u_0, v_0) consists of all $(u', v') \in L^2 \times L^2$ such that the solution of (1) which has initial value (u', v') tends to (u_0, v_0). We show that for every stationary solution $(u_0, v_0) \neq (0,0)$ the stable manifold $W^s(u_0, v_0)$ is given by

$$\begin{aligned} &W^s(u_0,v_0) := \\ &\Big\{(u',v') \in L^2 \times L^2 : \mu_{(u',v')} = \mu_{(u_0,v_0)},\ b_n^{(j)}(b')_m^{(k)} = (b')_n^{(j)} b_m^{(k)} \text{ for} \\ &\quad \text{all } (n,j),(m,k) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} \text{ with } \mu_n^{(j)} = \mu_m^{(k)} = \mu_{(u_0,v_0)}, \\ &\quad (b')_n^{(j)} = 0 \text{ for all } (n,j) \in \{1, \dots, p^-\} \times \{1,2\} \text{ such that } -\overline{\lambda} n^2 \pi^2 = \mu_{(u_0,v_0)}\Big\}, \end{aligned}$$

where $b_n^{(j)}$ should be the coefficients corresponding to (u_0, v_0) and $(b')_n^{(j)}$ the coefficients corresponding to (u', v').

Let $(u^p, v^p)\colon [0,\infty) \to L^2 \times L^2$ be a periodic solution of (1) with period index n. The corresponding periodic orbit is given by $\Gamma := \{(u^p, v^p)(t) : t \geq 0\}$. The stable manifold associated with this periodic orbit is defined by the initial values of all solutions of (1) which tend to Γ. We show that the stable manifold $W^s(\Gamma)$ is given by

$$W^s(\Gamma) := \Bigg\{ (u_0, v_0) \in L^2 \times L^2 : \mu_{(u_0,v_0)} = -\overline{\lambda} n^2 \pi^2, \\ \frac{b_m^{(j)}}{\sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2}} = \frac{(a^p)_m^{(j)}(0)}{\sqrt{\left((a^p)_n^{(1)}(0)\right)^2 + \left((a^p)_n^{(2)}(0)\right)^2}} \\ \text{for all } (m,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} \text{ with } \mu_m^{(j)} = -\overline{\lambda} n^2 \pi^2 \Bigg\}$$

where $(a^p)_m^{(j)}$ denote the coefficient functions corresponding to the periodic solution (u^p, v^p).

We show that there are diffusion constants $\lambda_1, \lambda_2 \in (0, \infty)$ such that a periodic orbit and a stationary solution are both stable. But we can also show that these cases are exceptional in the following sense:

We show that there are open and disjoint subsets P_0, P_s, P_p of $(0,\infty) \times (0,\infty) \setminus \{(\lambda, \lambda) : \lambda > 0\}$ such that the union $P_0 \cup P_s \cup P_p$ is dense in $(0,\infty) \times (0,\infty)$ and

- $P_0 \cup P_s \cup P_p$ consists only of non-critical diffusion constants (λ_1, λ_2),
- $(\lambda_1, \lambda_2) \in P_0$ implies that all solutions of (1) tend to the zero-solution,
- $(\lambda_1, \lambda_2) \in P_s$ implies that there is $(n, j) \in \{p^+(\lambda_1, \lambda_2), p^+(\lambda_1, \lambda_2) + 1, \dots\} \times \{1, 2\}$ such that

  $$W^s\left(\sqrt{g^{-1}(-\mu)} e_n^{(j)} \sqrt{2} \sin(n\pi \cdot)\right) \cup W^s\left(-\sqrt{g^{-1}(-\mu)} e_n^{(j)} \sqrt{2} \sin(n\pi \cdot)\right)$$

  is open and dense in $L^2 \times L^2$,
- $(\lambda_1, \lambda_2) \in P_p$ implies that there is one periodic solution with period index 1 and the stable manifold $W^s(\Gamma_1)$ of the corresponding periodic orbit Γ_1 is open and dense in $L^2 \times L^2$.

This means that in the case $(\lambda_1, \lambda_2) \in P_s$ all solutions of (1) which start in some open and dense subset of $L^2 \times L^2$ tend to one of the two (stable) stationary solutions; in the case $(\lambda_1, \lambda_2) \in P_p$ all solutions of (1) which start in an open and dense subset of $L^2 \times L^2$ tend to the periodic orbit Γ_1. Loosely speaking, this means that periodic motion dominates for $(\lambda_1, \lambda_2) \in P_p$ while convergence to a stationary solution dominates for $(\lambda_1, \lambda_2) \in P_s$ and, of course, for $(\lambda_1, \lambda_2) \in P_0$.

If we actually draw a picture of P_0, P_s, P_p, then we will see that P_p is located next to the subset $\{(\lambda_1, \lambda_2) \in (0,\infty) \times (0,\infty) : \lambda_1 = \lambda_2\}$ where both diffusion rates coincide, and P_s contains points (λ_1, λ_2) where the difference between λ_1 and λ_2 is large. Loosely speaking, this means that the different diffusion rates compensate the rotation forced by the rotation term

$$\begin{pmatrix} -v \\ u \end{pmatrix}$$

if the diffusion rates differ too much.

2. Existence of Solutions

Definition 1. (i) We set $L^2 := L^2((0,1))$, $H_0^1 := H_0^1((0,1))$ and $H^2 := H^2((0,1))$, where $H^k((0,1))$ should be the Sobolev space of all functions of L^2 which have distributional derivatives up to order k which are all quadratic integrable, and $H_0^k((0,1))$ should be the closure of $C_0^k((0,1))$ in the norm of H^k.

A function $(u,v)\colon [0,\infty) \to L^2 \times L^2$ is called a **solution** of the Dirichlet problem (1) with initial value $(u_0,v_0) \in L^2 \times L^2$ if

- $(u,v) \in C([0,\infty), L^2 \times L^2) \cap C^1((0,\infty), L^2 \times L^2)$,
- $(u,v)(t) \in (H_0^1 \cap H^2) \times (H_0^1 \cap H^2)$ for all $t > 0$,
- equation (1) is satisfied for all $t > 0$,
- $(u,v)(0) = (u_0,v_0)$.

(ii) A function $(u(\cdot,\cdot),v(\cdot,\cdot)) \in C((0,\infty)\times[0,1])$ is called a **classical solution** of the Dirichlet problem (1) with initial value $(u_0,v_0) \in L^2 \times L^2$ if

- $(u(t,\cdot),v(t,\cdot)) \in C^2([0,1])$ for all $t > 0$,
- $(u(\cdot,x),v(\cdot,x)) \in C^1((0,\infty))$ for all $x \in [0,1]$,
- equation (1) is satisfied (in the classical sense) for all $(t,x) \in (0,\infty)\times[0,1]$,
- $(u(t,0),v(t,0)) = (u(t,1),v(t,1)) = (0,0)$ for all $t > 0$,
- $||(u(t,\cdot),v(t,\cdot)) - (u_0,v_0)||_{L^2} \to 0 \qquad (t \searrow 0)$.

(iii) A solution $(u,v)\colon [0,\infty) \to L^2 \times L^2$ is called **stationary** if $(u,v)(t) = (u,v)(0)$ for all $t \geq 0$.

If (u,v) is a stationary solution, we call $(u,v)(0) \in L^2 \times L^2$ a fixed point of (1).

(iv) A solution $(u,v)\colon [0,\infty) \to L^2 \times L^2$ is called **periodic** if it is not stationary and there is $T > 0$ such that $(u,v)(t+T) = (u,v)(t)$ for all $t \geq 0$. If $T > 0$ is minimal with this property, which means that for every $T' \in (0,T)$ there is some $t > 0$ such that $(u,v)(t+T') \neq (u,v)(t)$, then we call T the period of (u,v).

Remark. (i) If there is a solution of (1) for every initial value $(u_0,v_0) \in L^2\times L^2$, then the solutions of (1) form a global semiflow on $L^2 \times L^2$.

(ii) If $(u,v)\colon [0,\infty) \to L^2 \times L^2$ is a periodic solution of (1) with period T, then $(\hat{u},\hat{v})\colon \mathbb{R} \to L^2 \times L^2$ defined by

$$(\hat{u},\hat{v})(t) := (u,v)(t+kT) \text{ for all } t \in (-kT, -(k-1)T],\ k \in \mathbb{N},$$

is a solution of (1) which coincides with (u,v) if we restrict $(\hat{u},\hat{v})$ to $[0,\infty)$. This means that we can define periodic solutions on the whole space $\mathbb{R}$. In order to have a short notation, we will just write (u,v) instead of $(\hat{u},\hat{v})$.

(ii) It is not trivial that each periodic solution (u,v) has a period. Since the set $\{\tau > 0 : (u,v)(t+\tau) = (u,v)(t) \quad \text{for all } t \geq 0\}$ is a non-empty and closed subset of $\mathbb{R}$ (which follows from the fact that (u,v) is continuous), the period is simply given by

$$T := \min\bigl\{\tau > 0 : (u,v)(t+\tau) = (u,v)(t) \text{ for all } t \geq 0\bigr\}.$$

Definition 2. (i) Let n be a positive integer. We set

$$A_n := \begin{pmatrix} -\lambda_1 n^2\pi^2 & -1 \\ 1 & -\lambda_2 n^2 \pi^2 \end{pmatrix}.$$

(ii) We set $\overline{\lambda} := (\lambda_1 + \lambda_2)/2$. For every positive integer n we define $\Delta\lambda_n := (\lambda_1 - \lambda_2)n^2\pi^2/2$.

(iii) For all $\lambda_1, \lambda_2 \in (0,\infty)$, $\lambda_1 \neq \lambda_2$, we introduce

$$p = p(\lambda_1,\lambda_2) := \frac{1}{\pi}\sqrt{\frac{2}{|\lambda_1-\lambda_2|}} \in (0,\infty).$$

In general, p will not be an integer. If $p = p(\lambda_1,\lambda_2)$ is an integer, then we call the diffusion constants (λ_1,λ_2) critical (or we just say that we deal with the critical case), otherwise non-critical. Furthermore, let $p^+ = p^+(\lambda_1,\lambda_2)$ be the smallest integer larger than p and $p^- = p^-(\lambda_1,\lambda_2)$ the largest integer smaller than p. This means that $p^+ - p^- = 1$ in the non-critical and $p^+ - p^- = 2$ in the critical case.

Remark. We note that $\lambda_1 \neq \lambda_2$ implies that $p \in (0,\infty)$ is well defined. It is clear that we would get $p^+ = p^- = \infty$ if we had $\lambda_1 = \lambda_2$. This is one reason why we exclude the case that both diffusion constants coincide.

Definition 3. (i) For every $n \in \{1,2,\ldots,p^-\}$ we define $p_n := \sqrt{1-\Delta\lambda_n^2}$ and

$$e_n^{(1)} := \begin{pmatrix} 1 \\ -\Delta\lambda_n \end{pmatrix}, \qquad e_n^{(2)} := \begin{pmatrix} 0 \\ -p_n \end{pmatrix}.$$

(ii) For every integer $n \geq p^+$ we introduce real numbers

$$\mu_n^{(1)} := -\overline{\lambda}n^2\pi^2 + \sqrt{\Delta\lambda_n^2 - 1},$$
$$\mu_n^{(2)} := -\overline{\lambda}n^2\pi^2 - \sqrt{\Delta\lambda_n^2 - 1}.$$

Furthermore, we set

$$\hat{e}_n^{(1)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left[-\Delta\lambda_n + \sqrt{\Delta\lambda_n^2 - 1}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

$$\hat{e}_n^{(2)} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \left[-\Delta\lambda_n - \sqrt{\Delta\lambda_n^2 - 1}\right]\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

and $e_n^{(j)} := \hat{e}_n^{(j)}/||\hat{e}_n^{(j)}||$ for $j = 1,2$.

(iii) If p is an integer, then we set $\mu_p = \mu_p^{(1)} = \mu_p^{(2)} := -\overline{\lambda}p^2\pi^2$ and

$$e_p^{(1)} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \qquad e_p^{(2)} := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Lemma 1. *(i) For every $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ we get $A_n e_n^{(j)} = \mu_n^{(j)} e_n^{(j)}$.*

(ii) For every $n \in \{1,2,\dots,p^-\}$ and $\alpha, \beta \in \mathbb{R}$ we get

$$A_n\left(\alpha e_n^{(1)} + \beta e_n^{(2)}\right) = \left[-\overline{\lambda} n^2\pi^2 \left(\alpha e_n^{(1)} + \beta e_n^{(2)}\right) + p_n\left(\beta e_n^{(1)} - \alpha e_n^{(2)}\right)\right].$$

(iii) If p is an integer, then we get for all $\alpha, \beta \in \mathbb{R}$

$$A_p\left(\alpha e_p^{(1)} + \beta e_p^{(2)}\right) = (2\beta + \mu_p \alpha) e_p^{(1)} + \mu_p \beta e_p^{(2)}.$$

Proof. The proof follows from an elementary computation. □

Lemma 2. *(i) We take $n \geq p^+$. Then the scalar product $(e_n^{(1)}, e_n^{(2)})$ satisfies*

$$(e_n^{(1)}, e_n^{(2)}) = \frac{1}{|\Delta\lambda_n|} = \frac{2}{|\lambda_1 - \lambda_2|}\frac{1}{\pi^2 n^2} \to 0 \qquad (n \to \infty).$$

(ii) We take real numbers $b_n^{(j)}$ for all $(n,j) \in \mathbb{N} \times \{1,2\}$. Then there is $C > 0$ (independent of the choice of the $b_n^{(j)}$'s) such that $\left(b_n^{(1)} e_n^{(1)} + b_n^{(2)} e_n^{(2)}\right)^2 \geq C\left[\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2\right]$ for all $n \in \mathbb{N}$. In particular, if $\sum_{n=p^+}^{\infty}\left(b_n^{(1)} e_n^{(1)} + b_n^{(2)} e_n^{(2)}\right)^2 < \infty$, then $\sum_{n=p^+}^{\infty}\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2$ converges.

Proof. (i) The assertion of (i) can be verified by an easy computation.

(ii) We have $\left(b_n^{(1)} e_n^{(1)} + b_n^{(2)} e_n^{(2)}\right)^2 \leq 2\left(b_n^{(1)}\right)^2 \left\|e_n^{(1)}\right\|^2 + 2\left(b_n^{(2)}\right)^2 \left\|e_n^{(2)}\right\|^2$ for $n < p^+$ and, for $n \geq p^+$,

$$\begin{aligned}
\left(b_n^{(1)} e_n^{(1)} + b_n^{(2)} e_n^{(2)}\right)^2 &= \left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2 + 2(e_n^{(1)}, e_n^{(2)}) b_n^{(1)} b_n^{(2)} \\
&\geq \left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2 - \frac{4}{|\lambda_1 - \lambda_2|}\frac{1}{\pi^2 n^2} b_n^{(1)} b_n^{(2)} \\
&\geq \left[\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2\right] \underbrace{\left(1 - \frac{2}{|\lambda_1 - \lambda_2|}\frac{1}{\pi^2 n^2}\right)}_{=:C_n}.
\end{aligned}$$

Since $C_n \to 1 \quad (n \to \infty)$ by (i), $C' := \inf\{C_n : n \geq p^+\}$ is a positive real number.

Thus, we may take $C := \min(\{C'\} \cup \{2\|e_n^{(j)}\| : n < p^+, j = 1,2\})$, and the assertion follows. □

Definition 4. For $w \in L^2$ and $n \in \mathbb{N}$ we define

$$c_n^{(w)} := \sqrt{2}\int_0^1 w \sin(n\pi\cdot).$$

We call $c_n^{(w)}$, $n \in \mathbb{N}$ the **coefficients of** w.

The following proposition is a well known result from Fourier analysis.

Proposition 1. *We take $w \in L^2$. Then we can write w in the form*

$$w = \sqrt{2} \sum_{n=1}^{\infty} c_n^{(w)} \sin(n\pi \cdot) .$$

Furthermore, $\sum_{n=1}^{\infty} (c_n^{(w)})^2$ *converges and we get*

$$||w||_{L^2}^2 = \sum_{n=1}^{\infty} \left(c_n^{(w)} \right)^2 .$$

Definition 5. We take $(u_0, v_0) \in L^2 \times L^2$. For all $(n, j) \in \mathbb{N} \times \{1, 2\}$ we set

$$b_n^{(j)} = b_n^{(j)}(u_0, v_0) := \frac{1}{||e_n^{(j)}||} \left(\begin{pmatrix} c_n^{(u_0)} \\ c_n^{(v_0)} \end{pmatrix}, e_n^{(j)} \right)$$

where $(\cdot, \cdot)$ denotes the scalar product in $\mathbb{R}^2$. We call $(b_n^{(j)})$ the **coefficients of** (u_0, v_0).

Proposition 2. *We take $(u_0, v_0) \in L^2 \times L^2$. Then we can write (u_0, v_0) in the form*

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \sqrt{2} \sum_{n=1}^{\infty} \left(b_n^{(1)} e_n^{(1)} + b_n^{(2)} e_n^{(2)} \right) \sin(n\pi \cdot) .$$

Furthermore, we get

$$\left|\left| \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right|\right|_{L^2}^2 = \sum_{n=1}^{\infty} \left(b_n^{(1)} e_n^{(1)} + b_n^{(2)} e_n^{(2)} \right)^2 .$$

Remark. We note that

$$\begin{pmatrix} u_0 \\ v_0 \end{pmatrix} = \sqrt{2} \sum_{n=1}^{\infty} \left(\beta_n^{(1)} e_n^{(1)} + \beta_n^{(2)} e_n^{(2)} \right) \sin(n\pi \cdot)$$

with real $\beta_n^{(j)}$ implies that $\beta_n^{(j)} = b_n^{(j)}$ for all $(n, j) \in \mathbb{N} \times \{1, 2\}$, i.e. the coefficients are uniquely determined.

Definition 6. We take $(u_0, v_0) \in L^2 \times L^2$. We consider the function $c_{(u_0,v_0)} : \mathbb{R} \to \mathbb{R}$,

$$c_{(u_0,v_0)}(t) := \begin{cases} e_p^{(1)} \left(b_p^{(1)} + 2t b_p^{(2)} \right) \exp(\mu_p t) + e_p^{(2)} b_p^{(2)} \exp(\mu_p t) & \text{if } p \text{ is an integer,} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$R_{(u_0,v_0)} : \mathbb{R}^2 \ni (t,y) \mapsto y^2 \sum_{k=1}^{p^-} \exp(-2\overline{\lambda} k^2 \pi^2 t) \Big[e_k^{(1)} \Big(b_k^{(1)} \cos(p_k t) - b_k^{(2)} \sin(p_k t) \Big) \\ + e_k^{(2)} \Big(b_k^{(1)} \sin(p_k t) + b_k^{(2)} \cos(p_k t) \Big) \Big]^2 \\ + y^2 \sum_{k=p^+}^{\infty} \Big[e_k^{(1)} b_k^{(1)} \exp(\mu_k^{(1)} t) + e_k^{(2)} b_k^{(2)} \exp(\mu_k^{(2)} t) \Big]^2 \\ + y^2 c^2_{(u_0,v_0)}(t) \in \mathbb{R}.$$

Then a solution $b = b_{(u_0,v_0)} \in C^1(\mathbb{R}, \mathbb{R})$ of the ODE

$$\frac{d}{dt} b = g(R_{(u_0,v_0)}(t,b))b \,, \qquad b(0) = 1$$

is called a **basic solution** (associated with (u_0, v_0)).

Lemma 3. *We take $(u_0, v_0) \in L^2 \times L^2$. Then there is a exactly one basic solution (associated with (u_0, v_0)).*

Proof. We consider the map $h \colon \mathbb{R}^2 \ni (t,y) \mapsto g(R(t,y))y \in \mathbb{R}$. Since g is continuously differentiable and $g(\eta)$ as well as $\eta g(\eta)$ are bounded for $\eta \in [0,\infty)$ by assumption,

$$\frac{\partial}{\partial y} h(t,y) = g(R(t,y)) + y g'(R(t,y)) \frac{\partial R}{\partial y}(t,y) \\ = g(R(t,y)) + 2g'(R(t,y))R(t,y)$$

is bounded uniformly for all $(t,y) \in \mathbb{R}^2$. Thus, the existence and uniqueness of the basic solution follows by standard arguments (Picard-Lindelöf). ☐

Definition 7. We take $(u_0, v_0) \in L^2 \times L^2$. We introduce functions $a_n^{(j)} : \mathbb{R} \to \mathbb{R}$, $(n,j) \in \mathbb{N} \times \{1,2\}$, as follows:

(i) For all $n \in \{1, 2, \ldots, p^-\}$ we set

$$\begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \end{pmatrix}(t) := b_{(u_0,v_0)}(t) \exp(-\overline{\lambda} n^2 \pi^2 t) \begin{pmatrix} \cos(p_n t) & -\sin(p_n t) \\ \sin(p_n t) & \cos(p_n t) \end{pmatrix} \begin{pmatrix} b_n^{(1)} \\ b_n^{(2)} \end{pmatrix}.$$

(ii) For all $n \geq p^+$, $j = 1, 2$, we set

$$a_n^{(j)}(t) := b_{(u_0,v_0)}(t) \exp(\mu_n^{(j)} t) b_n^{(j)} .$$

(iii) If p is an integer, then we set

$$a_p^{(1)}(t) := b_{(u_0,v_0)}(t) \exp(\mu_p t) \Big(b_n^{(1)} + 2t b_n^{(2)} \Big) \,, \\ a_p^{(2)}(t) := b_{(u_0,v_0)}(t) \exp(\mu_p t) b_n^{(2)} .$$

We call these functions $a_n^{(j)}$ the **coefficient functions associated with** (u_0, v_0).

Theorem 1. *We take* $(u_0, v_0) \in L^2 \times L^2$, *and define the coefficient functions associated with* (u_0, v_0) *as in Definition 7.*

(A) The sum $\sum_{n\in\mathbb{N}, j=1,2} a_n^{(j)}(t)e_n^{(j)} \sin(n\pi x)$ *converges (pointwise) for all* $(t, x) \in (0, \infty) \times [0, 1]$ *and uniformly on each set* $K \times [0, 1]$ *where* $K \subset (0, \infty)$ *is compact.*

(B) We introduce functions $u(\cdot,\cdot), v(\cdot,\cdot) : (0, \infty) \times [0, 1] \to \mathbb{R}$ *by*

$$\begin{pmatrix} u(\cdot,\cdot) \\ v(\cdot,\cdot) \end{pmatrix} := \sqrt{2} \sum_{\substack{n\in\mathbb{N} \\ j=1,2}} a_n^{(j)}(t)e_n^{(j)} \sin(n\pi x)\,.$$

Then $(u(t, x), v(t, x))$ *is a classical solution of* (1) *with initial value* (u_0, v_0). *Furthermore, we have* $u(t,\cdot), v(t,\cdot) \in C^\infty([0, 1])$ *for all* $t > 0$.

(C) We introduce the function $(u, v)\colon [0,\infty) \to L^2 \times L^2$ *by*

$$(u, v)(0) = (u_0, v_0)\,,$$
$$(u(t,\cdot), v(t,\cdot)) \text{ is a representative of } (u, v)(t) \text{ for all } t > 0.$$

Then (u, v) *is a solution of* (1) *with initial value* (u_0, v_0).

(D) The classical solution $(u(\cdot,\cdot), v(\cdot,\cdot))$ *with initial value* (u_0, v_0) *as well as the solution* (u, v) *with initial value* (u_0, v_0) *are uniquely determined.*

Proof. The assertion of Theorem 1 can be verified by a long but elementary computation, following, for example, the lines of the proofs given in Section 2 of [**2**].

3. Stationary Solutions

Definition 1. We take $(\lambda_1, \lambda_2) \in (0, \infty) \times (0, \infty)$, $\lambda_1 \neq \lambda_2$, and introduce

$$I := \begin{cases} \{p^+, p^+ + 1, \dots\} \times \{1, 2\} \cup \{(p, 1)\} & \text{if } (\lambda_1, \lambda_2) \text{ is critical,} \\ \{p^+, p^+ + 1, \dots\} \times \{1, 2\} & \text{if } (\lambda_1, \lambda_2) \text{ is non-critical.} \end{cases}$$

Lemma 1. *We take* $(u_0, v_0) \in L^2 \times L^2$. *Then* (u_0, v_0) *is a fixed point of* (1) *if and only if the coefficients associated with* (u_0, v_0) *satisfy*

(i) $b_n^{(j)} = 0$ *for all* $(n, j) \notin I$,

(ii) $b_n^{(j)} \neq 0$ *for some* $(n, j) \in I$ *implies that* $b_{(u_0,v_0)}(t) \equiv \exp(-\mu_n^{(j)} t)$.

Proof. By Definition 2.1, the coefficient-functions $a_n^{(j)}$ must be constant for all n, j. Since $a_n^{(j)}$, $n \le p^-$, can only be constant if $a_n^{(1)} \equiv a_n^{(2)} \equiv 0$, i.e. $b_n^{(j)} = 0$ for $n \le p^-$. In the critical case, $(a_p^{(1)}, a_p^{(2)})$ can only be constant if $b_p^{(2)} = 0$. Thus, (i) follows.

If we have $a_n^{(j)} \equiv c \neq 0$ for some $(n, j) \in I$, then $h(t) := b_{(u_0,v_0)}(t) \exp(-\mu_n^{(j)} t)$ is constant. Since $h(0) = 1$, (ii) is valid. □

As an easy consequence, we get

Corollary 1. *Let* $(u_0, v_0) \in L^2 \times L^2$, $(u_0, v_0) \neq (0,0)$, *be a fixed point of* (1). *Then there is* $\mu_c < 0$ *such that*

(i) $b_{(u_0,v_0)}(t) \equiv \exp(-\mu_c t)$,

(ii) $\mu_n^{(j)} = \mu_c$ *for all* $(n,j) \in I$ *with* $b_n^{(j)} \neq 0$.

We call $\mu_c = \mu_c(u_0, v_0)$ *the characteristic eigenvalue of* (u_0, v_0).

Proof. Since $(u_0, v_0) \neq (0,0)$, the set $E := \{(n,j) \in \mathbb{N} \times \{1,2\} : b_n^{(j)} \neq 0\}$ is non-empty. We take some $(m,\ell) \in E$ and set $\mu_c := \mu_m^{(\ell)} < 0$. By Lemma 1(i), we get $(m,\ell) \in I$. Furthermore, Lemma 1(ii) implies that $b_{(u_0,v_0)}(t) \equiv \exp(-\mu_m^{(\ell)} t) = \exp(-\mu_c t)$. This proves (i).

In order to show (ii) we take $(n,j) \in E$. Thus, Lemma 1(ii) gives $\exp(-\mu_c t) \equiv b_{(u_0,v_0)}(t) \equiv \exp(-\mu_m^{(\ell)} t)$ which implies that $\mu_n^{(j)} = \mu_c$. This proves (ii). □

Definition 2. A fixed point $(u_0, v_0) \in L^2 \times L^2$ of (1) is called a k-fixed point if the set

$$E_{(u_0,v_0)} := \{(n,j) \in \mathbb{N} \times \{1,2\} : b_n^{(j)} \neq 0\}$$

has exactly k elements, where $b_n^{(j)}$ should be the coefficients associated with (u_0, v_0). The set of all k-fixed points is denoted by F_k, the set of all fixed points by F.

Remark. We note that $E_{(u_0,v_0)} \subset I$ by Lemma 1.

Theorem 1. *We get* $F = F_0 \cup F_1 \cup F_2 \cup F_3$.

Proof. 1. It is clear that $F_k \subset F$ for all k. Thus, we only have to show that $F \subset F_0 \cup F_1 \cup F_2 \cup F_3$. Let $(u_0, v_0) \in L^2 \times L^2$ be a fixed point of (1). If $(u_0, v_0) = (0,0)$, then we have $(u_0, v_0) \in F_0$. In the case $(u_0, v_0) \neq (0,0)$ we proceed as follows. By Corollary 1, there exists a characteristic eigenvalue $\mu_c = \mu_c(u_0, v_0)$. It is sufficient to show that

$$E^1_{(u_0,v_0)} := \{(n,j) \in E_{(u_0,v_0)} : j = 1\}$$

has at most two and

$$E^2_{(u_0,v_0)} := \{(n,j) \in E_{(u_0,v_0)} : j = 2\}$$

has at most one element.

2. We examine the function

$$f_2 \colon [p, \infty) \ni x \mapsto -\overline{\lambda}\pi^2 x^2 - \left(\Delta\lambda_1^2 x^4 - 1\right)^{1/2} \in \mathbb{R}.$$

Since $[p, \infty) \ni x \mapsto x^2 \in \mathbb{R}$ is increasing, the function f_2 is (strictly) decreasing.

If $(n,2)$ is an element of $E^2_{(u_0,v_0)}$, then n satisfies $f_2(n) = \mu_n^{(2)} = \mu_c$. Since we have $f_2(x) = \mu_c$ for at most one $x \in [p, \infty)$, $E^2_{(u_0,v_0)}$ contains at most one element.

3. We examine the function

$$f_1 : [p, \infty) \ni x \mapsto -\overline{\lambda}\pi^2 x^2 + \left(\Delta\lambda_1^2 x^4 - 1\right)^{1/2} \in \mathbb{R}.$$

An elementary computation shows that $f_1'(x) = 0$ if and only if

$$x^4 = \frac{\overline{\lambda}^2}{\Delta\lambda_1^2 - \lambda_1\lambda_2}.$$

Thus, there is $x \geq p$ such that f_1 is strictly increasing on $[p, x]$ and strictly decreasing in $[x, \infty)$. In particular, $f_1^{-1}(\mu_c)$ and, thus, $E^1_{(u_0,v_0)}$ has at most two elements. □

It is clear that $F_0 = \{(0,0)\}$. Now we want to describe the sets F_k for $k = 1, 2, 3$.

Theorem 2. *Let $(u_0, v_0) \in L^2 \times L^2$ be a fixed point of* (1).

(i) *If $(u_0, v_0) \in F_1$, then there is $(n, j) \in I$ such that*

$$(u_0, v_0) = \pm\sqrt{g^{-1}(-\mu_n^{(j)})}\; e_n^{(j)}\sqrt{2}\sin(n\pi\cdot).$$

(ii) *If $(u_0, v_0) \in F_2$, then there are $(n, j), (m, \ell) \in I$ and $\varphi \in [0, 2\pi)$ such that $\mu_n^{(j)} = \mu_m^{(\ell)} =: \mu_c$ and*

$$(u_0, v_0) = \sqrt{g^{-1}(-\mu_c)}\left(e_n^{(j)}\cos\varphi\sin(n\pi\cdot) + e_m^{(\ell)}\sin\varphi\sin(m\pi\cdot)\right)\sqrt{2}.$$

Furthermore $\sqrt{g^{-1}(-\mu_c)}\left(e_n^{(j)}\cos\varphi\sin(n\pi\cdot) + e_m^{(\ell)}\sin\varphi\sin(m\pi\cdot)\right)\sqrt{2}$ *is a fixed point of* (1) *for all $\varphi \in [0, 2\pi)$.*

(iii) *If $(u_0, v_0) \in F_3$, then there are $(n, j), (m, \ell), (k, q) \in I$ and $\varphi \in [0, 2\pi)$, $\theta \in [-\pi/2, \pi/2]$ such that $\mu_n^{(j)} = \mu_m^{(\ell)} = \mu_k^{(q)} =: \mu_c$ and*

$$(u_0, v_0) = \sqrt{g^{-1}(-\mu_c)}\Big(e_n^{(j)}\cos\theta\cos\varphi\sin(n\pi\cdot) + e_m^{(\ell)}\cos\theta\sin\varphi\sin(m\pi\cdot) + e_k^{(q)}\sin\theta\sin(k\pi\cdot)\Big)\sqrt{2}.$$

Furthermore,

$$\sqrt{g^{-1}(-\mu_c)}\Big(e_n^{(j)}\cos\theta\cos\varphi\sin(n\pi\cdot) + e_m^{(\ell)}\cos\theta\sin\varphi\sin(m\pi\cdot) + e_k^{(q)}\sin\theta\sin(k\pi\cdot)\Big)\sqrt{2}$$

is a fixed point of (1) *for all $\varphi \in [0, 2\pi)$, $\theta \in [-\pi/2, \pi/2]$.*

Proof. Let $(u_0, v_0) \in L^2 \times L^2$ and denote the corresponding coefficients by $b_n^{(j)}$, $(n,j) \in \mathbb{N} \times \{1,2\}$.

(i) If $(u_0, v_0) \in F_1$, then there is $(n,j) \in \mathbb{N} \times \{1,2\}$ such that $b_n^{(j)} \neq 0$. Furthermore, we have $b_m^{(\ell)} = 0$ for all $(m,\ell) \neq (n,j)$. By Lemma 1, we get $(n,j) \in I$ and $b_{(u_0,v_0)}(t) = \exp(-\mu_n^{(j)} t)$. Thus, we have $R_{(u_0,v_0)}(t,y) = y^2 (b_n^{(j)})^2 \exp(2\mu_n^{(j)} t)$, and we get

$$R_{(u_0,v_0)}(t, b_{(u_0,v_0)}(t)) = \left(b_n^{(j)}\right)^2 .$$

Using the definition of the basic solution $b_{(u_0,v_0)}$, we get

$$\begin{aligned} -\mu_n^{(j)} b_{(u_0,v_0)}(t) = \frac{d}{dt} b_{(u_0,v_0)}(t) &= g\left(R_{(u_0,v_0)}(t, b_{(u_0,v_0)}(t))\right) b_{(u_0,v_0)}(t) \\ &= g\left(\left(b_n^{(j)}\right)^2\right) b_{(u_0,v_0)}(t) . \end{aligned}$$

Thus, we get $\left(b_n^{(j)}\right)^2 = g^{-1}(-\mu_n^{(j)})$, and (i) follows.

(ii) If $(u_0, v_0) \in F_2$, then there are $(n,j),(m,\ell) \in \mathbb{N} \times \{1,2\}$ such that $b_n^{(j)} \neq 0$, $b_m^{(\ell)} \neq 0$ and $b_\nu^{(\kappa)} = 0$ for all $(\nu,\kappa) \neq (n,j),(m,\ell)$. Using Lemma 1 and Corollary 1, we get $(n,j),(m,\ell) \in I$, $\mu_c := \mu_n^{(j)} = \mu_m^{(\ell)}$ and $b_{(u_0,v_0)}(t) = \exp(-\mu_c t)$. We note that $\mu_n^{(j)} = \mu_m^{(\ell)}$, $(n,j) \neq (m,\ell)$, implies that $n \neq m$. Thus, we have $R_{(u_0,v_0)}(t,y) = y^2[(b_n^{(j)})^2 + (b_m^{(\ell)})^2] \exp(2\mu_c t)$, which gives

$$R_{(u_0,v_0)}(t, b_{(u_0,v_0)}(t)) = \left(b_n^{(j)}\right)^2 + \left(b_m^{(\ell)}\right)^2 .$$

Using the definition of the basic solution $b_{(u_0,v_0)}$, we get

$$\begin{aligned} -\mu_c b_{(u_0,v_0)}(t) = \frac{d}{dt} b_{(u_0,v_0)}(t) &= g\left(R_{(u_0,v_0)}(t, b_{(u_0,v_0)}(t))\right) b_{(u_0,v_0)}(t) \\ &= g\left(\left(b_n^{(j)}\right)^2 + \left(b_m^{(\ell)}\right)^2\right) b_{(u_0,v_0)}(t) . \end{aligned}$$

Thus, we get

$$\left(b_n^{(j)}\right)^2 + \left(b_m^{(\ell)}\right)^2 = g^{-1}(-\mu_c) .$$

This means that there is $\varphi \in [0, 2\pi)$ such that $(b_n^{(j)}, b_m^{(\ell)}) = (\cos\varphi, \sin\varphi) g^{-1}(-\mu_c)$, i.e.

$$(u_0, v_0) = \sqrt{g^{-1}(-\mu_c)} \left(e_n^{(j)} \cos\varphi \sin(n\pi\cdot) + e_m^{(\ell)} \sin\varphi \sin(m\pi\cdot)\right) \sqrt{2} .$$

In particular, this means that $g^{-1}(-\mu_c)$ exists. On the other hand, an easy calculation shows that

$$\sqrt{g^{-1}(-\mu_c)} \left(e_n^{(j)} \cos\varphi \sin(n\pi\cdot) + e_m^{(\ell)} \sin\varphi \sin(m\pi\cdot)\right) \sqrt{2}$$

is a fixed point of (1) for all $\varphi \in [0, 2\pi)$. This proves (ii).

(iii) The proof of (iii) proceeds analogously to the proof of (ii). □

Theorem 2 implies that fixed points are either isolated (then they are contained in F_0 or F_1) or there exists a fixed point orbit, i.e. a connected subset of $L^2 \times L^2$ (a smooth manifold of dimension 2 or 3) which contains only fixed points of (1). Furthermore, all fixed points which belong to the same fixed point orbit have the same characteristic eigenvalue. This motivates

Definition 3. We define the set of all characteristic eigenvalues by

$$M := \{\mu_c(u_0, v_0) : (u_0, v_0) \text{ is a fixed point of (1)}\}\,.$$

For each $\mu_c \in M$ we set

$$F^{\mu_c} := \{(u_0, v_0) : (u_0, v_0) \text{ is a fixed point of (1) with } \mu_c(u_0, v_0) = \mu_c\}\,.$$

If we have $F^{\mu_c} \not\subset F_0 \cup F_1$, then we call F^{μ_c} the fixed point orbit associated with the characteristic eigenvalue μ_c.

It is clear that F^{μ_c} might have infinitely many elements (this will be the case when $F^{\mu_c} \not\subset F_0 \cup F_1$). Thus, the number of fixed points will not be finite, in general. But we can show that the number of isolated fixed points and the number of fixed point orbits are finite.

Theorem 3. *The set M has only finitely many elements.*

Proof. For every $\mu_c \in M$ there is a fixed point $(u_0, v_0) \neq (0,0)$ of (1) such that $\mu_c = \mu_c(u_0, v_0)$. By Lemma 1, there is a pair $(n, j) \in I$ such that $\mu_n^{(j)} = \mu_c$ and $b_{(u_0,v_0)}(t) \equiv \exp(-\mu_c t)$. Hence, we get

$$0 = \frac{d}{dt}(b_{(u_0,v_0)}(t)\exp(\mu_c t)) = (\mu_c + g(R(t, b_{(u_0,v_0)}(t)))b_{(u_0,v_0)}(t)\exp(\mu_c t)\,.$$

Thus, we have

$$-\mu_c = g(R(t, b_{(u_0,v_0)}(t)))$$

(note that $R(t, b_{(u_0,v_0)}(t)) = \left(\sum_{(n,j)\in I} e_n^{(j)} b_n^{(j)}\right)^2$ is independent of t). Since g is bounded from above by $g(0)$, we get $\mu_c > -g(0)$. Hence, the proof is complete if we can show that $\mu_n^{(j)} \to -\infty \quad (n \to +\infty)$ for $j = 1, 2$.

It is clear that for $n \geq p^+$

$$\mu_n^{(2)} = -\overline{\lambda} n^2\pi^2 - \left(\Delta\lambda_n^2 - 1\right)^{1/2} \to \infty \qquad (n \to \infty)\,.$$

Furthermore, we get for $n \geq p^+$

$$\begin{aligned}\mu_n^{(1)} &= -\overline{\lambda} n^2\pi^2 + \left(\Delta\lambda_n^2 - 1\right)^{1/2} = -\frac{\overline{\lambda}^2 n^4\pi^4 - \Delta\lambda_n^2 + 1}{\overline{\lambda} n^2\pi^2 + (\Delta\lambda_n^2 - 1)^{1/2}} \\ &= -\frac{n^2\pi^2\lambda_1\lambda_2 + 1}{\overline{\lambda} + \left(\left[\frac{\lambda_1-\lambda_2}{2}\right]^2 - 1/(n^4\pi^4)\right)^{1/2}} \quad \to -\infty \quad (n \to \infty)\,.\end{aligned}$$

This proves the theorem. □

In the proof of Theorem 3 we examine $\mu_n^{(j)}$ for large n. This result will be useful in the following sections. Hence we state it in

Corollary 2. *There is $c > 0$ such that for all $n \geq p^+$*

$$\mu_n^{(1)} < \mu_n^{(2)} < -cn^2$$

where the constant c depends only on λ_1 and λ_2.

4. Periodic Solutions

In this section we want to determine all periodic solutions.

Theorem 1. *Let $(u, v)\colon \mathbb{R} \to L^2 \times L^2$ be a periodic solution with period T. Then there is exactly one $n \in \{1, 2, \ldots, p^-\}$ such that the coefficient functions $a_n^{(1)}$, $a_n^{(2)}$ do not vanish identically. We call n the* **period index**.

Proof. 1. First we want to show that there is at least one $n \in \{1, 2, \ldots, p^-\}$ such that the coefficient functions $a_n^{(j)}$, $j = 1, 2$ do not vanish identically. We assume that $a_n^{(j)} \equiv 0$ for all $n \leq p^-$, $j = 1, 2$. If we had $a_n^{(j)} \equiv 0$ for all $(n, j) \in \mathbb{N} \times \{1, 2\}$, then $(u, v) = (0, 0)$ would be the zero-solution, which is a fixed point of (1), i.e. it is not periodic. Thus, there is at least one $(n, j) \in \mathbb{N} \times \{1, 2\}$ such that $a_n^{(j)} \not\equiv 0$. By assumption we have $n > p^-$.
Case 1. (λ_1, λ_2) is non-critical.
Thus, $n > p^-$ implies that $n \geq p^+$ and

$$0 \not\equiv a_n^{(j)}(t) = b_n^{(j)} \exp(\mu_n^{(j)} t) b_{(u,v)(0)}(t)\,.$$

In particular, we have $b_n^{(j)} \neq 0$. We set $\mu := \mu_n^{(j)}$. If there is $(m, \ell) \in \mathbb{N} \times \{1, 2\}$ such that $a_m^{(\ell)} \not\equiv 0$, then it follows that $m \geq p^+$, $b_m^{(\ell)} \neq 0$ and

$$\frac{a_m^{(\ell)}(t)}{a_n^{(j)}(t)} = \frac{b_m^{(\ell)}}{b_n^{(j)}} \exp((\mu - \mu_m^{(\ell)})t) \quad \text{for all } t \in \mathbb{R}.$$

(Note that $b_{(u,v)(0)}(t) \neq 0$ for all $t \in \mathbb{R}$.) Since $a_n^{(j)}(t+T) = a_n^{(j)}(t)$ and $a_m^{(\ell)}(t+T) = a_m^{(\ell)}(t)$, we get

$$\frac{b_m^{(\ell)}}{b_n^{(j)}} \exp((\mu - \mu_m^{(\ell)})(t+T)) = \frac{a_m^{(\ell)}(t+T)}{a_n^{(j)}(t+T)} = \frac{a_m^{(\ell)}(t)}{a_n^{(j)}(t)} = \frac{b_m^{(\ell)}}{b_n^{(j)}} \exp((\mu - \mu_m^{(\ell)})t)$$

which leads to $\mu_m^{(\ell)} = \mu$. Hence, $a_m^{(\ell)} \not\equiv 0$ implies that $m \geq p^+$, $b_m^{(\ell)} \neq 0$ and $\mu_m^{(\ell)} = \mu$. We set $c := \left(\sum_{m \geq p^+, \ell = 1,2} e_m^{(\ell)} b_m^{(\ell)}\right)^2$. Then we get

$$R_{(u_0,v_0)}(t, y) = y^2 \exp(2\mu t) c\,.$$

We introduce $h(t) := b_{(u,v)(0)}(t)\exp(\mu t)$. Since $a_n^{(j)}(t+T) = a_n^{(j)}(t)$ for all $t > 0$, we get $h(t+T) = h(t)$, too. Furthermore, h satisfies the ODE

$$\frac{d}{dt}h = (\mu + g(h^2 c))h\,, \quad h(0) = 1\,.$$

If we had $h(t_0) = 0$ for some t_0, then we would get $h \equiv 0$ which contradicts $h(0) = 1$. Thus, we have $h(t) > 0$ for all t. If $-\mu > g(0)$, then $dh/dt < 0$ for all t, i.e. h would be strictly decreasing which contradicts the fact that $h(t+T) = h(t)$ for all t. Hence, we get $-\mu \leq g(0)$. Thus, $h_0 := \sqrt{g^{-1}(-\mu)/c} \in [0,\infty)$ is well defined. If we have $h(t_0) = h_0$ for some t_0, then it follows that $h \equiv h_0$. This means that $h \not\equiv h_0$ implies that either $h < h_0$ or $h > h_0$ for all t. In the first case, h is strictly increasing, in the second case h is strictly decreasing. Since we have $h(t+T) = h(t)$ for all t, non of these two cases may occur, i.e. $h \equiv h_0$ is constant. This means that $a_m^{(\ell)}(t) = b_m^{(\ell)}h(t)$ is constant, too. Hence, all coefficient functions are constant, i.e. (u,v) is a fixed point. This contradicts the fact that (u,v) is a periodic solution.

Case 2. (λ_1, λ_2) is critical.

We assume that $b_p^{(2)} \neq 0$. Then $a_p^{(2)}(t) = b_p^{(2)} b_{(u_0,v_0)}(t)\exp(\mu_p t)$ is periodic. Furthermore, $a_p^{(1)}$ and, thus, the quotient $a_p^{(1)}/a_p^{(2)}$ is periodic, too. Since

$$\frac{a_p^{(1)}(t)}{a_p^{(2)}(t)} = \frac{b_p^{(1)}}{b_p^{(2)}} + 2t\,,$$

we see that the right side is not periodic, which is a contradiction.

Thus, we have $b_p^{(2)} = 0$, i.e. $b_n^{(j)} = 0$ for all $(n,j) \notin I$. Therefore, we can proceed as in Case 1.

2. We assume that there are $n, m \in \{1, 2, \ldots, p^-\}$, $m \neq n$, such that neither both, $a_n^{(1)}$ and $a_n^{(2)}$, nor both, $a_m^{(1)}$ and $a_m^{(2)}$, vanish identically. Therefore, we have $\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2 > 0$ for all $t \in \mathbb{R}$ and

$$q(t) := \frac{\left(a_m^{(1)}(t)\right)^2 + \left(a_m^{(2)}(t)\right)^2}{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2}$$

is well defined. Since (u,v) is periodic with period T, we get $q(t+T) = q(t)$ for all t. An elementary computation shows that $\frac{d}{dt}q = 2\overline{\lambda}\pi^2(n^2 - m^2)q$ for all t. Thus, we have $q(t) = q(0)\exp(2\overline{\lambda}\pi^2(n^2 - m^2)t)$, in particular $q(T) \neq q(0)$. This is a contradiction. $\square$

Theorem 2. *We take $n \in \{1,2,\dots,p^-\}$ such that $g(0) > \overline{\lambda}n^2\pi^2$. We set*

$$E_n := \left\{(k,j) \in I : \mu_k^{(j)} = -\overline{\lambda}n^2\pi^2\right\}.$$

Then E_n is a finite set which is either empty or contains one, two or three elements. We take real constants

$$\epsilon_{(k,j)} \in \mathbb{R} \quad \textit{for all } (k,j) \in E_n.$$

(a) There is a periodic solution $(u,v)\colon \mathbb{R} \to L^2 \times L^2$ of (1) such that the associated coefficient functions satisfy

(i) $a_m^{(\ell)} \equiv 0$ *for all* $(m,\ell) \notin (E_n \cup \{(n,1),(n,2)\})$,

(ii)

$$\frac{a_k^{(j)}(t)}{\sqrt{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2}} = \epsilon_{(k,j)} \quad \textit{for all } t \in \mathbb{R},$$

(iii) $a_n^{(1)}(0) > 0$, $a_n^{(2)}(0) = 0$.

(b) We denote the periodic solution which we have constructed in (a) by (u^p, v^p). If $(u,v)\colon \mathbb{R} \to L^2 \times L^2$ is a periodic solution of (1) such that conditions (i) and (ii) of (a) are satisfied, then there is $\tau \in \mathbb{R}$ such that

$$(u,v)(t) = (u^p,v^p)(t+\tau) \quad \textit{for all } t \in \mathbb{R}.$$

Since the periodic solution (u^p,v^p) is uniquely determined by n and $\epsilon_{(k,j)}$ for $(k,j) \in E_n$, we will denote (u^p,v^p) by

$$(u,v)_{n;\epsilon_{(k,j)}:(k,j)\in E_n}.$$

Before we prove Theorem 2, we show

Lemma 1. *Let (u,v) be a periodic solution of (1) with period index $n \in \{1,2,\dots,p^-\}$. Furthermore, we define E_n as in Theorem 2. Then*

$$q_k^{(j)} : \mathbb{R} \ni t \mapsto \frac{\left(a_k^{(j)}(t)\right)^2}{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2} \in \mathbb{R}$$

is well defined and constant for all $(k,j) \in E_n$.

Proof. By Theorem 1, we have $(a_n^{(1)}(0), a_n^{(2)}(0)) \neq (0,0)$. Thus, we have $\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2 > 0$ for all t, i.e. $q_k^{(j)}$ is well defined for all $(k,j) \in E_n$. An elementary computation shows that

$$\begin{aligned}\frac{d}{dt}q_k^{(j)} = 2\big[&(-\overline{\lambda}n^2\pi^2 + g(R_{(u,v)(0)}(t,b_{(u,v)(0)})))\\ &- (-\overline{\lambda}n^2\pi^2 + g(R_{(u,v)(0)}(t,b_{(u,v)(0)})))\big] = 0\,.\end{aligned}$$

Thus, the assertion follows. $\square$

Proof of Theorem 2. First, we note that the fact that E_n has at most three elements follows analogously to Theorem 3.1. Furthermore, we note that $(k,j) \in E_n$ implies that $(k, 2-j) \notin E_n$ because $\mu_k^{(1)} \neq \mu_k^{(2)}$ for all $k \geq p^+$.

1. We consider the function

$$P\colon \mathbb{R} \ni t \mapsto \left(e_n^{(1)} \cos(p_n t) + e_n^{(2)} \sin(p_n t)\right)^2 \in [0,\infty)\,.$$

Then P is periodic with period $T = 2\pi/p_n$. We examine solutions of the ODE

$$\frac{d}{dt}h = \left[-\overline{\lambda}n^2\pi^2 + g\left(\left(P + \sum_{(k,j)\in E_n} \epsilon^2_{(k,j)}\right) h^2\right)\right] h\,, \quad h(0) = h_0 > 0\,. \tag{5}$$

We show that there is exactly one initial value $h_0 > 0$ such that $h(t+T) = h(t)$ for all $t \in \mathbb{R}$.

If we had $h(t_0) = 0$ for some $t_0 \in \mathbb{R}$, then we would get $h \equiv 0$ which contradicts $h(0) > 0$. Thus, $h(t)$ is positive for all $t \in \mathbb{R}$. Since $g(0) > \overline{\lambda}n^2\pi^2 > 0$ by assumption, there is $h_- > 0$ such that

$$-\overline{\lambda}n^2\pi^2 + g\left(\left(P(t) + \sum_{(k,j)\in E_n} \epsilon^2_{(k,j)}\right) h_-^2\right) > 0 \quad \text{for all } t \in \mathbb{R}.$$

(note that P is bounded because it is periodic). Since $g^- \leq 0$ and $-\overline{\lambda}n^2\pi^2 < 0$, there is $h_+ > h_-$ such that

$$-\overline{\lambda}n^2\pi^2 + g\left(\left(P(t) + \sum_{(k,j)\in E_n} \epsilon^2_{(k,j)}\right) h_+^2\right) < 0 \quad \text{for all } t \in \mathbb{R}.$$

We show that $h(0) \in [h_-, h_+]$ implies that $h(t) \in [h_-, h_+]$ for all $t \geq 0$. We assume that this is not true. Then $t_0 := \inf\{t \geq 0 : h(t) \notin [h_-, h_+]\} \in [0,\infty)\}$ exists, and we get $h(t_0) \in \{h_-, h_+\}$. If $h(t_0) = h_+$, then it follows that

$$\frac{d}{dt}h(t_0) = \left[-\overline{\lambda}n^2\pi^2 + g\left(\left(P(t) + \sum_{(k,j)\in E_n} \epsilon^2_{(k,j)}\right) h_+^2\right)\right] h_+ < 0\,.$$

Thus, there is $\tau > 0$ such that $h(t) \leq h_+$ for all $t \in [t_0, t_0 + \tau)$. This contradicts the definition of t_0. Analogously, we get $dh/dt(t_0) > 0$ in the case $h(t_0) = h_-$, which leads to a contradiction, too.

We introduce the period map

$$\Pi : (0,\infty) \ni h_0 \mapsto h(T) \in (0,\infty)$$

where h is the solution of (5) which is uniquely determined by the initial value $h(0) = h_0$. Thus, Π is continuous and maps the compact interval $[h_-, h_+]$ into itself. Hence, there is at least one $h_p \in [h_-, h_+]$ such that $\Pi(h_p) = h_p$.

We show that Π has no fixed point besides h_p. We assume that there is $h_0 \in (0, \infty)$, $h_0 \neq h_p$, such that $\Pi(h_0) = h_0$. We denote the solution of (5) which has initial value h_p by $h^{(p)}$, the solution of (5) with initial value h_0 by $h^{(0)}$. If $h^{(0)}(t_0) = h^{(p)}(t_0)$ for some $t_0 \in \mathbb{R}$, then we get $h^{(0)} \equiv h^{(p)}$. Thus, $h_0 \neq h_p$ implies that either $h^{(0)}(t) < h^{(p)}(t)$ or $h^{(0)}(t) > h^{(p)}(t)$ for all $t \in \mathbb{R}$.
Case 1. $h^{(0)}(t) < h^{(p)}(t)$ for all $t \in \mathbb{R}$. Then we get

$$\begin{aligned}
0 = \log h^{(0)}(T) - \log h^{(0)}(0) &= \int_0^T \frac{dh^{(0)}/dt}{h^{(0)}}\, dt \\
&= \int_0^T \left[\mu + g\left(\left(P(t) + \sum_{(k,j)\in E_n} \epsilon^2_{(k,j)} \right) \left(h^{(0)}(t) \right)^2 \right) \right] dt \\
&> \int_0^T \left[\mu + g\left(\left(P(t) + \sum_{(k,j)\in E_n} \epsilon^2_{(k,j)} \right) \left(h^{(p)}(t) \right)^2 \right) \right] dt \\
&= \int_0^T \frac{dh^{(p)}/dt}{h^{(p)}}\, dt = \log h^{(p)}(T) - \log h^{(p)}(0) = 0\,.
\end{aligned}$$

This is a contradiction.
Case 2. $h^{(0)}(t) > h^{(p)}(t)$ for all $t \in \mathbb{R}$.

Then we get a contradiction analogously to Case 1.

Hence, we have proved that Π has exactly one fixed point h_p.

2. In this step we construct a periodic solution of (1) which has properties (i), (ii), (iii). In order to do so, we introduce

$$\begin{aligned}
\begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \end{pmatrix}(t) &:= h^{(p)}(t) \begin{pmatrix} \cos(p_n t) \\ \sin(p_n t) \end{pmatrix} \\
a_k^{(j)}(t) &:= \epsilon_{(k,j)} h^{(p)}(t) \quad \text{for all } (k,j) \in E_n, \\
a_k^{(j)}(t) &:= 0 \quad \text{for all } (k,j) \notin (E_n \cup \{(n,1),(n,2)\}).
\end{aligned}$$

Obviously, (iii) is satisfied. An elementary computation shows that $(u,v)\colon \mathbb{R} \to L^2 \times L^2$ defined by

$$\begin{pmatrix} u \\ v \end{pmatrix}(t) := \left(a_n^{(1)}(t) e_n^{(1)} + a_n^{(2)}(t) e_n^{(2)} \right) \sqrt{2}\sin(n\pi\cdot) + \sum_{(k,j)\in E_n} a_k^{(j)}(t) e_k^{(j)} \sqrt{2}\sin(k\pi\cdot)$$

is a periodic solution of (1). Thus, (i), (ii) are valid and (a) follows.

3. In order to prove (b), we take an arbitrary periodic solution $(u, v)\colon \mathbb{R} \to L^2 \times L^2$ of (1) which has period index n and satisfies (i), (ii).

Since (u, v) as period index n, it follows that $(b_n^{(1)}, b_n^{(2)}) \neq (0, 0)$. Thus, there is exactly one $\tau \in [0, 2\pi)$ such that

$$\begin{pmatrix} b_n^{(1)} \\ b_n^{(2)} \end{pmatrix} = \sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2} \begin{pmatrix} \cos(-\tau) \\ \sin(-\tau) \end{pmatrix}.$$

Then we get $a_n^{(2)}(\tau) = 0$, $a_n^{(1)}(\tau) = \sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2} > 0$ and

$$\begin{aligned} \begin{pmatrix} a_n^{(1)} \\ a_n^{(2)} \end{pmatrix}(t) &= b_{(u,v)(\tau)}(t-\tau) \exp(-\overline{\lambda} n^2 \pi^2 (t-\tau)) \\ &\quad \times \sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2} \begin{pmatrix} \cos(p_n(t-\tau)) \\ \sin(p_n(t-\tau)) \end{pmatrix}. \end{aligned}$$

We set $h(t) := b_{(u,v)(\tau)}(t) \exp(-\overline{\lambda} n^2 \pi^2 t) \sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2}$. Then h satisfies

$$\frac{d}{dt} h(t) = \left(-\overline{\lambda} n^2 \pi^2 + g\left(||(u, v)(t+\tau)||_{L^2}^2\right)\right) h(t).$$

We take $(k, j) \in E_n$. By (ii), $\left(a_k^{(j)}(t)\right)^2$ has the form

$$\left(a_k^{(j)}(t)\right)^2 = \epsilon_{(k,j)}^2 \left[\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2\right] = \epsilon_{(k,j)}^2 h(t)^2 \quad \text{for all } t \in \mathbb{R}.$$

Since $||(u, v)(t+\tau)||_{L^2}^2$ is given by

$$\begin{aligned} &||(u, v)(t+\tau)||_{L^2}^2 \\ &\quad = \left(\left(a_n^{(1)}(t+\tau)\right)^2 e_n^{(1)} + \left(a_n^{(2)}(t+\tau)\right)^2 e_n^{(2)}\right)^2 + \sum_{(k,j) \in E_n} \left(a_n^{(j)}(t+\tau)\right)^2 \\ &\quad = h^2(t) \left(e_n^{(1)} \cos(p_n t) + e_n^{(2)} \sin(p_n t)\right)^2 + h^2(t) \sum_{(k,j) \in E_n} \epsilon_{(k,j)}^2 \\ &\quad = h^2(t) P(t) + h^2(t) \sum_{(k,j) \in E_n} \epsilon_{(k,j)}^2, \end{aligned}$$

h is a solution of ODE (5).

Since (u, v) is periodic by assumption, $a_n^{(1)}, a_n^{(2)}$ are periodic, too. Since $a_n^{(2)}(t) = 0$ if and only if $t = 2k\pi/p_n + \tau$ where k is an integer, the period must be an integer multiple of $2\pi/p_n = T$. We assume that the period of (u,v) is given by $T' = Tp'$, $p' \in \mathbb{N}$. Thus, it follows that h is T'-periodic, too.

By Step 1, we know that $h(0) < h_p$ implies that $h(kT) > h((k-1)T)$ for all positive integers k. Thus, we would have $h(T') > h(0)$, which gives a contradiction. Analogously, we would get $h(T') < h(0)$ if we had $h(0) > h_p$. Hence, it follows that $h(0) = h_p$. Since the solution of (5) is uniquely determined by $h(0) = h_p$, we get $h = h^{(p)}$ and

$$(u,v)(t) = (u^p, v^p)(t-\tau)\,.$$

This completes the proof. □

Corollary 1. *We define the set $\mathcal{P}$ of all periodic solutions by*

$$\mathcal{P} := \big\{(u_0,v_0) \in L^2 \times L^2 : \textit{there is a periodic solution } (u,v) : \mathbb{R} \to L^2 \times L^2 \\ \textit{of } (1) \textit{ with initial value } (u,v)(0) = (u_0,v_0)\big\}\,.$$

Then it follows that

$$\mathcal{P} = \Big\{(u,v)_{n;\epsilon_{(k,j)}:(k,j)\in E_n}(t) : t \in \mathbb{R},\ n \le p^-,\ n < \pi^{-1}\sqrt{g(0)/\overline{\lambda}}, \\ \epsilon_{(k,j)} \in \mathbb{R} \textit{ for } (k,j) \in E_n\Big\}\,.$$

Proof. 1. By Theorem 2(a), $(u,v)_{n;\epsilon_{(k,j)}:(k,j)\in E_n}$ is a periodic solution of (1) for all $n \le p^-$ with $n^2 < g(0)\overline{\lambda}^{-1}\pi^{-2}$ and all $\epsilon_{(k,j)} \in \mathbb{R}$ for $(k,j) \in E_n$. It only remains to show that every periodic solution of (1) is of the form $(u,v)_{n;\epsilon_{(k,j)}:(k,j)\in E_n}(\cdot+\tau)$

We take $(u_0,v_0) \in \mathcal{P}$, i.e. there is a periodic solution (u,v) of (1) such that $(u,v)(0) = (u_0,v_0)$. We denote its period index by n.

We assume that there is $(k,j) \notin E_n \cup \{(n,1),(n,2)\}$ such that $a_k^{(j)} \not\equiv 0$. By Theorem 1, we get $k > p^-$.

Case 1. (λ_1,λ_2) is non-critical.

Thus, we get

$$a_k^{(j)}(t) = b_k^{(j)} \exp(\mu_k^{(j)} t) b_{(u_0,v_0)}(t)\,.$$

Since $a_k^{(j)} \not\equiv 0$, we have $b_k^{(j)} \neq 0$. Furthermore, an easy calculation shows that

$$\frac{d}{dt}\frac{\left(a_k^{(j)}(t)\right)^2}{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2} = 2(\mu_k^{(j)} + \overline{\lambda}n^2\pi^2)\frac{\left(a_k^{(j)}(t)\right)^2}{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2}\,.$$

If $\mu_k^{(j)} \neq -\overline{\lambda}n^2\pi^2$, then this quotient is either (strictly) increasing or decreasing, but it cannot be periodic, which gives a contradiction since all coefficient functions associated with (u,v) have to be periodic. Since $\mu_k^{(j)} = -\overline{\lambda}n^2\pi^2$, $k > p$, implies that $(k,j) \in E_n$, we get a contradiction.

Case 2. (λ_1, λ_2) is critical.

We assume that $b_p^{(2)} \neq 0$. Then $a_p^{(2)}(t) = b_p^{(2)} b_{(u_0,v_0)}(t) \exp(\mu_p t)$ is periodic. Furthermore, $a_p^{(1)}$ and, thus, the quotient $a_p^{(1)}/a_p^{(2)}$ is periodic, too. Since

$$\frac{a_p^{(1)}(t)}{a_p^{(2)}(t)} = \frac{b_p^{(1)}}{b_p^{(2)}} + 2t\,,$$

we see that the right side is not periodic, which is a contradiction.

Thus, we have $b_p^{(2)} = 0$, i.e. $b_n^{(j)} = 0$ for all $(n,j) \notin I$. Therefore, we can proceed as in Case 1.

2. We have proved that $a_k^{(j)} \equiv 0$ for all $(k,j) \notin E_n \cup \{(n,1),(n,2)\}$. We set

$$\epsilon_{(k,j)} := \frac{a_k^{(j)}(0)}{\sqrt{\left(a_n^{(1)}(0)\right)^2 + \left(a_n^{(2)}(0)\right)^2}} \quad \text{for all } (k,j) \in E_n.$$

Thus, it follows from Lemma 1 that

$$\frac{a_k^{(j)}(t)}{\sqrt{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2}} = \epsilon_{(k,j)} \quad \text{for all } t \in \mathbb{R}.$$

Hence, (u,v) satisfies conditions (i) and (ii) mentioned in Theorem 2 (a). Therefore, Theorem 2 (b) shows that there is $\tau \in \mathbb{R}$ such that

$$(u,v)(t) = (u,v)_{n;\epsilon_{(k,j)}(t):(k,j)\in E_n}(t+\tau) \quad \text{for all } t \in \mathbb{R}.$$

Thus, we have $(u_0,v_0) = (u,v)(0) = (u,v)_{n;\epsilon_{(k,j)}(t):(k,j)\in E_n}(\tau) \in \mathcal{P}$.

This completes the proof. $\square$

5. Limiting Behaviour of Solutions

In this section we determine ω-limit-set for each $(u_0,v_0) \in L^2 \times L^2$, which means that we determine the limiting behaviour of the solutions of (1) for every initial value $(u_0,v_0) \in L^2 \times L^2$.

We will see that each solution (u,v) tends either to a fixed point of the system or to a periodic solution, i.e. the ω-limit-set of each element of $L^2 \times L^2$ is contained in $F \cup \mathcal{P}$.

Definition 1. (i) For all $(u_0,v_0) \in L^2 \times L^2$, $(u_0,v_0) \neq (0,0)$ we set

$$\mu'_{(u_0,v_0)} := \max\Big(\Big\{\mu_n^{(j)} : n \geq p^+, b_n^{(j)} \neq 0\Big\} \cup \Big\{-\overline{\lambda} n^2\pi^2 : n \leq p^-, \left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2 \neq 0\Big\}\Big)\,.$$

We note that this maximum does actually exist since $\mu_n^{(j)}$ tends to $-\infty$ for $n \to \infty$ (see Corollary 3.2). Furthermore, we set

$$\mu_{(u_0,v_0)} := \begin{cases} \mu_p & \text{if } (\lambda_1,\lambda_2) \text{ is critical, } \mu_p > \mu'_{(u_0,v_0)} \text{ and } (b_p^{(1)}, b_p^{(2)}) \neq (0,0), \\ \mu'_{(u_0,v_0)} & \text{otherwise.} \end{cases}$$

We call $\mu_{(u_0,v_0)}$ the **dominant multiplier**.

(ii) We take $(u_0,v_0) \in L^2 \times L^2$, $(u_0,v_0) \neq (0,0)$.

(a) If (λ_1,λ_2) is critical and $\mu_{(u_0,v_0)} = \mu_p$ and $b_p^{(2)} \neq 0$, then we call system (1) a system with **critical dominant behaviour**. Furthermore, we set $N_{(u_0,v_0)} = N^s_{(u_0,v_0)} := \{(p,1)\}$, $N^o_{(u_0,v_0)} = N^p_{(u_0,v_0)} := \emptyset$ if $-\mu_p < g(0)$, and $N_{(u_0,v_0)} = N^o_{(u_0,v_0)} := \{(p,1)\}$, $N^s_{(u_0,v_0)} = N^p_{(u_0,v_0)} := \emptyset$ if $-\mu_p \geq g(0)$.

(b) If (λ_1,λ_2) is non-critical or if, in the critical case, $\mu_{(u_0,v_0)} > \mu_p$ or $b_p^{(2)} = 0$, then we call system (1) a system with **non-critical dominant behaviour**. In this case, we define

$$N^p_{(u_0,v_0)} := \Big\{(n,1),(n,2) : n \leq p^-, \left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2 \neq 0, -\overline{\lambda} n^2\pi^2 = \mu_{(u_0,v_0)}, \\ \overline{\lambda} n^2\pi^2 < g(0)\Big\}$$

$$N^s_{(u_0,v_0)} := \Big\{(n,j) : n \geq p^+, j = 1,2, b_n^{(j)} \neq 0, \mu_n^{(j)} = \mu_{(u_0,v_0)}, -\mu_n^{(j)} < g(0)\Big\}$$

$$N^0_{(u_0,v_0)} := \Big\{(n,j) : n \geq p^+, j = 1,2, b_n^{(j)} \neq 0, \mu_n^{(j)} = \mu_{(u_0,v_0)}, -\mu_n^{(j)} \geq g(0)\Big\} \\ \cup \Big\{(n,1),(n,2) : n \leq p^-, \left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2 \neq 0, -\overline{\lambda} n^2\pi^2 = \mu_{(u_0,v_0)}, \\ \overline{\lambda} n^2\pi^2 \geq g(0)\Big\}$$

$$N_{(u_0,v_0)} := N^p_{(u_0,v_0)} \cup N^s_{(u_0,v_0)} \cup N^0_{(u_0,v_0)}.$$

(iii) For $(u_0,v_0) = (0,0)$ we set $N_{(0,0)} = N^s_{(0,0)} = N^p_{(0,0)} = N^0_{(0,0)} := \emptyset$.

We call $N_{(u_0,v_0)}$ the set of dominant indices.

Lemma 1. *We take $(u_0,v_0) \in L^2 \times L^2$. Let $(u,v)\colon [0,\infty) \to L^2 \times L^2$ be the solution of (1) which has initial value $(u,v)(0) = (u_0,v_0)$. Then the coefficient functions $a_n^{(j)} : \mathbb{R} \to \mathbb{R}$ are bounded on $[0,\infty)$ for all $(n,j) \in N_{(u_0,v_0)}$ and we have*

$$\left\| (u,v)(t) - \sqrt{2} \sum_{(n,j) \in N_{(u_0,v_0)}} a_n^{(j)} e_n^{(j)} \sin(n\pi\cdot) \right\|^2_{L^2} \to 0 \qquad (t \to \infty).$$

Proof. The assertion is trivial if $(u_0,v_0) = (0,0)$. Thus, we only deal with the case $(u_0,v_0) \neq (0,0)$. Hence, we have $N_{(u_0,v_0)} \neq \emptyset$.

1. We take $(n, j) \in N_{(u_0,v_0)}$. We have

$$||(u,v)||_{L^2}^2 \geq \left(a_n^{(1)} e_k^{(1)} + a_n^{(1)} e_k^{(1)}\right)^2 \geq \left(a_n^{(j)}\right)^2 \underbrace{\sin^2 \angle(e_n^{(1)}, e_n^{(2)})}_{=:c_n>0} .$$

We set

$$\eta := \begin{cases} g^{-1}(-\mu_{(u_0,v_0)}) & \text{if } g(0) > -\mu_{(u_0,v_0)}, \\ 0 & \text{otherwise.} \end{cases}$$

If we have $c_n \left(a_n^{(j)}(t)\right)^2 > \eta$, the we get

$$\begin{aligned} \frac{d}{dt}\left(a_n^{(j)}(t)\right)^2 &= 2\left(a_n^{(j)}(t)\right)^2 \left(\mu_{(u_0,v_0)} + g(||(u,v)(t)||_{L^2}^2)\right) \\ &< 2\left(a_n^{(j)}(t)\right)^2 \left(\mu_{(u_0,v_0)} + g(\eta)\right) = 0 . \end{aligned}$$

Hence, $\left(a_n^{(j)}(t)\right)^2$ decreases whenever it is larger than η/c_n which means that $\left(a_n^{(j)}(t)\right)^2$ is bounded on $[0,\infty)$.

2. Case 1. We have critical dominant behaviour.

Since $a_p^{(1)}(t) = b_{(u_0,v_0)}(t)\exp(\mu_p t)\left(b_p^{(1)} + 2tb_p^{(2)}\right)$ is bounded on $[0,\infty)$ by Step 1, we obtain that $b_{(u_0,v_0)}(t)\exp(\mu_p t) \to 0 \quad (t \to \infty)$. Thus, we get

$$\begin{aligned} \left\|(u,v)(t) - \sqrt{2}a_p^{(1)} e_p^{(1)} \sin(p\pi\cdot)\right\|_{L^2}^2 \\ = \sum_{n\leq p^-} \left(a_n^{(1)}(t)e_n^{(1)} + a_n^{(2)}(t)e_n^{(2)}\right)^2 + \left(a_p^{(2)}(t)\right)^2 \\ + \sum_{n\geq p^+} \left(a_n^{(1)}(t)e_n^{(1)} + a_n^{(2)}(t)e_n^{(2)}\right)^2. \end{aligned} \tag{6}$$

Since $\left(a_n^{(1)}(t) + a_n^{(2)}(t)\right)^2 \leq \left(b_n^{(1)} + b_n^{(2)}\right)^2 b_{(u_0,v_0)}^2(t)\exp(2\mu_p t) \to 0 \quad (t \to \infty)$ for all $n \leq p^-$, the first sum of (6) tends to 0. Furthermore, we get (using Lemma 2.2)

$$\begin{aligned} &\sum_{n\geq p^+} \left(a_n^{(1)}(t)e_n^{(1)} + a_n^{(2)}(t)e_n^{(2)}\right)^2 \\ &= b_{(u_0,v_0)}^2(t)\exp(2\mu_p t) \\ &\quad \times \sum_{n\geq p^+} \left(b_n^{(1)} e_n^{(1)} \exp((\mu_n^{(1)} - \mu_p)t) + b_n^{(2)} e_n^{(2)} \exp((\mu_n^{(1)} - \mu_p)t)\right)^2 \end{aligned}$$

$$\leq 2C \underbrace{b^2_{(u_0,v_0)}(t)\exp(2\mu_p t)}_{\to 0 \quad (t\to\infty)}$$
$$\times \underbrace{\left(\sum_{n\geq p^+}\left(b_n^{(1)}\right)^2 \exp(2(\mu_n^{(1)}-\mu_p)t) + \sum_{n\geq p^+}\left(b_n^{(1)}\right)^2 \exp(2(\mu_n^{(1)}-\mu_p)t)\right)}_{\text{bounded for } t\to\infty}.$$

Thus, the last sum of (6) tends to 0, too. Since $a_p^{(1)}(t)$ is bounded for $t \to \infty$, we get

$$a_p^{(2)}(t) = a_p^{(1)}(t)\,\underbrace{\frac{b_p^{(2)}}{b_p^{(1)} + 2tb_p^{(2)}}}_{\to 0 \quad (t\to\infty)} \to 0 \qquad (t\to\infty)\,.$$

Thus, the assertion is proved.

3. Case 2. We have non-critical dominant behaviour.

We take $(n,j) \in N_{(u_0,v_0)}$. Since $a_{(n,j)}$ is bounded on $[0,\infty)$ by Step 1, it follows that $b_{(u_0,v_0)}\exp(\mu_{(u_0,v_0)}t)$ is bounded on $[0,\infty)$, too.

By Corollary 3.2, we have $\mu_n^{(j)} \to -\infty \quad (n\to\infty)$. Hence, there is $\Delta\mu > 0$ such that

$$\begin{aligned}
\mu_n^{(j)} &< \mu_{(u_0,v_0)} - \Delta\mu \quad \text{for all } (n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} \setminus N_{(u_0,v_0)} \\
&\qquad\qquad\qquad\quad \text{with } b_n^{(j)} \neq 0, \\
-\overline{\lambda} n^2\pi^2 &< \mu_{(u_0,v_0)} - \Delta\mu \quad \text{for all } (n,j) \in \{1,2,\dots,p^-\} \times \{1,2\} \setminus N_{(u_0,v_0)}.
\end{aligned}$$

If (λ_1,λ_2) is non-critical, then we get

$$\begin{aligned}
(7) \qquad & \left\| (u,v)(t) - \sqrt{2} \sum_{(n,j)\in N_{(u_0,v_0)}} a_n^{(j)} e_n^{(j)} \sin(n\pi\cdot) \right\|_{L^2}^2 \\
&= \left\| \sqrt{2} \sum_{(n,j)\notin N_{(u_0,v_0)}} a_n^{(j)} e_n^{(j)} \sin(n\pi\cdot) \right\|_{L^2}^2 \\
&\leq b^2_{(u_0,v_0)}(t)\exp(2\mu_{(u_0,v_0)}t)\underbrace{\exp(-2\Delta\mu t)}_{\to 0 \quad (t\to\infty)}
\end{aligned}$$

$$\Bigg[\sum_{\substack{(n,j)\notin \\ N_{(u_0,v_0)}, \\ n\le p^-}} \underbrace{\Big(e_n^{(1)}\Big(b_n^{(1)}\cos(p_nt)-b_n^{(2)}\sin(p_nt)\Big)+e_n^{(2)}\Big(b_n^{(1)}\sin(p_nt)+b_n^{(2)}\cos(p_nt)\Big)\Big)^2}_{\text{bounded}}$$

$$+\sum_{\substack{(n,j)\notin \\ N_{(u_0,v_0)}, \\ n\ge p^+}} \Big(e_n^{(1)}b_n^{(1)}\exp((\mu_n^{(1)}-\mu_{(u_0,v_0)}-\Delta\mu)t)$$

$$+e_n^{(2)}b_n^{(2)}\exp((\mu_n^{(2)}-\mu_{(u_0,v_0)}-\Delta\mu)t)\Big)^2\Bigg].$$

The boundedness of the last sum follows as in Step 2 using Lemma 2.2. Thus, the assertion follows if (λ_1,λ_2) is non-critical.

If (λ_1,λ_2) is critical, we proceed as follows: If we have $\mu_p=\mu_{(u_0,v_0)}$, then we set $b_p^{(2)}=0$ (because we have non-critical dominant behaviour). Therefore, (7) is valid, and the assertion follows.

If we have $\mu_p<\mu_{(u_0,v_0)}$, then we only have to add the term

$$c'(t):=b^2_{(u_0,v_0)}(t)\exp(2\mu_pt)\Big(\Big(b_p^{(1)}+2tb_p^{(2)}\Big)e_p^{(1)}+b_p^{(2)}e_p^{(2)}\Big)^2$$

on the right side of (7). Since $b^2_{(u_0,v_0)}(t)\exp(2\mu_{(u_0,v_0)}t)$ is bounded for $t\to\infty$, we get $c'(t)\to 0 \quad (t\to\infty)$, and the proof is complete. □

The following result shows that the knowledge of $b_n^{(j)}$ for all $(n,j)\in N_{(u_0,v_0)}$, is sufficient in order to determine the limiting behaviour of the solution of (1) with initial value (u_0,v_0). This is the reason why we call $N_{(u_0,v_0)}$ the set of dominant indices.

Theorem 1. *We take $(u_0,v_0)\in L^2\times L^2$. Let $(u,v)\colon[0,\infty)\to L^2\times L^2$ be the solution of* (1) *with initial value (u_0,v_0).*

(i) *If $N^p_{(u_0,v_0)}=N^s_{(u_0,v_0)}=\emptyset$, then we get (in the L^2-sense)*

$$(u,v)(t)\to 0 \qquad (t\to+\infty).$$

(ii) *If $N^p_{(u_0,v_0)}=\emptyset$ and $N^s_{(u_0,v_0)}\ne\emptyset$, then we get (for $t\to\infty$)*

$$(u,v)(t)\to\sqrt{\frac{g^{-1}(-\mu_{(u_0,v_0)})}{\sum_{(n,j)\in N^s_{(u_0,v_0)}}\Big(b_n^{(j)}\Big)^2}}\left(\sqrt{2}\sum_{(n,j)\in N^s_{(u_0,v_0)}} b_n^{(j)}e_n^{(j)}\sin(n\pi\cdot)\right)\in F.$$

(iii) *If $N^p_{(u_0,v_0)}\ne\emptyset$, then there is $n\le p^-$ such that $N^p_{(u_0,v_0)}=\{(n,1),(n,2)\}$, and we get*

$$\Big|\Big|(u,v)(t)-(u,v)_{n;\epsilon_{(k,j)}:(k,j)\in E_n}(t+\tau)\Big|\Big|_{L^2}\to 0 \qquad (t\to\infty),$$

where the real constants $\epsilon_{(k,j)}$ *are given by*

$$\epsilon_{(k,j)} := \frac{b_k^{(j)}}{\sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2}} \qquad \textit{for all } (k,j) \in E_n,$$

and $\tau \in \mathbb{R}$ *fulfills* $\tan(p_n \tau) = -b_n^{(2)}/b_n^{(1)}$.

Proof. (i) If $N^o_{(u_0,v_0)} = \emptyset$, then we have $(u_0, v_0) = (0,0)$, and the assertion is trivial. Thus, we assume that $N^o_{(u_0,v_0)} \neq \emptyset$.

Using Lemma 1, we only have to show that $a_n^{(j)}(t) \to 0 \quad (t \to \infty)$ for all $(n,j) \in N^0_{(u_0,v_0)}$. We take $(n,j) \in N^0_{(u_0,v_0)}$. If $(n,j) \neq (p,1)$ (which is always the case when we have non-critical dominant behaviour), then we get

$$\frac{d}{dt} a_n^{(j)} = \left(\mu_n^{(j)} + g(||(u,v)(t)||^2_{L^2})\right) a_n^{(j)} .$$

If we had $a_n^{(j)}(t_0) = 0$ for some t_0, then we would get $a_n^{(j)} \equiv 0$ which contradicts our assumption. Thus, $a_n^{(j)}$ is either positive or negative, w.l.o.g. we assume that $a_n^{(j)}(t) > 0$ for all t. Then we get

$$\frac{d}{dt} a_n^{(j)} < \left(\mu_n^{(j)} + g(0)\right) a_n^{(j)} \leq 0$$

which implies that $a_n^{(j)}(t) \searrow c_0 \quad (t \to \infty)$ converges to some $c_0 \geq 0$. If we had $c_0 > 0$, then there would be some constant $c_1 > 0$ (depending on $e_n^{(1)}$ and $e_n^{(2)}$) such that

$$\limsup_{t \to \infty} \frac{d}{dt} a_n^{(j)} \leq (\mu_n^{(j)} + g(c_1 c_0^2)) c_0 < 0$$

which contradicts $a_n^{(j)} \to c_0$. Thus, we have $c_0 = 0$, i.e., $a_n^{(j)} \to 0 \quad (t \to \infty)$

If $(n,j) = (p,1)$ (which means, in particular, that we have dominant critical behaviour), then $a_p^{(2)}(t) \to 0 \quad (t \to \infty)$ follows as above. Thus, we have

$$\frac{d}{dt} a_p^{(1)} = \left(\mu_p + g(||(u,v)(t)||^2_{L^2})\right) a_p^{(1)} + \underbrace{2a_p^{(2)}}_{\to 0},$$

and we obtain $a_p^{(1)}(t) \to 0 \quad (t \to \infty)$ with considerations similar to those above.

This proves (i).

(ii) Since we have $N^s_{(u_0,v_0)} \neq \emptyset$, it follows that $g(0) > -\mu_{(u_0,v_0)}$. Since $N^p_{(u_0,v_0)} = \emptyset$, this implies that $N_{(u_0,v_0)} = N^s_{(u_0,v_0)}$. Thus, $g^{-1}(-\mu_{(u_0,v_0)})$ exists.

Case 1. We have non-critical dominant behaviour.

We set $b'(t) := b_{(u_0,v_0)}(t)\exp(\mu_{(u_0,v_0)}t)$. In particular, we have $b'(t) > 0$ for all $t \geq 0$. Furthermore, we get $a_n^{(j)}(t) = b_n^{(j)} b'(t)$ for all $(n,j) \in N_{(u_0,v_0)} = N^s_{(u_0,v_0)}$. We want to show that $b'(t)$ converges to

$$b'_\infty := \sqrt{\frac{g^{-1}(-\mu_{(u_0,v_0)})}{\sum_{(n,j)\in N^s_{(u_0,v_0)}} \left(b_n^{(j)}\right)^2}}\,.$$

This will show that $a_n^{(j)}(t) \to b'_\infty b_n^{(j)} \quad (t \to \infty)$ if we have $(n,j) \in N^s_{(u_0,v_0)}$. Using Lemma 1, (ii) follows. Therefore, we only have to show that $b'(t) \to b'_\infty \quad (t \to \infty)$. Using Lemma 1, we get

$$\begin{aligned} ||(u,v)(t)||^2_{L^2} - (b'(t))^2 &\sum_{(n,j)\in N^s_{(u_0,v_0)}} \left(b_n^{(j)}\right)^2 \\ &= ||(u,v)(t)||^2_{L^2} - \sum_{(n,j)\in N^s_{(u_0,v_0)}} \left(a_n^{(j)}(t)\right)^2 \qquad \to 0 \qquad (t\to\infty)\,. \end{aligned}$$

We take $\delta > 0$. Then there are $\kappa', t_0 > 0$ such that for all $t \geq t_0$ the inequality $b'(t) \leq b'_\infty - \delta$ implies that

$$\mu_{(u_0,v_0)} + g(||(u,v)(t)||^2_{L^2}) \geq \kappa' > 0\,.$$

Thus, we have $\frac{d}{dt}b'(t) \geq \kappa' b'(t)$ if $b'(t) \leq b'_\infty - \delta$. This gives $\liminf_{t\to\infty} b'(t) \geq b'_\infty - \delta$ for all $\delta > 0$ which implies that $\liminf_{t\to\infty} b'(t) \geq b'_\infty$. Analogously, we get $\limsup_{t\to\infty} b'(t) \leq b'_\infty$. Hence, we have $\lim_{t\to\infty} b'(t) = b'_\infty$, and (ii) is proved.

Case 2. We have critical dominant behaviour. In this case, we have $N^s_{(u_0,v_0)} = \{(p,1)\}$. Using Lemma 1, we only have to show that $\left(a_p^{(1)}(t)\right)^2$ tends to $g^{-1}(-\mu_{(u_0,v_0)})$. Similar to Case 1, we get $||(u,v)(t)||^2_{L^2} - \left(a_p^{(1)}(t)\right)^2 \to 0 \quad (t \to \infty)$. We take $\delta > 0$. As in Case 1, we can find $\kappa', t_0 > 0$ such that $\left(a_p^{(1)}(t)\right)^2 \leq g^{-1}(-\mu_{(u_0,v_0)}) - \delta$ implies that $\mu_{(u_0,v_0)} + g(||(u,v)(t)||^2_{L^2}) \geq \kappa' > 0$. We get

$$\frac{d}{dt}\left(a_p^{(1)}(t)\right)^2 = 2\left(a_p^{(1)}(t)\right)^2\left(\mu_{(u_0,v_0)} + g(||(u,v)(t)||^2_{L^2})\right) + 4a_p^{(1)}(t)a_p^{(2)}(t)\,.$$

We note that $a_p^{(2)}(t)/a_p^{(1)}(t) \to 0 \quad (t\to\infty)$. Thus, there is $t'_0 \geq t_0$ such that

$$\frac{d}{dt}\left(a_p^{(1)}(t)\right)^2 \geq \frac{\kappa'}{2}\left(a_p^{(1)}(t)\right)^2$$

for $t \geq t_0'$ if $\left(a_p^{(1)}(t)\right)^2 \leq g^{-1}(-\mu_{(u_0,v_0)}) - \delta$. Thus, we can proceed as in Case 1, and we get $\lim_{t\to\infty}\left(a_p^{(1)}(t)\right)^2 = g^{-1}(-\mu_{(u_0,v_0)})$.

This proves (ii).

(iii) Since $N^p_{(u_0,v_0)} \neq \emptyset$, we have non-critical dominant behaviour. We take $(n,j) \in N^p_{(u_0,v_0)}$. By definition of $N^p_{(u_0,v_0)}$ we have $n \leq p^-$ and $-\mu_{(u_0,v_0)} = \overline{\lambda} n^2\pi^2 < g(0)$. Furthermore, $a_n^{(j)} \not\equiv 0$, $n \leq p^-$, implies that $a_n^{(1)}, a_n^{(2)} \not\equiv 0$, i.e. $(n,1),(n,2) \in N^p_{(u_0,v_0)}$. We note that $(m,j) \in N^p_{(u_0,v_0)}$ gives $\overline{\lambda} m^2\pi^2 = -\mu_{(u_0,v_0)} = \overline{\lambda} n^2\pi^2$ which leads to $m = n$. Thus, we have

$$N^p_{(u_0,v_0)} = \{(n,1),(n,2)\}\,.$$

Since $-\mu_{(u_0,v_0)} < g(0)$, it follows that $N^0_{(u_0,v_0)} = \emptyset$, but it may be that $N^s_{(u_0,v_0)}$ is not empty. By Theorem 3.3, $N^s_{(u_0,v_0)}$ has at most three elements, in particular, it is a finite set.

As in the previous step, we set $b'(t) := b_{(u_0,v_0)}\exp(\mu_{(u_0,v_0)}t)$. Thus, we get $a_k^{(j)}(t) = b'(t) b_k^{(j)}$ for all $(k,j) \in N^s_{(u_0,v_0)}$ and

$$a_n^{(1)}(t) = b'(t)(b_n^{(1)}\cos(p_n t) - b_n^{(2)}\sin(p_n t))$$
$$a_n^{(2)}(t) = b'(t)(b_n^{(1)}\sin(p_n t) + b_n^{(2)}\cos(p_n t))\,.$$

We introduce

$$\begin{aligned}\nu(t) := (b'(t))^2\Big[&e_n^{(1)}\left(b_n^{(1)}\cos(p_n t) - b_n^{(2)}\sin(p_n t)\right)\\ &+ e_n^{(2)}\left(b_n^{(1)}\sin(p_n t) + b_n^{(2)}\cos(p_n t)\right)\Big]^2 + (b'(t))^2 \sum_{(k,j)\in N^s_{(u_0,v_0)}} \left(b_k^{(j)}\right)^2 .\end{aligned}$$

Then Lemma 1 gives

$$||(u,v)(t)||^2_{L^2} - \nu(t) \to 0 \qquad (t\to\infty)\,. \tag{8}$$

We set $T := 2\pi/p_n$ and take $\tau \in [0,T)$ such that $a_n^{(2)}(\tau) = 0$ and $a_n^{(1)}(\tau) > 0$. We note that $a_n^{(2)}(\tau) = 0$ implies that $\tan(p_n\tau) = -b_n^{(2)}/b_n^{(1)}$. Furthermore, we set

$$\epsilon_{(k,j)} := \frac{b_k^{(j)}}{a_n^{(1)}(\tau)} \quad \text{for all } (k,j) \in N^s_{(u_0,v_0)}.$$

We define $P(t) := (e_n^{(1)}\cos(p_n t) + e_n^{(2)}\sin(p_n t))^2$ as in the proof of Theorem 3.3. Then we get

$$\begin{aligned}\nu(t+\tau) &= (b'(t+\tau))^2\left(\left(a_n^{(1)}(\tau)\right)^2 P^2(t) + \sum_{(k,j)\in N^s_{(u_0,v_0)}}\left(b_k^{(j)}\right)^2\right)\\ &= \left(a_n^{(1)}(\tau) b'(t+\tau)\right)^2\left(P^2(t) + \sum_{(k,j)\in N^s_{(u_0,v_0)}} \epsilon^2_{(k,j)}\right).\end{aligned} \tag{9}$$

If we define h by $h(t) := a_n^{(1)}(\tau)b'(t+\tau)$, then h is positive for all t, differentiable and satisfies

$$\frac{d}{dt}h(t) = \left(\mu_{(u_0,v_0)} + g(||(u,v)(t+\tau)||_{L^2}^2)\right) h(t)\,.$$

Given n and $\epsilon_{(k,j)}$ for all $(k,j) \in N^s_{(u_0,v_0)}$ (we note that $N^s_{(u_0,v_0)} = E_n$ in the notation of Theorem 4.2), we introduce $h_p \in (0,\infty)$ and the T-periodic function $h^{(p)}$ as in the proof of Theorem 4.2. The proof of (iii) is complete if we can show that $h(t) - h^{(p)}(t) \to 0$ for $t \to \infty$ which means that

$$q\colon [0,\infty) \ni t \mapsto \frac{h(t)}{h^{(p)}(t)} \in (0,\infty)$$

tends to 1 for $t \to \infty$.

For every $\delta > 0$ there is $\delta' = \delta'(\delta) > 0$ such that $h(t) \le h^{(p)}(t) - \delta'$ implies that $q(t) \le 1 - \delta$ (note that $h^{(p)}$ is periodic and, thus, bounded away from 0 and ∞). We set

$$H(t) := \left(h^{(p)}\right)^2(t)\Big(P^2(t) + \sum_{(k,j)\in N^s_{(u_0,v_0)}} \epsilon^2_{(k,j)}\Big) \quad \text{for all } t \in \mathbb{R}.$$

Using (8) and (9), there are $t_0 = t_0(\delta) > 0$ and $\kappa = \kappa(\delta) > 0$ such that

$$||(u,v)(t+\tau)||_{L^2}^2 \le H(t) - \kappa \quad \text{for all } t \ge t_0.$$

We note that $\varepsilon := \min_{t\in[0,T]} g(H(t)-\kappa) - g(H(t))$ is positive because g is strictly decreasing. Thus, it follows that, if we have $q(t) \le 1-\delta$ and $t \ge t_0$, then

$$\frac{d}{dt}q(t) = \left[g\left(||(u,v)(t+\tau)||_{L^2}^2\right) - g(H(t))\right] q(t) \ge \varepsilon q(t)\,.$$

This implies that $\liminf_{t\to\infty} q(t) \ge 1-\delta$. Since this result holds for all $\delta > 0$, we get $\liminf_{t\to\infty} q(t) \ge 1$. Analogously, it follows that $\limsup_{t\to\infty} q(t) \le 1$ which leads to $\lim_{t\to\infty} q(t) = 1$. This proves (iii). □

Theorem 1 shows that every solution $(u,v)\colon [0,\infty) \to L^2 \times L^2$, $(u,v)(0) = (u_0,v_0)$, of (1) tends either to a fixed point (i.e. a stationary solution) or to a periodic solution. In order to determine the limit set, it is sufficient to know the set $N_{(u_0,v_0)}$ of dominant indices and the (dominant) coefficients $b_n^{(j)}$ for all $(n,j) \in N_{(u,v)(0)}$.

Using this result, we get information about the stability and attractivity of all fixed points and periodic solutions.

6. Stability and Attractivity in the Non-critical Case

In this section we ask whether the fixed points and the periodic solutions, which we have constructed in Sections 3 and 4 and which all together attract all solutions as a result of Section 5, are stable and attractive.

In the whole section we assume that (λ_1, λ_2) is non-critical. In the critical case we can prove similar results but the proofs become (much) more technical although they use the same ideas. Furthermore, we note that nearly all $(\lambda_1, \lambda_2) \subset (0, \infty) \times (0, \infty)$ are non-critical.

Definition 1. (a) A fixed point $(u_0, v_0) \in F$ is said to be stable if and only if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that every solution $(u, v)\colon [0, \infty) \to L^2 \times L^2$ of (1) which satisfies $||(u, v)(0) - (u_0, v_0)||_{L^2} < \delta$ fulfills

$$||(u, v)(t) - (u_0, v_0)||_{L^2} < \varepsilon \quad \text{for all } t \geq 0.$$

(b) A periodic solution $(u^p, v^p)\colon \mathbb{R} \to L^2 \times L^2$ with period $T > 0$ is said to be stable if and only if for every $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) > 0$ such that every solution $(u, v)\colon [0, \infty) \to L^2 \times L^2$ of (1) which satisfies $\text{dist}_{L^2}\{(u, v)(0), \Gamma\} < \delta$, where $\Gamma := \{(u^p, v^p)(t) : 0 \leq t \leq T\}$ is the periodic orbit associated with (u^p, v^p), fulfills

$$\text{dist}_{L^2}\{(u, v)(t), \Gamma\} < \varepsilon \quad \text{for all } t \geq 0.$$

Definition 2. (a) A fixed point $(u_0, v_0) \in F$ is said to be attractive if and only if there is $\delta > 0$ such that every solution $(u, v) : [0, \infty) \to L^2 \times L^2$ of (1) which satisfies $||(u, v)(0) - (u_0, v_0)||_{L^2} < \delta$ fulfills

$$||(u, v)(t) - (u_0, v_0)||_{L^2} \to 0 \qquad (t \to \infty)\,.$$

(b) A periodic solution $(u^p, v^p) : \mathbb{R} \to L^2 \times L^2$ with period $T > 0$ is said to be attractive if and only if there is $\delta > 0$ such that every solution $(u, v) : [0, \infty) \to L^2 \times L^2$ of (1) which satisfies $\text{dist}_{L^2}\{(u, v)(0), \Gamma\} < \delta$, where Γ is defined as in Definition 1, fulfils

$$\text{dist}_{L^2}\{(u, v)(0), \Gamma\} \to 0 \qquad (t \to \infty)\,.$$

Definition 3. We define

$$\mu := \max\Big(\{\mu_n^{(1)} : n \geq p^+\} \cup \{-\overline{\lambda}\pi^2 : \text{provided that } p^- \geq 1\}\Big)$$

and call it the **maximal multiplier**.

Remark. The maximal multiplier is well defined, since $\mu_n^{(1)} \to -\infty \quad (n \to \infty)$.

Theorem 1. *Let μ be the maximal multiplier. The trivial fixed point $(0,0)$ is stable and attractive if $-\mu \geq g(0)$. On the other hand, $(0,0)$ is neither stable nor attractive if $-\mu < g(0)$.*

Proof. 1. We assume that $-\mu \geq g(0)$. Let $(u,v)\colon [0,\infty) \to L^2 \times L^2$ be a solution of (1) and denote the corresponding coefficient functions by $a_n^{(j)}$.

If there is $n \geq p^+$ such that $a_n^{(j)}(0) \neq 0$. Then we have $a_n^{(j)}(t) \neq 0$ for all $t \geq 0$. It follows that $||(u,v)(t)||_{L^2} > 0$ for all $t \geq 0$. Thus, we have

$$g\left(||(u,v)(t)||_{L^2}^2\right) < g(0) \leq -\mu \qquad \text{for all } t \geq 0.$$

Thus, it follows that

$$\frac{d}{dt}\left(a_n^{(j)}\right)^2 = 2a_n^{(j)}\left(\mu_n^{(j)} + g\left(||(u,v)||_{L^2}^2\right)\right)a_n^{(j)} < 2\left(a_n^{(j)}\right)^2 \underbrace{(\mu + g(0))}_{<0} < 0$$

for all $t \geq 0$. Hence, it follows that for all $n \geq p^+$ the function $[0,\infty) \ni t \mapsto \left(a_n^{(j)}(t)\right)^2 \in \mathbb{R}$ is either identical to zero or it is strictly decreasing.

Analogously, it follows that the function $[0,\infty) \ni t \mapsto \left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(1)}(t)\right)^2 \in \mathbb{R}$, $n \leq p^-$, is either zero or strictly decreasing.

2. We assume that $-\mu \geq g(0)$. We take $(u_0,v_0) \in L^2 \times L^2$ and denote the corresponding coefficients by $b_n^{(j)}$. Furthermore, let (u,v) be the solution of (1) with initial value (u_0,v_0) and denote the coefficient functions by $a_n^{(j)}$. We set $\delta := ||(u_0,v_0)||_{L^2}$ and

$$\varepsilon := \left(2\sum_{\substack{n>p\\ j=1,2}}\left(b_n^{(j)}\right)^2 + 2\sum_{n\leq p}\left[\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2\right]\max\left\{\left(e_n^{(1)}\right)^2, \left(e_n^{(2)}\right)^2\right\}\right)^{1/2}.$$

Using Step 1, we get $\left(a_n^{(j)}(t)\right)^2 \leq \left(b_n^{(j)}\right)^2$ for all $n \geq p^+$, $t \geq 0$, and $\left(a_n^{(1)}\right)^2 + \left(a_n^{(2)}\right)^2 \leq \left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2$ for all $n \leq p^-$, $t \geq 0$. Therefore, we have $||(u,v)(t)||_{L^2} \leq \varepsilon$ for all $t \geq 0$.

We define C as in Lemma 2.2, and set $m_+ := \max\left\{\left(e_n^{(j)}\right)^2 : n \leq p, j = 1,2\right\}/2$. Thus, we obtain

$$\varepsilon^2 \leq 2\max\left\{\left(e_n^{(j)}\right)^2 : n \leq p^-, j = 1,2\right\}\sum_{n\in\mathbb{N},j=1,2}\left(b_n^{(j)}\right)^2$$

$$= m_+\sum_{n\in\mathbb{N},j=1,2}\left(b_n^{(j)}\right)^2 \leq \frac{m_+}{C}||(u_0,v_0)||_{L^2}^2 = \frac{m_+}{C}\delta^2\,.$$

This means that, given $\varepsilon_0 > 0$, there is

$$\delta = \delta(\varepsilon_0) := \sqrt{\frac{C}{m_+}\varepsilon_0}$$

such that $||(u_0, v_0)||_{L^2} \leq \delta$ implies that $||(u, v)(t)||_{L^2} \leq \varepsilon_0$ for all $t \geq 0$, i.e., $(0, 0)$ is stable.

3. If $-\mu \geq g(0)$, then $N^p_{(u_0,v_0)} = N^s_{(u_0,v_0)} = \emptyset$ for all $(u_0, v_0) \in L^2 \times L^2$ by Definition 5.1. Thus, Theorem 5.1(i) shows that the solution starting at (u_0, v_0) tends to the zero-solution for all $(u_0, v_0) \in L^2 \times L^2$, i.e. $(0, 0)$ is attractive.

4. We assume that $-\mu < g(0)$. Then there is $n \in \mathbb{N}$ such that

$$\mu = \begin{cases} -\overline{\lambda}\pi^2 & \text{if } n \leq p^-, \\ \mu_n^{(1)} & \text{if } n \geq p^+. \end{cases}$$

We consider $(u_\delta, v_\delta) := \delta\sqrt{2}\sin(n\pi\cdot)e_n^{(j)}/||e_n^{(j)}||$ for all $\delta > 0$. Thus, we have $N^p_{(u_\delta,v_\delta)} \cup N^s_{(u_\delta,v_\delta)} \neq \emptyset$, and the solution of (1) with initial value (u_δ, v_δ) tends either to a fixed point or to a periodic orbit for $t \to \infty$ by Theorem 5.1.

Hence, (u_0, v_0) is neither stable nor attractive. □

Now we are going to determine stability and atractiveness of all other fixed points and all periodic orbits.

Lemma 1. *Let $(u_0, v_0) \in F$, $(u_0, v_0) \neq (0, 0)$, be a fixed point of (1), and let μ be the maximal multiplier. If $\mu_{(u_0,v_0)} < \mu$, then (u_0, v_0) is neither stable nor attractive.*

Proof. By definition of the maximal multiplier, there is $n \in \mathbb{N}$ such that $\mu = -\overline{\lambda}n^2\pi^2$ in the case $n \leq p^-$ or $\mu = \mu_n^{(1)}$ in the case $n \geq p^+$. We note that $-\mu < g(0)$ since otherwise no fixed point besides the trivial fixed point $(0, 0)$ exists by Theorem 1.

Thus, the solution of (1) with initial value $(u_\delta, v_\delta) := \delta\sqrt{2}\sin(n\pi\cdot)e_n^{(j)}/||e_n^{(j)}||$, $\delta > 0$, tends either to a fixed point $(\hat{u}, \hat{v}) \neq (u_0, v_0)$ or to a periodic orbit by Theorem 5.1. Since we can take δ arbitrary small, (u_0, v_0) is neither stable nor attractive. □

Lemma 2. *Let $(u, v) := (u, v)_{n;\epsilon_{(k,j)}:(k,j)\in E_n}$, be a periodic solution of (1) with period T, and denote the maximal multiplier by μ. If $-\overline{\lambda}n^2\pi^2 < \mu$, then (u, v) is neither stable nor attractive.*

Proof. The proof proceeds analogously to the proof of Lemma 1. □

Remark. By Definition 3, we get $\mu \geq -\overline{\lambda}\pi^2$. Thus, we get $-\overline{\lambda}n^2\pi^2 < \mu$ for all $n > 1$, which means that $(u, v)_{n;\epsilon_{(k,j)}:(k,j)\in E_n}$ is neither stable nor attractive for all $n > 1$.

Up to this point we know that fixed points and periodic solutions can only be stable or attractive if the corresponding dominant multiplier coincides with the maximal multiplier. The following results show that it is sufficient for stability that the corresponding dominant multiplier coincides with the maximal multiplier, but it is not sufficient for attractivity.

Lemma 3. *Let $(u_0, v_0) \in F$, $(u_0, v_0) \neq (0,0)$, be a fixed point of* (1), *and denote the maximal multiplier by μ. If we have $\mu_{(u_0,v_0)} = \mu$, then (u_0, v_0) is stable.*

Proof. 1. Let $b_n^{(j)}$ be the coefficients associated with (u_0, v_0). Thus, we have $b_n^{(j)} \neq 0$ if and only if $(n,j) \in N^s_{(u_0,v_0)}$. Furthermore, $(n,j) \in N^s_{(u_0,v_0)}$ implies that $(n, 2-j) \notin N^s_{(u_0,v_0)}$. Thus, $e_n^{(j)} \sin(n\pi\cdot), e_k^{(\ell)} \sin(k\pi\cdot)$ are orthogonal for all $(n,j),(k,\ell) \in N^s_{(u_0,v_0)}$, $(n,j) \neq (k,\ell)$. Thus, we have

$$||(u_0, v_0)||^2_{L^2} = \sum_{(n,j)\in N^s_{(u_0,v_0)}} \left(b_n^{(j)}\right)^2 .$$

Since (u_0, v_0) is a fixed point, we have $g(||(u_0, v_0)||^2_{L^2}) = -\mu$. We define C as in Lemma 2.2 and set

$$M := 1 + \frac{\sqrt{2}}{C} \max_{n \leq p, j=1,2} ||e_n^{(j)}|| .$$

2. We set $D := \sqrt{g^{-1}(-\mu)}/M$. For every $\delta \in (0, D)$ we define the compact interval I_δ by

$$I_\delta := \left[\underbrace{\frac{1}{1+\delta\frac{M}{\sqrt{g^{-1}(-\mu)}}}}_{=:\delta_-}, \underbrace{\frac{1}{1-\delta\frac{M}{\sqrt{g^{-1}(-\mu)}}}}_{=:\delta_+}\right] \subset (0,\infty) .$$

Let $(u,v)\colon [0,\infty) \to L^2 \times L^2$ be a solution of (1) which satisfies $||(u,v)(0) - (u_0,v_0)||_{L^2} < \delta$. Furthermore, we introduce the function $b'\colon \mathbb{R} \ni t \mapsto b_{(u,v)(0)}(t)\exp(\mu t) \in (0,\infty)$. We note that $b'(0) = 1$, i.e. $b'(0) \in I_\delta$. We want to show that $b'(t) \in I_\delta$ for all $t \geq 0$.

Since $b'(0) \in I_\delta$, it is sufficient to show that $b'(t) < \delta_-$ for some $t > 0$ implies that $\frac{d}{dt}b'(t) > 0$, and $b'(t) > \delta_+$ implies that $\frac{d}{dt}b'(t) < 0$. Then it follows that $b'(t)$ cannot leave I_δ, i.e. $b'(t) \in I_\delta$ for all $t \geq 0$.

We assume that $b'(t) < \delta_-$ for some $t > 0$. Then we get

$$
\begin{aligned}
&||(u,v)(t)||_{L^2}\\
&\quad= \left|\left| b'(t)\sqrt{2} \sum_{(n,j)\in\mathbb{N}\times\{1,2\}} a_n^{(j)}(0)\exp((\mu_n^{(j)}-\mu)t)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}\\
&\quad\le \left|\left| b'(t)\sqrt{2} \sum_{(n,j)\in\mathbb{N}\times\{1,2\}} a_n^{(j)}(0)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}\\
&\qquad+ \left|\left| b'(t)\sqrt{2} \sum_{(n,j)\in\mathbb{N}\times\{1,2\}} a_n^{(j)}(0)\left(1-\exp((\mu_n^{(j)}-\mu)t)\right)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}.
\end{aligned}
$$

The first expression satisfies

$$
\begin{aligned}
&\left|\left| b'(t)\sqrt{2} \sum_{(n,j)\in\mathbb{N}\times\{1,2\}} a_n^{(j)}(0)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}\\
&\qquad= b'(t)||(u,v)(0)||_{L^2} < b'(t)||(u_0,v_0)||_{L^2} + b'(t)\delta = b'(t)\sqrt{g^{-1}(-\mu)} + b'(t)\delta
\end{aligned}
$$

and the second satisfies

$$
\begin{aligned}
&\left|\left| b'(t)\sqrt{2} \sum_{(n,j)\in\mathbb{N}\times\{1,2\}} a_n^{(j)}(0)\left(1-\exp((\mu_n^{(j)}-\mu)t)\right)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}\\
&\le b'(t)\left|\left| \sqrt{2} \sum_{(n,j)\in\mathbb{N}\times\{1,2\},\mu_n^{(j)}<\mu} a_n^{(j)}(0)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}\\
&\le b'(t)\left(2\sum_{(n,j)\in\mathbb{N}\times\{1,2\},\mu_n^{(j)}<\mu}\left(a_n^{(j)}(0)\right)^2||e_n^{(j)}||^2\right)^{1/2}\\
&\le \sqrt{2}b'(t)\max_{n\le p,j=1,2}||e_n^{(j)}||\left(\sum_{(n,j)\in\mathbb{N}\times\{1,2\},\mu_n^{(j)}<\mu}\left(a_n^{(j)}(0)\right)^2\right)^{1/2}\\
&= (M-1)cb'(t)\left(\sum_{(n,j)\in\mathbb{N}\times\{1,2\},\mu_n^{(j)}<\mu}\left(a_n^{(j)}(0)\right)^2\right)^{1/2}\\
&\le (M-1)cb'(t)\left(\sum_{(n,j)\in\mathbb{N}\times\{1,2\}}\left(a_n^{(j)}(0)-b_n^{(j)}\right)^2\right)^{1/2}
\end{aligned}
$$

$$\leq (M-1)cb'(t)\left(\sum_{n\in\mathbb{N}} c^{-2}\left|\left|\left(a_n^{(1)}(0)-b_n^{(1)}\right)e_n^{(1)}+\left(a_n^{(2)}(0)-b_n^{(1)}\right)e_n^{(2)}\right|\right|^2\right)^{1/2}$$

$$= (M-1)b'(t)||(u,v)(0)-(u_0,v_0)||_{L^2} < (M-1)\delta b'(t)\,.$$

Thus, $b'(t) < \delta_-$ implies that

$$||(u,v)(t)||_{L^2} < b'(t)\left(\sqrt{g^{-1}(-\mu)}+M\delta\right) < \delta_-\left(\sqrt{g^{-1}(-\mu)}+M\delta\right) = \sqrt{g^{-1}(-\mu)}\,.$$

Hence, we get

$$\frac{d}{dt}b'(t) = \left(\mu + g\left(||(u,v)(t)||_{L^2}^2\right)\right)b'(t) > 0\,.$$

Analogously, $b'(t) > \delta_+$ implies that $\frac{d}{dt}b'(t) < 0$. Thus, we have proved that $b'(t) \in I_\delta$ for all $t \geq 0$.

3. We assume that (u,v) is a solution of (1) with $||(u,v)(0)-(u_0,v_0)||_{L^2} < \delta$, $\delta > 0$. Then we get

$$||(u,v)(t)-(u_0,v_0)||_{L^2} \leq \underbrace{\left|\left|\sqrt{2}\sum_{(n,j)\in N^s_{(u_0,v_0)}}\left(a_n^{(j)}(t)-b_n^{(j)}\right)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}}_{=:S_1(t)}$$

$$+\underbrace{\left|\left|\sqrt{2}\sum_{(n,j)\notin N^s_{(u_0,v_0)}} a_n^{(j)}(t)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}}_{=:S_2(t)}\,.$$

Thus, we get (note that $(n,j) \in N^s_{(u_0,v_0)}$ implies that $||e_n^{(j)}|| = 1$)

$$S_1^2(t) \leq 2\sum_{(n,j)\in N^s_{(u_0,v_0)}}\left(a_n^{(j)}(t)-b_n^{(j)}\right)^2 = 2\sum_{(n,j)\in N^s_{(u_0,v_0)}}\left(a_n^{(j)}(0)b'(t)-b_n^{(j)}\right)^2$$

$$\leq 4\left[\sum_{(n,j)\in N^s_{(u_0,v_0)}}\left(a_n^{(j)}(0)-b_n^{(j)}\right)^2 + \sum_{(n,j)\in N^s_{(u_0,v_0)}}(1-b'(t))^2\left(a_n^{(j)}(t)\right)^2\right]$$

$$\leq 4\delta^2/c^2 + (1-b'(t))^2\sum_{(n,j)\in N^s_{(u_0,v_0)}}\left(a_n^{(j)}(t)\right)^2\,.$$

Since $\sum_{(n,j)\in N^s_{(u_0,v_0)}}\left(a_n^{(j)}(t)\right)^2$ is bounded (because of $\delta < D$) and $1-b'(t) = O(\delta)$ (because of $b'(t) \in I_\delta$ for all $t \geq 0$ by Step 2), there is some constant $c_1 > 0$ such that $S_1^2(t) \le c_1^2\delta^2$ for all $t \ge 0$. Using the results of Step 1 and 2, we get

$$\begin{aligned}
S_2^2(t) &\le 2 \max_{n\le p^-, j=1,2} ||e_n^{(j)}||^2 \sum_{(n,j)\notin N^s_{(u_0,v_0)}} \left(a_n^{(j)}(t)\right)^2 \\
&= 2(b'(t))^2 \max_{n\le p^-, j=1,2} ||e_n^{(j)}||^2 \sum_{(n,j)\notin N^s_{(u_0,v_0)}} \left(a_n^{(j)}(0)\right)^2 \underbrace{\exp((\mu_n^{(j)}-\mu)t)}_{\le 1} \\
&\le 2\delta_+^2 \max_{n\le p^-, j=1,2} ||e_n^{(j)}||^2 \sum_{(n,j)\notin N^s_{(u_0,v_0)}} \left(a_n^{(j)}(0)\right)^2 \\
&< 2\delta_+^2 \max_{n\le p^-, j=1,2} ||e_n^{(j)}||^2 \frac{\delta^2}{c^2}
\end{aligned}$$

Thus, it follows that there is some constant $c_2 > 0$ such that $S_2^2(t) \le c_2^2\delta^2$ for all $t \ge 0$. If we put these two results together, we get

$$||(u,v)(t) - (u_0,v_0)||_{L^2} \le (c_1 + c_2)\delta \quad \text{for all } t \ge 0.$$

Thus, for every given $\varepsilon > 0$ there is $\delta = \delta(\varepsilon) \in (0, D)$ such that $(c_1 + c_2)\delta < \varepsilon$. Hence, (u_0, v_0) is stable.

Lemma 4. *Let $(u_0,v_0) \in F$, $(u_0,v_0) \neq (0,0)$, be a fixed point of* (1), *and denote the maximal multiplier by μ. If we have $\mu_{(u_0,v_0)} = \mu$ and the set*

$$\begin{gathered}
\{(n,j) \in \{1,\dots,p^-\} \times \{1,2\} : -\overline{\lambda} n^2\pi^2 = \mu\} \\
\cup \{(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} : \mu_n^{(j)} = \mu\}
\end{gathered}$$

has only one element, then (u_0, v_0) is attractive.

Proof. Since $\mu_{(u_0,v_0)} = \mu$, there is $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ such that $\mu_n^{(j)} = \mu$, $b_n^{(j)} \neq 0$. Thus, there is $\delta > 0$ such that $||(u,v)(0) - (u_0,v_0)||_{L^2} < \delta$ implies that $a_n^{(j)}(0) \neq 0$. Using the assumption, we obtain that $N_{(u,v)(0)} = \{(n,j)\}$. Thus, Theorem 5.1(ii) shows that $(u,v)(t)$ tends to (u_0,v_0) for $t \to \infty$, i.e. (u_0,v_0) is attractive. □

Lemma 5. *Let $(u_0,v_0) \in F$, $(u_0,v_0) \neq (0,0)$, be a fixed point of* (1), *and denote the maximal multiplier by μ. If $\mu_{(u_0,v_0)} = \mu$ and the set*

$$\begin{gathered}
\{(n,j) \in \{1,\dots,p^-\} \times \{1,2\} : -\overline{\lambda} n^2\pi^2 = \mu\} \\
\cup \{(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} : \mu_n^{(j)} = \mu\}
\end{gathered}$$

has at least two elements, then (u_0, v_0) is not attractive.

Proof. Case 1. There are $(n_1,j_1), (n_2,j_2) \in N^s_{(u_0,v_0)}$ such that $(n_1,j_1) \neq (n_2,j_2)$. We note that this implies that $n_1 \neq n_2$. Then, for every $\delta > 0$, there is $\varphi \in (0, \pi/2)$ such that

$$2(1-\cos\varphi)\left(\left(b_{n_1}^{(j_1)}\right)^2 + \left(b_{n_2}^{(j_2)}\right)^2\right) < \delta^2 .$$

We set

$$c_n^{(j)} := \begin{cases} b_{n_1}^{(j_1)} \cos\varphi - b_{n_2}^{(j_2)} \sin\varphi & \text{for } (n,j) = (n_1, j_1) \\ b_{n_1}^{(j_1)} \sin\varphi + b_{n_2}^{(j_2)} \cos\varphi & \text{for } (n,j) = (n_2, j_2) \\ b_n^{(j)} & \text{otherwise} \end{cases} .$$

Then $(u_0', v_0') := \sqrt{2} \sum_{(n,j) \in \mathbb{N} \times \{1,2\}} c_n^{(j)} e_n^{(j)} \sin(n\pi\cdot)$ is a also fixed point of (1). Since

$$\begin{aligned} \|(u_0', v_0') - (u_0, v_0)\|_{L^2}^2 &= \sum_{(n,j) \in N^s_{(u_0,v_0)}} \left(c_n^{(j)} - b_n^{(j)} \right)^2 \\ &= \left[b_{n_1}^{(j_1)} (1 - \cos\varphi) + b_{n_2}^{(j_2)} \sin\varphi \right]^2 + \left[-b_{n_1}^{(j_1)} \sin\varphi + b_{n_2}^{(j_2)} (1 - \cos\varphi) \right]^2 \\ &= 2(1 - \cos\varphi) \left[\left(b_{n_1}^{(j_1)} \right)^2 + \left(b_{n_2}^{(j_2)} \right)^2 \right] < \delta \,, \end{aligned}$$

there is a fixed point of (1) in every δ-neighbourhood of (u_0, v_0), i.e. (u_0, v_0) is not attractive.

Case 2. The set $N^s_{(u_0,v_0)}$ has only one element.

Then there is $n \in \{1, \ldots, p^-\}$ such that $\mu_{(u_0,v_0)} = -\overline{\lambda} n^2 \pi^2$. Since $\mu_{(u_0,v_0)} = \mu$ by assumption and $\mu \geq -\overline{\lambda}\pi^2$, it follows that $n = 1$. For every $\delta > 0$ we set

$$(u_0', v_0') := (u_0, v_0) + \frac{\delta}{2} \frac{e_1^{(1)}}{\|e_1^{(1)}\|} \sqrt{2} \sin(\pi\cdot) \in L^2 \times L^2 \,.$$

By Theorem 5.1(iii), the solution (u, v) of (1) with initial value (u_0', v_0') tends to a periodic solution and, thus, does not tend to (u_0, v_0). Since $\|(u_0', v_0') - (u_0, v_0)\|_{L^2} = \delta/2 < \delta$, the assertion follows. □

Now we deal with periodic solutions.

Lemma 6. *If we have $\mu = -\overline{\lambda}\pi^2$, then each periodic solution of the form $(u^p, v^p) := (u, v)_{1;\epsilon_{(k,j)}:(k,j)\in E_1}$ is stable.*

Proof. 1. Let $\Gamma := \{(u^p, v^p) : 0 \leq t \leq T\}$ be the periodic orbit. Furthermore, we note that (u^p, v^p) has period $T = 2\pi/p_1$. We denote the coefficient functions associated with (u^p, v^p) by $(a^p)_n^{(k)} : \mathbb{R} \to \mathbb{R}$. By definition of $(u^p, v^p) := (u, v)_{1;\epsilon_{(k,j)}:(k,j)\in E_1}$ in Section 4, we get $(a^p)_n^{(k)} \equiv 0$ for all $n \in \{2, \ldots, p^-\}$ and for all $(n, k) \in \{p^+, p^+ + 1, \ldots\} \times \{1, 2\} \setminus E_1$. Furthermore, we have $(a^p)_1^{(1)}(0) > 0$, $(a^p)_1^{(2)}(0) = 0$. We consider the function

$$h^p : \mathbb{R} \ni t \mapsto (a^p)_1^{(1)}(0) \exp(\mu t) b_{(u^p,v^p)(0)}(t) \in \mathbb{R}.$$

As shown in the proof of Theorem 4.2, h^p is positive and periodic with period T. Thus, we get $h^p_+ := \max\{h^p(t) : t \in [0,T]\}$, $h^p_- := \min\{h^p(t) : t \in [0,T]\} \in (0,\infty)$.

2. We take $\delta \in (0,1)$. Let $(u,v)\colon [0,\infty) \to L^2 \times L^2$ be a solution of (1) such that $\operatorname{dist}_{L^2}\{(u,v)(0),\Gamma\} < \delta$. This means that there is $\tau \in [0,T)$ such that $||(u,v)(0) - (u^p,v^p)(\tau)||_{L^2} < \delta$. Let $a_n^{(j)}$ be the coefficient functions associated with (u,v). We set

$$\begin{aligned} h&\colon [0,\infty) \ni t \mapsto \sqrt{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2}\exp(\mu t) b_{(u,v)(0)}(t) \quad \in (0,\infty)\,, \\ q&\colon [0,\infty) \ni t \mapsto \frac{h(t)}{h^p(t+\tau)} \quad \in (0,\infty)\,. \end{aligned}$$

It is easy to see that

$$\hat{h}\colon \mathbb{R} \ni t \mapsto \sqrt{\left((a^p)_1^{(1)}(\tau)\right)^2 + \left((a^p)_1^{(2)}(\tau)\right)^2}\exp(\mu t) b_{(u^p,v^p)(\tau)}(t) \in \mathbb{R}$$

is also a periodic solution of the ODE (5) (see proof of Theorem 4.2) which defines h^p. Furthermore, we have $\hat{h}(0) = h^p(\tau)$. Since the solution of (5) is unique, we get $\hat{h}(t) = h(t+\tau)$ for all $t \in \mathbb{R}$. Thus, we get

$$q(t) = \frac{\sqrt{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2}}{\sqrt{\left((a^p)_1^{(1)}(\tau)\right)^2 + \left((a^p)_1^{(2)}(\tau)\right)^2}} \frac{b_{(u,v)(0)}(t)}{b_{(u^p,v^p)(\tau)}(t)}\,.$$

If we have $E_1 \neq \emptyset$, then we define $\epsilon'_{(n,j)}$ for all $(n,j) \in E_1$ by

$$\epsilon'_{(n,j)} := \frac{a_n^{(j)}(0)}{\sqrt{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2}}\,.$$

Then we get

$$\begin{aligned} &\left|\left|\begin{pmatrix} u \\ v \end{pmatrix}(t)\right|\right|_{L^2} \\ &\quad\le \underbrace{\left|\left|\left(a_1^{(1)}(t)e_1^{(1)} + a_1^{(2)}(t)e_1^{(2)}\right)\sqrt{2}\sin(\pi\cdot) + \sqrt{2}\sum_{(n,j)\in E_1} a_n^{(j)}(t)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}}_{=:S_1} \\ &\qquad + \underbrace{\left|\left|\sqrt{2}\sum_{(n,j)\notin E_1, n>1} a_n^{(j)}(t)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}}_{=:S_3}\,. \end{aligned}$$

Since $(n, j) \in E_1$ implies that $(n, 2-j) \notin E_1$, we get

$$S_1^2 = \underbrace{\left|\left|\left(a_1^{(1)}(t)e_1^{(1)} + a_1^{(2)}(t)e_1^{(2)}\right)\right|\right|^2}_{S_{11}^2} + \underbrace{\sum_{(n,j)\in E_1} \left(a_n^{(k)}(t)\right)^2}_{S_{12}^2}.$$

Given $r > 0$, $O(\delta^r)$ is said to be an expression which does only depend on δ such that $O(\delta^r)/\delta^r$ is bounded for $\delta \searrow 0$. Using the fact that $||(u,v)(0) - (u^p, v^p)(\tau)||_{L^2} < \delta$, we obtain

$$\left|(a^p)_n^{(j)}(\tau)\right| - O(\delta) \le \left|(a_n^{(j)}(0)\right| \le \left|(a^p)_n^{(j)}(\tau)\right| + O(\delta) \quad \text{for all } (n,j), \tag{10}$$

(of course, we get different expressions $O(\delta)$ for different (n, j)). Hence, we get

$$\begin{aligned}
S_{11}^2 &= \left|\left|\left(a_1^{(1)}(t)e_1^{(1)} + a_1^{(2)}(t)e_1^{(2)}\right)\right|\right|^2 \\
&= \frac{h^2(t)}{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2}\Big|\Big|((a_1^{(1)}(0)\cos(p_1t) - a_1^{(2)}(0)\sin(p_1t))e_1^{(1)} \\
&\qquad + (a_1^{(1)}(0)\sin(p_1t) + a_1^{(2)}(0)\cos(p_1t))e_1^{(2)})\Big|\Big|^2 \\
&= q^2(t)(h^p(t+\tau))^2\frac{\left((a^p)_1^{(1)}(\tau)\right)^2 + \left((a^p)_1^{(2)}(\tau)\right)^2}{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2} \\
&\quad \times \Big|\Big|((a_1^{(1)}(0)\cos(p_1t) - a_1^{(2)}(0)\sin(p_1t))e_1^{(1)} \\
&\qquad + (a_1^{(1)}(0)\sin(p_1t) + a_1^{(2)}(0)\cos(p_1t))e_1^{(2)})\Big|\Big|^2 \\
&\le q^2(t)(h^p(t+\tau))^2(1+O(\delta)) \\
&\quad \times \Big(\Big|\Big|(((a^p)_1^{(1)}(\tau)\cos(p_1t) - (a^p)_1^{(2)}(\tau)\sin(p_1t))e_1^{(1)} \\
&\qquad + ((a^p)_1^{(1)}(\tau)\sin(p_1t) + (a^p)_1^{(2)}(\tau)\cos(p_1t))e_1^{(2)})\Big|\Big|^2 + O(\delta)\Big) \\
&= q^2(t)\Big|\Big|(((a^p)_1^{(1)}(t+\tau)\cos(p_1t) - (a^p)_1^{(2)}(t+\tau)\sin(p_1t))e_1^{(1)} \\
&\qquad + ((a^p)_1^{(1)}(t+\tau)\sin(p_1t) + (a^p)_1^{(2)}(t+\tau)\cos(p_1t))e_1^{(2)})\Big|\Big|^2 + q^2(t)O(\delta).
\end{aligned}$$

Using the fact that E_1 is a finite set and that

$$\left|\epsilon_{(n,j)} - \epsilon'_{(n,j)}\right| = \left|\frac{a_n^{(j)}(0)}{\sqrt{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2}} - \frac{(a^p)_n^{(j)}(\tau)}{\sqrt{\left((a^p)_1^{(1)}(\tau)\right)^2 + \left((a^p)_1^{(2)}(\tau)\right)^2}}\right|$$

$$= O(\delta)$$

by (10), we obtain that

$$\begin{aligned}
S_{12}^2 &= \sum_{(n,j)\in E_1} \left(a_n^{(j)}(t)\right)^2 = h^2(t) \sum_{(n,j)\in E_1} (\epsilon'_{(n,j)})^2 \\
&\le q^2(t)(h^p(t+\tau))^2 \sum_{(n,j)\in E_1} \left[(\epsilon_{(n,j)})^2 + O(\delta)\right] \\
&= q^2(t) \sum_{(n,j)\in E_1} \left((a^p)_n^{(j)}(t+\tau)\right)^2 + q^2(t)O(\delta)\,.
\end{aligned}$$

Let C be defined as in Lemma 2.2. Furthermore, we get

$$\left|\left|\alpha e_n^{(1)} + \beta e_n^{(2)}\right|\right|^2 \le \left(\alpha^2+\beta^2\right) \underbrace{2\left(\max_{n\le p^-, j=1,2} ||e_n^{(j)}||\right)^2}_{=:m} \quad \text{for all } \alpha,\beta\in\mathbb{R}.$$

Thus, we get

$$\begin{aligned}
S_2^2 &= \left|\left|\sqrt{2}\sum_{(n,j)\notin E_1, n>1} a_n^{(j)}(t)e_n^{(j)}\sin(n\pi\cdot)\right|\right|_{L^2}^2 \\
&\le m \sum_{(n,j)\notin E_1, n>1} \left(a_n^{(j)}(t)\right)^2 \\
&= m\frac{h^2(t)}{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2} \sum_{(n,j)\notin E_1, n>1} \left(a_n^{(j)}(0)\right)^2 \\
&= m\frac{h^2(t)}{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2} \sum_{(n,j)\notin E_1, n>1} \left(a_n^{(j)}(0) - \underbrace{(a^p)_n^{(j)}(\tau)}_{=0}\right)^2 \\
&\le m\frac{h^2(t)}{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2} \sum_{(n,j)\in\mathbb{N}\times\{1,2\}} \left(a_n^{(j)}(0) - (a^p)_n^{(j)}(\tau)\right)^2
\end{aligned}$$

$$
\begin{aligned}
&\le m \frac{h^2(t)}{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2} C^{-2} ||(u,v)(0) - (u^p, v^p)(\tau)||_{L^2}^2 \\
&< \frac{m}{C^2} q^2(t) (h^p(t+\tau))^2 \frac{\left((a^p)_1^{(1)}(\tau)\right)^2 + \left((a^p)_1^{(2)}(\tau)\right)^2}{\left(a_1^{(1)}(0)\right)^2 + \left(a_1^{(2)}(0)\right)^2} \delta^2 \\
&\le \frac{m}{C^2} (h_+^p)^2 q^2(t) (1 + O(\delta)) \delta^2 \\
&= q^2(t) O(\delta^2)\,.
\end{aligned}
$$

It we put these three results together, we get

$$
||(u,v)(t)||_{L^2} \le q(t) ||(u^p, v^p)(t+\tau)||_{L^2} + q(t) O(\delta)\,.
$$

Analogously, we get $||(u,v)(t)||_{L^2} \ge q(t)||(u^p,v^p)(t+\tau)||_{L^2} - q(t)O(\delta)$. Furthermore, we have $1 - O(\delta) \le q(0) \le 1 + O(\delta)$. We set $m_- := \min\{||(u^p,v^p)(t)||_{L^2} : 0 \le t \le T\} > 0$. Then it follows that there are $\gamma, \delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$

$$
0 < \frac{1}{1 + \frac{\gamma}{m_-}\delta} \le q(0) \le \frac{1}{1 - \frac{\gamma}{m_-}\delta} < \infty\,,
$$

and for all $t \ge 0$

$$
-\gamma\delta q(t) + q(t)||(u^p,v^p)(t+\tau)||_{L^2} \le ||(u,v)(t)||_{L^2} \le q(t)||(u^p,v^p)(t+\tau)||_{L^2} + \gamma\delta q(t)\,.
$$

3. We introduce the compact interval

$$
I_\delta := \left[\frac{1}{1 + \frac{\gamma}{m_-}\delta}, \frac{1}{1 - \frac{\gamma}{m_-}\delta}\right] \subset (0, \infty)\,.
$$

By Step 2, we have $q(0) \in I_\delta$. We want to show that $q(t) \in I_\delta$ for all $t \ge 0$.

If we have

$$
q(t) < \frac{1}{1 + \frac{\gamma}{m_-}\delta} < 1
$$

for some $t > 0$, then an elementary computation shows that

$$
\frac{q(t)}{1 - q(t)} < \frac{m_-}{\delta\gamma}\,.
$$

By definition of m_-, we get $(1 - q(t))||(u^p,v^p)(t)||_{L^2} > \delta\gamma q(t)$. Using Step 2, it follows that

$$
||(u,v)(t)||_{L^2} \le q(t)||(u^p,v^p)(t+\tau)||_{L^2} + \gamma\delta q(t) < ||(u^p,v^p)(t+\tau)||_{L^2}\,.
$$

Analogously,

$$q(t) > \frac{1}{1 - \frac{\gamma}{m_-}\delta} > 1$$

implies that $||(u,v)(t)||_{L^2} > ||(u^p,v^p)(t+\tau)||_{L^2}$. Since we get

$$\frac{d}{dt}q(t) = q(t)\left[g\left(||(u,v)(t)||_{L^2}^2\right) - g\left(||(u^p,v^p)(t+\tau)||_{L^2}^2\right)\right],$$

the function q is increasing whenever it is smaller than $1/(1+\frac{\gamma}{m_-}\delta)$ and decreasing whenever it is larger than $1/(1-\frac{\gamma}{m_-}\delta)$. Since we have $q(0) \in I_\delta$, it follows that $q(t) \in I_\delta$ for all $t \geq 0$.

4. By definition of I_δ, there is $\eta > 0$ such that $I_\delta \subset [1-\eta\delta, 1+\eta\delta]$ for all $\delta \in (0,\delta_0)$. Thus, Step 3 gives $1-\eta\delta \leq q(t) \leq 1+\eta\delta$ for all $\delta \in (0,\delta_0)$.

We set $m_+ := \max\{||(u^p,v^p)(t)||_{L^2} : 0 \leq t \leq T\} > 0$. Then Step 2 implies that for all $t \geq 0$

$$\begin{aligned}
||(u,v)(t) - (u^p,v^p)(t+\tau)||_{L^2} &\leq \Big|||(u,v)(t)||_{L^2} - ||(u^p,v^p)(t+\tau)||_{L^2}\Big| \\
&\leq ||(u^p,v^p)(t+\tau)||_{L^2}|1-q(t)| + \gamma\delta q(t) \\
&\leq \eta\delta m_+ + \gamma\delta(1+\eta\delta) \\
&< \underbrace{(\eta m_+ + \gamma + \gamma\eta\delta_0)}_{=:C_0}\,\delta\,.
\end{aligned}$$

Since C_0 does neither depend on δ nor on the solution (u,v), we can set

$$\delta(\varepsilon) := \min\{\delta_0, \varepsilon/C_0\} \quad \text{for any given } \varepsilon > 0.$$

Then, $||(u,v)(0) - (u^p,v^p)(\tau)||_{L^2} < \delta(\varepsilon)$ implies that $||(u,v)(t) - (u^p,v^p)(t+\tau)||_{L^2} < \varepsilon$ for all $t \geq 0$. Since $\mathrm{dist}_{L^2}\{(u,v)(0),\Gamma\} < \delta(\varepsilon)$ implies that there is $\tau \in [0,T)$ such that $||(u,v)(0) - (u^p,v^p)(\tau)||_{L^2} < \delta(\varepsilon)$ holds (by Step 2), the assertion follows. $\square$

Lemma 7. *If we have $\mu = -\overline{\lambda}\pi^2$ and $\mu_n^{(j)} \neq \mu$ for all $(n,j) \in \{p^+, p^+{+}1, \dots\} \times \{1,2\}$, then each periodic solution of the form $(u^p,v^p) := (u,v)_{1;\epsilon_{(k,j)}:(k,j)\in E_1}$ is attractive.*

Proof. We denote the period of (u^p,v^p) by T and set $\Gamma := \{(u^p,v^p)(t) : 0 \leq t \leq T\}$. Furthermore, we denote the coefficient functions associated with (u^p,v^p) by $(a^p)_n^{(j)}$, $(n,j) \in \mathbb{N} \times \{1,2\}$. Since the period index of (u^p,v^p) is 1, we get $\left((a^p)_1^{(1)}(t), (a^p)_1^{(2)}(t)\right) \neq (0,0)$ for all $t \in \mathbb{R}$. Thus, we have

$$\delta := \min\left\{\left|\left|(a^p)_1^{(1)}(t)e_1^{(1)} + (a^p)_1^{(2)}(t)e_1^{(2)}\right|\right| : t \in [0,T]\right\} > 0\,.$$

Let $(u,v)\colon [0,\infty) \to L^2 \times L^2$ be a solution of (1) which satisfies $\operatorname{dist}_{L^2}\{(u,v)(0),\Gamma\} < \delta$. By definition of δ, we have $\left(a_1^{(1)}(0), a_1^{(2)}(0)\right) \neq (0,0)$. Therefore we get $N_{(u,v)(0)} = \{(1,1),(1,2)\}$. By Theorem 5.1(iii), it follows that $\operatorname{dist}_{L^2}\{(u,v)(t),\Gamma\} \to 0 \quad (t\to\infty)$, which completes the proof. $\square$

Lemma 8. *Let $(u^p,v^p) := (u,v)_{1;\epsilon_{(k,j)}:(k,j)\in E_1}$, be a periodic solution of (1), and denote the maximal multiplier by μ. If $\mu = -\overline{\lambda}\pi^2$ and $\mu_n^{(\ell)} = \mu$ for at least one $(n,\ell) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$, then (u^p,v^p) is not attractive.*

Proof. By assumption, there is $(n,\ell) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ such that $\mu_n^{(\ell)} = \mu = -\overline{\lambda}\pi^2$. This means that $(n,\ell) \in E_1$. For every $\delta > 0$ we set

$$(u_\delta, v_\delta) := (u^p,v^p)(0) + \frac{\delta}{2} e_n^{(\ell)} \sqrt{2}\sin(n\pi\cdot) \in L^2 \times L^2 .$$

Let $(\hat{u},\hat{v})$ be the solution of (1) with initial value (u_0,v_0). By Theorem 5.1(iii), (u,v) tends to a periodic solution of the form $(u,v)_{1;\epsilon'_{(k,j)}:(k,j)\in E_1}(\tau + \cdot)$ with $\epsilon'_{(k,j)} := \epsilon_{(k,j)}$ for $(k,j) \in E_1 \setminus \{(n,\ell)\}$ and

$$\begin{aligned} \epsilon'_{(n,\ell)} &:= \frac{(a^p)_n^{(j)}(0) + \delta/2}{\sqrt{\left((a^p)_1^{(1)}(0)\right)^2 + \left((a^p)_1^{(1)}(0)\right)^2}} \\ &= \epsilon_{(n,\ell)} + \frac{\delta/2}{\sqrt{\left((a^p)_1^{(1)}(0)\right)^2 + \left((a^p)_1^{(1)}(0)\right)^2}} \end{aligned}$$

where $(a^p)_k^{(j)}$ should be the coefficient functions associated with (u^p,v^p).

Since we have $\epsilon'_{(n,\ell)} \neq \epsilon'_{(n,\ell)}$, $(\hat{u},\hat{v})$ does not tend to $\Gamma := \{(u^p,v^p)(t) : 0 \le t \le T\}$ where T is the period of (u^p,v^p). Since $||(u,v)(0) - (u^p,v^p)(0)||_{L^2} = \delta/2 < \delta$, there is an element (u_0,v_0) of $L^2 \times L^2$ in every neighbourhood of $(u^p,v^p)(0)$ such that the solution of (1) with initial value (u_δ,v_δ) does not tend to Γ.

This shows that the periodic orbit Γ is not attractive. $\square$

We can put all these results together and get the following

Conclusion. Let μ be the maximal multiplier.

(a) A fixed point $(u_0,v_0) \in F$ is stable if and only if $\mu_{(u_0,v_0)} = \mu$.

(b) A fixed point $(u_0,v_0) \in F$ is attractive if and only if $\mu_{(u_0,v_0)} = \mu$ and $N_{(u_0,v_0)}$ has only one element.

(c) A periodic solution $(u^p,v^p)\colon \mathbb{R} \to L^2 \times L^2$ is stable if and only if the period index is 1 and $\mu = -\overline{\lambda}\pi^2$.

(d) A periodic solution $(u^p,v^p)\colon \mathbb{R} \to L^2 \times L^2$ is attractive if and only if the period index is 1 and $N_{(u^p,v^p)(0)} = \{(1,1),(1,2)\}$ (which implies that $\mu = -\overline{\lambda}\pi^2$).

7. Stable Manifolds in the Non-Critical Case

In this section we use the results of the previous sections in order to determine the stable manifolds for all fixed points and periodic solutions in the non-critical case.

Definition 1. For $(u_0, v_0) \in F$ we define the stable manifold by

$$\begin{aligned} W^s(u_0,v_0) := \big\{(u',v') \in L^2 \times L^2 : {} & \text{solution } (u,v)\colon [0,\infty) \to L^2 \times L^2 \text{ of } (1) \\ & \text{with } (u,v)(0) = (u',v') \\ & \text{fulfills } (u,v)(t) \to (u_0,v_0) \quad (t \to \infty)\big\}\,. \end{aligned}$$

Let $(u^p, v^p)\colon \mathbb{R} \to L^2 \times L^2$ be a periodic solution with period $T > 0$ and denote the periodic orbit by $\Gamma := \{(u^p,v^p)(t) : 0 \le t \le T\}$. Then the stable manifold is given by

$$\begin{aligned} W^s(\Gamma) := \big\{(u',v') \in L^2 \times L^2 : {} & \text{solution } (u,v)\colon [0,\infty) \to L^2 \times L^2 \text{ of } (1) \\ & \text{with } (u,v)(0) = (u',v') \\ & \text{fulfills } \operatorname{dist}_{L^2}\{(u,v)(t),\Gamma\} \to 0 \quad (t \to \infty)\big\}\,. \end{aligned}$$

Remark 1. We note that Definition 1 does not imply that W^s is actually a manifold although we call it the 'stable manifold'. But we will soon see that W^s is a manifold so that the denotation is motivated.

Theorem 1. *We take $(u_0, v_0) \in F$ and denote the corresponding coefficients by $b_n^{(j)}$. Furthermore, we set $E := \{(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} : \mu_n^{(j)} = \mu_{(u_0,v_0)}\}$. Then we get*

$$\begin{aligned} W^s(u_0,v_0) = \big\{(u',v') \in L^2 \times L^2 : {} & \mu_{(u',v')} = \mu_{(u_0,v_0)} \\ & \mathit{sgn}(b_n^{(j)}) = \mathit{sgn}((b')_n^{(j)}) \textit{ for all } (n,j) \in E, \\ & b_n^{(j)} (b')_m^{(k)} = (b')_n^{(j)} b_m^{(k)} \textit{ for all } (n,j),(m,k) \in E, \\ & (b')_n^{(j)} = 0 \textit{ for all } (n,j) \in \{1,\dots,p^-\} \times \{1,2\} \\ & \textit{such that } -\overline{\lambda} n^2 \pi^2 = \mu_{(u_0,v_0)}\big\}\,. \end{aligned}$$

Proof. 1. We take $(u',v') \in L^2 \times L^2$ such that $\mu_{(u',v')} = \mu_{(u_0,v_0)}$, $\operatorname{sgn}(b_n^{(j)}) = \operatorname{sgn}((b')_n^{(j)})$ for all $(n,j) \in E$, $b_n^{(j)} (b')_m^{(k)} = (b')_n^{(j)} b_m^{(k)}$ for all $(n,j),(m,k) \in E$, and $(b')_n^{(j)} = 0$ for all $(n,j) \in \{1,\dots,p^-\} \times \{1,2\}$ such that $-\overline{\lambda} n^2 \pi^2 = \mu_{(u_0,v_0)}$. Then Theorem 5.1(ii) shows that the solutions of (1) which have initial values (u_0,v_0) respectively (u',v') tend to the same fixed point which is, thus, (u_0,v_0). Hence, we get $(u',v') \in W^s(u_0,v_0)$.

2. We take $(u', v') \in W^s(u_0, v_0)$. Let (u, v) be the solution of (1) with initial value (u', v'). If we had $\mu_{(u',v')} \neq \mu_{(u_0,v_0)}$, then (u, v) does not tend to (u_0, v_0) for $t \to \infty$ by Theorem 5.1. Thus, we get $\mu_{(u',v')} = \mu_{(u_0,v_0)}$.

If we take $(n, j), (m, k) \in E$, then the coefficient functions associated with (u, v) satisfy

$$\begin{aligned} a_n^{(j)}(t) &:= (b')_n^{(j)} b_{(u',v')}(t) \exp(\mu_{(u',v')} t) \\ a_m^{(k)}(t) &:= (b')_m^{(k)} b_{(u',v')}(t) \exp(\mu_{(u',v')} t) \,. \end{aligned}$$

If we had $b_n^{(j)} (b')_m^{(k)} \neq (b')_n^{(j)} b_m^{(k)}$, then we would get $(b')_n^{(j)} \neq 0$ or $(b')_m^{(k)} \neq 0$. W.l.o.g. we assume that $(b')_n^{(j)} \neq 0$. Thus,

$$q(t) := \frac{a_m^{(k)}(t)}{a_n^{(j)}(t)} = \frac{(b')_m^{(k)}}{(b')_n^{(j)}}$$

is constant (i.e. independent of t). Since $(u', v') \in W^s(u_0, v_0)$ implies that $q(t) \to b_m^{(k)}/b_n^{(j)}$ $(t \to \infty)$ and $b_m^{(k)}/b_n^{(j)} \neq (b')_m^{(k)}/(b')_n^{(j)}$, we get a contradiction. Thus, we have $b_n^{(j)} (b')_m^{(k)} = (b')_n^{(j)} b_m^{(k)}$ for all $(n, j), (m, k) \in E$.

If there was $(n, j) \in \{1, \dots, p^-\} \times \{1, 2\}$ such that $(b')_n^{(j)} \neq 0$ and $-\overline{\lambda} n^2 \pi^2 = \mu_{(u',v')}$, then we would get $(n, j) \in N^p_{(u',v')}$ which implies that (u, v) tends to a periodic solution by Theorem 5.1(iii). Thus, $(u', v') \in W^s(u_0, v_0)$ implies that $(b')_n^{(j)} = 0$ for all $(n, j) \in \{1, \dots, p^-\} \times \{1, 2\}$ with $-\overline{\lambda} n^2 \pi^2 = \mu_{(u_0,v_0)}$.

Using Theorem 5.1(ii), we see that $\operatorname{sgn}(b_n^{(j)}) \neq \operatorname{sgn}((b')_n^{(j)})$ for some $(n, j) \in E$ implies that (u', v') does not tend to (u_0, v_0). Therefore, we have

$$\begin{aligned} W^s(u_0, v_0) \subset \big\{ (u', v') \in L^2 \times L^2 : \mu_{(u',v')} &= \mu_{(u_0,v_0)} \\ \operatorname{sgn}(b_n^{(j)}) &= \operatorname{sgn}((b')_n^{(j)}) \text{ for all } (n, j) \in E, \\ b_n^{(j)} (b')_m^{(k)} &= (b')_n^{(j)} b_m^{(k)} \text{ for all } (n, j), (m, k) \in E, \\ (b')_n^{(j)} &= 0 \text{ for all } (n, j) \in \{1, \dots, p\} \times \{1, 2\} \\ &\text{such that } -\overline{\lambda} n^2 \pi^2 = \mu_{(u_0,v_0)} \big\} \,. \end{aligned}$$

and the proof is complete. □

Remark 2. If $(u_0, v_0) \in F$ is a fixed point such that $-\overline{\lambda} n^2 \pi^2 \neq \mu_{(u_0,v_0)}$ for all $n \in \{1, \dots, p^-\}$ and $\mu_n^{(j)} = \mu_{(u_0,v_0)}$ for only one $(n, j) \in \{p^+, p^+ + 1, \dots\} \times \{1, 2\}$, then we get

$$W^s(u_0, v_0) = \left\{ (u', v') \in L^2 \times L^2 : \mu_{(u',v')} = \mu_{(u_0,v_0)}, \operatorname{sgn}(b_n^{(j)}) = \operatorname{sgn}((b')_n^{(j)}) \right\} \,.$$

As a consequence of Theorem 1, we get

Theorem 2. *We take* $(u_0, v_0) \in F$. *Then* $W^s(u_0, v_0)$ *is open if and only if* $\mu_{(u_0,v_0)}$ *coincides with the maximal multiplier* μ, *we have* $\mu \neq -\overline{\lambda}\pi^2$ *and* $\mu = \mu_n^{(j)}$ *for only one* $(n, j) \in \{p^+, p^+ + 1, \dots\} \times \{1, 2\}$.

Proof. 1. We assume that $\mu_{(u_0,v_0)} = \mu$, $\mu \neq -\overline{\lambda} n^2\pi^2$ and $\mu = \mu_n^{(j)}$ for only one $(n, j) \in \{p^+, p^+ + 1, \dots\} \times \{1, 2\}$. We take $(n, j) \in \{p^+, p^+ + 1, \dots\} \times \{1, 2\}$ such that $\mu_n^{(j)} = \mu$. By Remark 2, we have

$$W^s(u_0, v_0) = \left\{(u', v') \in L^2 \times L^2 : \mu_{(u',v')} = \mu_{(u_0,v_0)},\ \operatorname{sgn}(b_n^{(j)}) = \operatorname{sgn}((b')_n^{(j)})\right\} .$$

We take $(u', v') \in W^s(u_0, v_0)$. Let $(b')_k^{(\ell)}$ be the coefficients associated with (u', v'). Then $(u', v') \in W^s(u_0, v_0)$ implies that $\operatorname{sgn}((b')_n^{(j)}) = \operatorname{sgn}(b_n^{(j)}) \neq 0$. We set $\delta := |(b')_n^{(j)}| > 0$. If we take $(u'', v'') \in L^2 \times L^2$ with $||(u'', v'') - (u', v')||_{L^2} < \delta$, then we get $\operatorname{sgn}(b_n^{(j)}) = \operatorname{sgn}((b'')_n^{(j)})$, which means that $\mu_{(u'',v'')} = \mu_{(u_0,v_0)}$ and, thus, $(u'', v'') \in W^s(u_0, v_0)$. Hence, $W^s(u_0, v_0)$ is open.

2. We assume that $W^s(u_0, v_0)$ is an open set. We take $(n, j) \in \mathbb{N} \times \{1, 2\}$ such that $\mu = \mu_n^{(j)}$ if $n \geq p^+$ or $\mu = -\overline{\lambda} n^2\pi^2$ if $n \leq p^-$. If we had $\mu_{(u_0,v_0)} < \mu$, then for every $(u', v') \in W^s(u_0, v_0)$ and $\delta > 0$

$$(u'', v'') := (u', v') + \frac{\delta}{2} \frac{e_n^{(j)}}{||e_n^{(j)}||} \sqrt{2} \sin(n\pi \cdot)$$

would not be contained in $W^s(u_0, v_0)$ because of $\mu_{(u'',v'')} = \mu > \mu_{(u_0,v_0)}$. Hence, $W^s(u_0, v_0)$ would not be open. Thus, we have $\mu_{(u_0,v_0)} = \mu$.

We take $(u', v') \in W^s(u_0, v_0)$. If we had $-\overline{\lambda}\pi^2 = \mu$, then

$$(u'', v'') := (u', v') + \frac{\delta}{2} \frac{e_1^{(1)}}{||e_1^{(1)}||} \sqrt{2} \sin(\pi \cdot)$$

would not be contained in $W^s(u_0, v_0)$ for any $\delta > 0$ since the solution of (1) with initial value (u'', v'') tends to a periodic orbit by Theorem 5.1(iii). Thus, we get a contradiction as above.

We take $(n, j) \in N^s_{(u',v')}$. We assume that there is $(m, k) \in \{p^+, p^+ + 1, \dots\} \times \{1, 2\}$ such that $(n, j) \neq (m, k)$ and $\mu_m^{(k)} = \mu$. Given $\delta > 0$, we consider

$$(u'', v'') := (u', v') + \frac{\delta}{2} \frac{e_m^{(k)}}{||e_m^{(k)}||} \sqrt{2} \sin(m\pi \cdot) .$$

We get a contradiction as above if we can show that (u'', v'') is not contained in $W^s(u_0, v_0)$ for every $\delta > 0$. We note that the corresponding coefficients satisfy $(b'')_q^{(\ell)} = (b')_q^{(\ell)}$ for all $(q, \ell) \neq (m, k)$ and $(b'')_m^{(k)} \neq (b')_m^{(k)}$. Thus, it follows that

$$\sum_{(q,\ell)\in N^s_{(u'',v'')}} \left((b'')_q^{(\ell)}\right)^2 \neq \sum_{(q,\ell)\in N^s_{(u',v')}} \left((b')_q^{(\ell)}\right)^2 .$$

Hence, we have $(b'')_n^{(j)} = (b')_n^{(j)}$ and

$$\frac{(b'')_n^{(j)}}{\sqrt{\sum_{(q,\ell)\in N^s_{(u'',v'')}} \left((b'')_q^{(\ell)}\right)^2}} \neq \frac{(b')_n^{(j)}}{\sqrt{\sum_{(q,\ell)\in N^s_{(u',v')}} \left((b')_q^{(\ell)}\right)^2}}$$

which shows that the solutions of (1) with initial values (u', v') respectively (u'', v'') tend to different elements of F by Theorem 5.1(ii). This completes the proof. □

Theorem 3. *Let $(u^p, v^p)\colon \mathbb{R} \to L^2 \times L^2$ be a periodic solution of (1). We denote the period by T, the periodic orbit by Γ and the period index by n. Furthermore, we denote the corresponding coefficient functions by $(a^p)_m^{(j)}$, and set $E := \{(m,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} : \mu_m^{(j)} = -\overline{\lambda} n^2\pi^2\}$. Then we get*

$$\begin{aligned} W^s(\Gamma) = \Bigg\{ &(u_0, v_0) \in L^2 \times L^2 : \mu_{(u_0,v_0)} = -\overline{\lambda} n^2\pi^2, \\ &\frac{b_m^{(j)}}{\sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2}} = \frac{(a^p)_m^{(j)}(0)}{\sqrt{\left((a^p)_n^{(1)}(0)\right)^2 + \left((a^p)_n^{(2)}(0)\right)^2}} \\ &\textit{for all } (m,j) \in E \Bigg\}. \end{aligned}$$

Proof. 1. We take $(u_0, v_0) \in L^2 \times L^2$ such that $\mu_{(u_0,v_0)} = -\overline{\lambda} n^2\pi^2$, and

$$\frac{b_m^{(j)}}{\sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2}} = \frac{(a^p)_m^{(j)}(0)}{\sqrt{\left((a^p)_n^{(1)}(0)\right)^2 + \left((a^p)_n^{(2)}(0)\right)^2}} \quad \text{for all } (m,j) \in E.$$

Then Theorem 5.1 shows that the solutions of (1) with initial values (u_0, v_0) respectively $(u, v)(0)$ tend to the same periodic orbit which is, thus, Γ. Hence, we get $(u_0, v_0) \in W^s(\Gamma)$.

2. We take $(u_0, v_0) \in L^2 \times L^2$ and denote the solution of (1) with initial value (u_0, v_0) by (u, v). If we had $\mu_{(u_0,v_0)} \neq -\overline{\lambda} n^2\pi^2$, then $(u, v)(t)$ would not tend to Γ for $t \to \infty$ by Theorem 5.1. Thus, we have $\mu_{(u_0,v_0)} = -\overline{\lambda} n^2\pi^2$.

We take $(m, j) \in E$. Then the coefficient functions associated with (u, v) satisfy

$$\frac{a_m^{(j)}(t)}{\sqrt{\left(a_n^{(1)}(t)\right)^2 + \left(a_n^{(2)}(t)\right)^2}} = \frac{b_m^{(j)}}{\sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2}} \quad \text{for all } t \geq 0.$$

Since

$$\mathbb{R} \ni t \mapsto \frac{(a^p)_m^{(j)}(t)}{\sqrt{\left((a^p)_n^{(1)}(t)\right)^2 + \left((a^p)_n^{(2)}(t)\right)^2}} \in \mathbb{R}$$

is a constant function, $\mathrm{dist}_{L^2}\{(u,v)(t),\Gamma\} \to 0 \quad (t \to \infty)$ implies that

$$\frac{b_m^{(j)}}{\sqrt{\left(b_n^{(1)}\right)^2 + \left(b_n^{(2)}\right)^2}} = \frac{(a^p)_m^{(j)}(0)}{\sqrt{\left((a^p)_n^{(1)}(0)\right)^2 + \left((a^p)_n^{(2)}(0)\right)^2}} .$$

This completes the proof. $\square$

Theorem 4. *Let $(u^p, v^p)\colon \mathbb{R} \to L^2 \times L^2$ be a periodic solution of* (1) *with period T and periodic orbit Γ. Then $W^s(\Gamma)$ is open if and only if the period index is* 1*, the maximal multiplier μ coincides with $-\overline{\lambda}\pi^2$ and $\mu \neq \mu_n^{(j)}$ for all $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$.*

Proof. 1. We assume that the maximal index is 1, the maximal multiplier μ coincides with $-\overline{\lambda}\pi^2$ and $\mu \neq \mu_n^{(j)}$ for all $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$. Thus, Theorem 3 gives

$$W^s(\Gamma) = \left\{(u_0, v_0) \in L^2 \times L^2 : \mu_{(u_0,v_0)} = -\overline{\lambda}\pi^2\right\} .$$

We take $(u_0, v_0) \in W^s(\Gamma)$. Then the corresponding coefficients satisfy $\left(b_1^{(1)}, b_1^{(2)}\right)$ $\neq (0,0)$. Thus, there is $\delta > 0$ such that the coefficients $(b')_n^{(j)}$ of $(u', v') \in L^2 \times L^2$ with $||(u',v') - (u_0,v_0)||_{L^2} < \delta$ satisfy $\left((b')_1^{(1)}, (b')_1^{(2)}\right) \neq (0,0)$. This means that $\mu_{(u',v')} = -\overline{\lambda}\pi^2$ and, thus, $(u',v') \in W^s(u_0,v_0)$. Hence, there is a neighbourhood of (u_0,v_0) which is contained in $W^s(\Gamma)$, i.e. $W^s(\Gamma)$ is open.

2. We assume that $W^s(\Gamma)$ is an open set. If the period index was larger than 1, then for every $\delta > 0$ the solution (u,v) of (1) with initial value

$$(u',v') := (u^p,v^p)(0) + \frac{\delta}{2}\frac{e_1^{(1)}}{||e_1^{(1)}||}\sqrt{2}\sin(\pi\cdot)$$

would not tend to Γ by Theorem 5.1(iii). Thus, the period index must be 1.

If there was $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ such that $\mu_n^{(j)} > -\overline{\lambda}\pi^2$, then the solution (u,v) of (1) with initial value

$$(u',v') := (u^p,v^p)(0) + \frac{\delta}{2}e_n^{(j)}\sqrt{2}\sin(n\pi\cdot)$$

would tend to a fixed point for every $\delta > 0$ by Theorem 5.1. Thus, there is no such (n,j).

If there was $(n,j) \in \times\{1,2\}$ such that $\mu_n^{(j)} = -\overline{\lambda}\pi^2$, then the solution (u,v) of (1) with initial value

$$(u',v') := (u^p,v^p)(0) + \frac{\delta}{2}e_n^{(j)}\sqrt{2}\sin(n\pi\cdot)$$

would tend to a periodic orbit which does not coincide with Γ by Theorem 5.1(iii) for every $\delta > 0$.

This gives a contradiction as above and completes the proof. $\square$

Thus, we have determined all stable manifolds. Using Theorems 1 and 3, we see that these stable manifolds are indeed manifolds. We also get the codimensions of these manifolds just by looking at the formulas given in Theorems 1 and 3.

We want to construct an open and dense subset of $L^2 \times L^2$ in which the dynamics can easily be described.

Theorem 5. *Let μ be the maximal multiplier and set*

$$E := \left\{(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\} : \mu_n^{(j)} = \mu\right\}.$$

We introduce

$$A^f := \bigcup_{\substack{(u_0,v_0)\in F,\\ \mu_{(u_0,v_0)}=\mu}} W^s(u_0,v_0),$$

$$A^p := \begin{cases} \bigcup_{\epsilon_{(n,j)}\in\mathbb{R} \text{ for } (n,j)\in E} W^s(\{(u,v)_{1;\epsilon_{(n,j)}:(n,j)\in E}(t) : t \in \mathbb{R}\}) & \text{if } -\overline{\lambda}\pi^2 = \mu, \\ \emptyset & \text{otherwise.} \end{cases}$$

Then $A := A^f \cup A^p$ is open and dense in $L^2 \times L^2$. Furthermore, we get

$$A = \{(u',v') \in L^2 \times L^2 : \mu_{(u',v')} = \mu\}.$$

Proof. We set $B := \{(u',v') \in L^2 \times L^2 : \mu_{(u',v')} = \mu\}$. It is clear that B is open and dense in $L^2 \times L^2$. Thus, it only remains to show that $A = B$. Using Theorems 1 and 3, it follows that $A \subset B$.

In order to show that $B \subset A$, we take $(u',v') \in B$ and denote the solution of (1) with initial value (u',v') by (u,v). By Theorem 5.1, (u,v) tends either to an element (u_0,v_0) of F or to a periodic orbit. In the first case, Theorem 5.1 ensures that $\mu_{(u_0,v_0)} = \mu_{(u',v')} = \mu$. Thus, we have

$$(u',v') \in \bigcup_{\substack{(u_0,v_0)\in F\\ \mu_{(u_0,v_0)}=\mu}} W^s(u_0,v_0) = A^f \subset A.$$

In the second case, (u,v) tends to a periodic orbit $\{(u,v)_{m;\epsilon_{(n,j)}:(n,j)\in E_m}(t) : t \in \mathbb{R}\}$ with $m \in \{1,\dots,p^-\}$. Since $\mu_{(u',v')} = \mu$, we have $m = 1$ and $\mu = -\overline{\lambda}\pi^2$. Thus, we have $E_m = E$, and we get $(u',v') \in A^p \subset A$. $\square$

As an easy, but interesting consequence, we get

Corollary 1. *We assume that the maximal multiplier μ satisfies $-\mu < g(0)$.*

(i) *We assume that $\mu \neq -\overline{\lambda}\pi^2$ and there is only one $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ such that $\mu_n^{(j)} = \mu$. Then*

$$W^s\left(\sqrt{g^{-1}(-\mu)}e_n^{(j)}\sqrt{2}\sin(n\pi\cdot)\right) \cup W^s\left(-\sqrt{g^{-1}(-\mu)}e_n^{(j)}\sqrt{2}\sin(n\pi\cdot)\right)$$

is open and dense in $L^2 \times L^2$.

(ii) *We assume that $\mu = -\overline{\lambda}\pi^2$ and there is no $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ such that $\mu_n^{(j)} = \mu$. Then E_1 defined as in Theorem 5.1(iii) is empty, $(u,v)_1$ is a periodic solution with corresponding periodic orbit Γ_1, and the stable manifold $W^s(\Gamma_1)$ is open and dense in $L^2 \times L^2$.*

By Corollary 1, we have a good description of the dynamics in an open and dense set provided that we are in the non-critical case (as always in this section) and the assumptions of Corollary 1 are satisfied. Since these assumptions depend only on the diffusion constants λ_1, λ_2 (because the maximal multiplier does only depend on λ_1, λ_2), we can examine the set of all non-critical diffusion constants such that the assumptions of Corollary 1 are valid. We will show that this set is open and dense in $(0,\infty) \times (0,\infty)$.

Lemma 1. *There is a set $C \subset (0,\infty) \times (0,\infty)$, which is open and dense in $(0,\infty) \times (0,\infty)$, such that each $(\lambda_1, \lambda_2) \in C$ is non-critical and for all $(\lambda_1, \lambda_2) \in C$ the maximal multiplier $\mu = \mu(\lambda_1, \lambda_2)$ coincides only with one element of*

$$M_{(\lambda_1,\lambda_2)} := \{-\overline{\lambda}n^2\pi^2 : 0 \leq n \leq p^-(\lambda_1, \lambda_2)\} \cup \{\mu_n^{(j)} : n \geq p^+(\lambda_1, \lambda_2), j = 1, 2\}.$$

(We note that $\overline{\lambda}$ as well as $\mu_n^{(j)}$ for $n \geq p^+(\lambda_1, \lambda_2)$, depend (only) on (λ_1, λ_2).)

Proof. 1. Let $C' \subset (0,\infty) \times (0,\infty)$ be the set of all non-critical diffusion constants. It is easy to verify that C' is open and dense. Thus, it is sufficient to show that C is an open and dense subset of C'. Given $(\lambda_1, \lambda_2) \in C'$, we introduce

$$f_{(\lambda_1,\lambda_2)} : [p^+(\lambda_1, \lambda_2), \infty) \ni x \mapsto -\overline{\lambda}\pi^2 x^2 + \sqrt{\left(\frac{\lambda_1 - \lambda_2}{2}\right)^2 \pi^4 x^4 - 1} \in \mathbb{R}.$$

An elementary computation shows that $f_{(\lambda_1,\lambda_2)}$ has exactly one (absolute) maximum and $f_{(\lambda_1,\lambda_2)}(x) \to -\infty \quad (x \to \infty)$.

2. In order to show that C lies open in C', we take $(\lambda_1, \lambda_2) \in C$. We denote the corresponding maximal multiplier by $\mu = \mu(\lambda_1, \lambda_2)$. Furthermore, we set $p^+ := p^+(\lambda_1, \lambda_2)$ and $p^- := p^-(\lambda_1, \lambda_2)$.

Case 1. There is $n \in \{1, 2, \dots, p^-\}$ such that $\mu = -\overline{\lambda}n^2\pi^2$.

Since μ is the maximal multiplier, it follows that $\mu = -\overline{\lambda}\pi^2$. By definition of C, this means that $f_{(\lambda_1,\lambda_2)}(n) < \mu$ for all $n \in \{p^+, p^+ + 1, \dots\}$. Since each

$f_{(\lambda_1,\lambda_2)}(n)$ depends continuously on (λ_1,λ_2) and $f_{(\lambda_1,\lambda_2)}$ is decreasing in some interval $(x_{(\lambda_1,\lambda_2)},\infty)$, there is a neighbourhood Q of (λ_1,λ_2) in $(0,\infty)\times(0,\infty)$ such that $f_{(\lambda_1',\lambda_2')}(n) < -\pi^2(\lambda_1'+\lambda_2')/2$ for all $n \in \{p^+(\lambda_1',\lambda_2'), p^+(\lambda_1',\lambda_2')+1,\dots\}$, $(\lambda_1',\lambda_2') \in Q$. Thus, we have $Q \cap C' \subset C$.

Case 2. There is $(n,j) \in \{p^+, p^+ + 1, \dots\} \times \{1,2\}$ such that $\mu_n^{(j)} = \mu$.

Since μ is the maximal multiplier, we get $j = 1$. Furthermore, we have $-\overline{\lambda}\pi^2 < \mu$ (by definition of C). Using again the fact that $f_{(\lambda_1,\lambda_2)}(n)$ depends continuously on (λ_1,λ_2), we get a neighbourhood Q of (λ_1,λ_2) in $(0,\infty)\times(0,\infty)$ such that for all $(\lambda_1',\lambda_2') \in Q$

(i) $p^+(\lambda_1',\lambda_2') \geq n$,

(ii) $f_{(\lambda_1',\lambda_2')}(n) > -\pi^2(\lambda_1' + \lambda_2')/2$,

(iii) $f_{(\lambda_1',\lambda_2')}(m) < f_{(\lambda_1',\lambda_2')}(n)$ for all $m \in \{p^+(\lambda_1',\lambda_2'), p^+(\lambda_1',\lambda_2') + 1, \dots\}$.

Thus, we have $Q \cap C' \subset C$.

Hence, C is an open subset of C'.

2. We want to show that C lies dense in C'. We take $(\lambda_1,\lambda_2) \in C' \setminus C$. For all $\delta > 0$ we set $(\lambda_1^\delta, \lambda_2^\delta) := (\lambda_1,\lambda_2) + (\delta,\delta)$. Thus, we get $\lambda_1^\delta - \lambda_2^\delta = \lambda_1 - \lambda_2$ and $(\lambda_1^\delta + \lambda_2^\delta)/2 = \overline{\lambda} + \delta$. Therefore, we have $(\lambda_1^\delta,\lambda_2^\delta) \in C'$ for all $\delta > 0$ and $p(\lambda_1^\delta,\lambda_2^\delta) = p(\lambda_1,\lambda_2) =: p$, $p^+(\lambda_1^\delta,\lambda_2^\delta) = p^+(\lambda_1,\lambda_2) =: p^+$, $p^-(\lambda_1^\delta,\lambda_2^\delta) = p^-(\lambda_1,\lambda_2) =: p^-$. Furthermore, we have

$$(\mu^\delta)_n^{(1)} = \mu_n^{(1)} - \delta n^2\pi^2 \quad \text{for all } n \in \{p^+, p^+ + 1, \dots\}.$$

Case 1. There is at most one $n \in \{p^+, p^+ + 1, \dots\}$ such that $\mu_n^{(1)} = \mu$.

Since we have $(\lambda_1,\lambda_2) \notin C$, it follows that $\mu = -\overline{\lambda}\pi^2$ and there is $n \geq p^+$ with $\mu_n^{(1)} = \mu$ (because μ coincides with at least two elements of M). We know that $f_{(\lambda_1,\lambda_2)}$ has exactly one maximum; we take $x_{(\lambda_1,\lambda_2)} \in [p^+,\infty)$ such that $f_{(\lambda_1,\lambda_2)}(x_{(\lambda_1,\lambda_2)})$ is this maximum.

Since $\mu_n^{(1)} > \mu_m^{(1)}$ for all $m \neq n$, it follows that $x_{(\lambda_1,\lambda_2)} \in (n-1, n+1)$. We can take $\delta_0 > 0$ so small that $x_{(\lambda_1^\delta,\lambda_2^\delta)} \in (n-1,n+1)$ for all $\delta \in (0,\delta_0)$ because $f_{(\lambda_1',\lambda_2')}$ depends continuously on (λ_1',λ_2'). Thus, we know that

$$(\mu^\delta)_m^{(1)} < \max\{(\mu^\delta)_k^{(1)} : p^+ \leq k \leq n+1\} \quad \text{for all } m \geq n+2.$$

This means that the maximal multiplier μ^δ associated with $(\lambda_1^\delta,\lambda_2^\delta)$ does not coincide with $(\mu^\delta)_m^{(1)}$ for all $m \geq n+2$.

Since the functions $(0,\delta_0) \ni \delta \mapsto (\mu^\delta)_m^{(1)} \in \mathbb{R}$, $m \in \{p^+, p^+ + 1, \dots, n+1\}$, are continuous, there is $\delta_1 \leq \delta_0$ such that

$$(\mu^\delta)_m^{(1)} < (\mu^\delta)_n^{(1)} \quad \text{for all } \delta \in (0,\delta_1),\ m \in \{p^+, p^+ + 1, \dots, n+1\} \setminus \{n\}.$$

Furthermore, we have

$$-\frac{\lambda_1^\delta + \lambda_2^\delta}{2}\pi^2 = -\overline{\lambda}\pi^2 - \delta\pi^2 = \mu - \delta\pi^2 = \mu_n^{(1)} - \delta\pi^2 = (\mu^\delta)_n^{(1)} + \delta\pi^2(n^2-1) > (\mu^\delta)_n^{(1)}\,.$$

This gives $\mu^\delta = -\frac{\lambda_1^\delta+\lambda_2^\delta}{2}\pi^2$ and $\mu^\delta > (\mu^\delta)_m^{(1)}$ for all $m \geq p^+$. This means that

$$(\lambda_1^\delta, \lambda_2^\delta) \in C \quad \text{for all } \delta \in (0, \delta_1).$$

Case 2. There are $n, m \in \{p^+, p^+ + 1, \dots\}$, $n \neq m$, such that $\mu_n^{(1)} = \mu_m^{(1)} = \mu$.

Since $f_{(\lambda_1,\lambda_2)}$ has only one maximum at $x_{(\lambda_1,\lambda_2)}$ (see Case 1), it follows that $x_{(\lambda_1,\lambda_2)} \in (\min\{n,m\}, \max\{n,m\})$ and, thus, $|n-m| = 1$. Hence, we may assume that $m = n+1$ and $x_{(\lambda_1,\lambda_2)} \in (n, n+1)$. Analogously to Case 1, we get $x_{(\lambda_1^\delta,\lambda_2^\delta)} \in (n, n+1)$ for all $\delta \in (0, \delta_0)$, and, furthermore, there is $\delta_1 \leq \delta_0$ such that

$$(\mu^\delta)_k^{(1)} < (\mu^\delta)_n^{(1)} \quad \text{for all } \delta \in (0, \delta_1),\ k \in \{p^+, p^+ + 1, \dots\} \setminus \{n, n+1\}.$$

Thus, we have for all $\delta \in (0, \delta_1)$

$$(\mu^\delta)_n^{(1)} = \mu_n^{(1)} - \delta n^2\pi^2 = \mu_{n+1}^{(1)} - \delta n^2\pi^2 > \mu_{n+1}^{(1)} - \delta(n+1)^2\pi^2 = (\mu^\delta)_{n+1}^{(1)},$$

i.e. $(\mu^\delta)_n^{(1)} > (\mu^\delta)_m^{(1)}$ for all $m \in \{p^+, p^+ + 1, \dots\}$, $m \neq n$. If we have $-\overline{\lambda}\pi^2 = \mu$, then we get

$$-\frac{\lambda_1^\delta + \lambda_2^\delta}{2}\pi^2 = -\overline{\lambda}\pi^2 - \delta\pi^2 = \mu - \delta\pi^2 = \mu_n^{(1)} - \delta\pi^2 = (\mu^\delta)_n^{(1)} + \delta\pi^2(n^2-1) > (\mu^\delta)_n^{(1)},$$

and it follows that $(\lambda_1^\delta, \lambda_2^\delta) \in C$ for all $\delta \in (0, \delta_1)$. If we have $-\overline{\lambda}\pi^2 < \mu$, then there is $\delta_2 \leq \delta_1$ such that

$$-\frac{\lambda_1^\delta + \lambda_2^\delta}{2}\pi^2 < (\mu^\delta)_n^{(1)} \qquad \text{for all } \delta \in (0, \delta_2).$$

Thus, we get $(\lambda_1^\delta, \lambda_2^\delta) \in C$ for all $\delta \in (0, \delta_2)$.

Anyway, we have shown that in every neighbourhood of $(\lambda_1, \lambda_2) \in C' \setminus C$ there is at least one element of C, i.e. C is dense in C'.

This completes the proof. □

Theorem 6. *There are open sets $P_0, P_s, P_p \subset (0,\infty) \times (0,\infty)$ such that*

(i) *$P_0 \cup P_s \cup P_p$ is open and dense in $(0,\infty) \times (0,\infty)$,*
(ii) *P_0, P_s, P_p are pairwise disjoint,*
(iii) *$(\lambda_1, \lambda_2) \in P_0 \cup P_s \cup P_p$ implies that (λ_1, λ_2) is non-critical,*
(iv) *$(\lambda_1, \lambda_2) \in P_0$ implies that all solutions of (1) tend to the zero solution,*
(v) *$(\lambda_1, \lambda_2) \in P_s$ implies that there is $(n, j) \in \{p^+(\lambda_1,\lambda_2), p^+(\lambda_1,\lambda_2)+1, \dots\} \times \{1, 2\}$ such that*

$$\bigcup_{j\in\{0,1\}} W^s\left((-1)^j \sqrt{g^{-1}(-\mu(\lambda_1,\lambda_2))}\, e_n^{(j)} \sqrt{2}\sin(n\pi\cdot)\right)$$

is open and dense in $L^2 \times L^2$,
(vi) *$(\lambda_1, \lambda_2) \in P_p$ implies that $(u, v)_1$ is a periodic solution of (1) and $W^s\big(\{(u,v)_1(t) : t \in \mathbb{R}\}\big)$ is open and dense in $L^2 \times L^2$.*

Proof. 1. Let C be defined as in Lemma 6. Then we set

$$P_0 := \{(\lambda_1, \lambda_2) \in C : -\mu(\lambda_1, \lambda_2) \geq g(0)\} ,$$
$$P_s := \left\{(\lambda_1, \lambda_2) \in C : -\mu(\lambda_1, \lambda_2) < g(0), \mu(\lambda_1, \lambda_2) = -\overline{\lambda}\pi^2\right\} ,$$
$$P_p := \left\{(\lambda_1, \lambda_2) \in C : -\mu(\lambda_1, \lambda_2) < g(0), \mu(\lambda_1, \lambda_2) \neq -\overline{\lambda}\pi^2\right\} .$$

By Lemma 1, (i), (ii) and (iii) are valid.

2. We take $(\lambda_1, \lambda_2) \in P_0$. Then we have $N_{(u_0,v_0)} = N^0_{(u_0,v_0)}$ for all $(u_0, v_0) \in L^2 \times L^2$, and (iv) follows by Theorem 5.1.

3. We take $(\lambda_1, \lambda_2) \in P_s$. By definition of C, there is exactly one pair $(n, j) \in \{p^+(\lambda_1, \lambda_2), p^+(\lambda_1, \lambda_2)+1, \dots\} \times \{1, 2\}$ such that $\mu = \mu_n^{(j)}$. Thus, (v) follows from Corollary 1(i).

4. We take $(\lambda_1, \lambda_2) \in P_p$. Then (vi) follows from Corollary 1(ii). □

This means that we have an easy description of the dynamics of (1) on an open and dense subset of $L^2 \times L^2$ for diffusion constants which are contained in an open and dense subset of the parameter space $(0, \infty) \times (0, \infty)$.

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