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OPTIMAL NATURAL DUALITIES FOR SOME QUASIVARIETIES OF DISTRIBUTIVE DOUBLE p–ALGEBRAS

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Abstract. Quasivarieties of distributive double p-algebras generated by an ordinal sum $\underline{M}$ of two Boolean lattices are considered. Globally minimal failsets within $\mathbb{S}(\underline{M}^2)$ are completely determined; from them all the optimal dualities for these quasivarieties are obtained.

1. Introduction

This paper concerns natural duality theory, as developed in [**3**] and [**1**]. The objective of this theory is to find a concrete representation, as a set of functions, of each algebra in a given quasivariety $\mathcal{A} = \mathbb{ISP}(\underline{M})$, where $\underline{M}$ is a finite algebra. Such representations have been obtained for a considerable number of quasivarieties, in particular of varieties of algebras having a lattice reduct, and it is of interest, and of practical importance to the study of such varieties, to find workable dualities.

Given a quasivariety $\mathcal{A} = \mathbb{ISP}(\underline{M})$ of algebras generated by a finite algebra $\underline{M}$, let R be a set of **algebraic relations** on $\underline{M}$, i.e., a set of relations on $\underline{M}$ such that each of them is the underlying set of a subalgebra of a power of $\underline{M}$. Define $\underset{\sim}{M} := (M; R, \tau)$ to be the topological relational structure on the underlying set M of $\underline{M}$ in which τ is the discrete topology. Given $\underset{\sim}{M}$, we define the category $\mathcal{X} := \mathbb{IS}_c\mathbb{P}(\underset{\sim}{M})$ in which objects are all isomorphic copies of closed substructures of powers of $\underset{\sim}{M}$ and in which morphisms are the continuous R-preserving maps. Let D and E be the natural hom functors $\mathcal{A}(-, \underline{M})\colon \mathcal{A} \to \mathcal{X}$ and $\mathcal{X}(-, \underset{\sim}{M})\colon \mathcal{X} \to \mathcal{A}$ respectively. The set R **yields a duality on** $\mathcal{A}$ if $\underline{A}$ is isomorphic to its second dual $ED(\underline{A})$, for every $\underline{A} \in \mathcal{A}$ (see [**6**], [**3**], [**1**] for further details). Even when the dualising set R is finite there are cases where R is extremely large. This can occurs, for example, when R can be taken to be $\mathbb{S}(\underline{M}^2)$ as it is the case of $\underline{M}$ having a lattice reduct. In these cases it would be useful to know whether such a

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duality is **optimal** in the sense that no proper subset of R yields a duality. In [**5**] B. A. Davey and H. A. Priestley present a general theory of optimal dualities. For a given finite dualising set Ω of finitary algebraic relations on $\underline{M}$, they prove that the subsets R of Ω which yield optimal dualities are precisely the transversals of the globally minimal failsets in Ω, as defined at the beginning of Section 4. By a **failset** we mean a set of the form

$$\mathrm{Fail}_{\underline{r}}(u) := \{\, s \in \Omega \mid u \text{ does not preserve } s \,\};$$

here r ranges over Ω and u over the set of maps from $D(\underline{r})$ to M which do not preserve r. The **globally minimal failsets** are those failsets which are minimal with respect to set inclusion.

Here we have a dual motivation. Our first aim is to obtain optimal dualities for the varieties under consideration. The second, and potentially the more important, is to use these varieties to explore the structure of globally minimal failsets in a non-trivial case, and so to gain a better understanding of the general theory. At the same time we present some techniques which can be used to determine failsets. In our examples $\Omega = \mathbb{S}(\underline{M}^2)$, and so Ω includes in particular the graphs of the (non-extendable) partial endomorphisms of $\underline{M}$. In general, failsets which do not contain partial endomorphisms are easier to analyse than those that do. We are able in our examples to describe certain globally minimal failsets containing as minimal elements non-extendable partial endomorphisms of $\underline{M}$. In this work we apply the theory to certain quasivarieties of distributive double p-algebras. These algebras have as reducts a pseudocomplemented distributive lattice and a dual pseudocomplemented distributive lattice. In [**5**], B. A. Davey and H. A. Priestley have already applied the theory to the quasivarieties $\mathcal{B}_n$ of pseudocomplemented distributive lattices.

For this work we have used some of the Pascal programs that B. A. Davey and H. A. Priestley have used for studying optimality in [**5**], and which have as a basis the backtracking algorithm presented in [**8**].

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2. Preliminaries

An algebra $\underline{A} = (A; \vee, \wedge, ^*, 0, 1)$ of type $(2,2,1,0,0)$ is a **distributive p-algebra** if $(A; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and * is a unary operation defined by

$$x \wedge a = 0 \text{ if and only if } x \leqslant a^*,$$

i.e., x^* is the pseudocomplement of x.

An algebra $\underline{A} = (A; \vee, \wedge, ^*, ^+, 0, 1)$ of type $(2,2,1,1,0,0)$ is a **distributive double p-algebra** if $(A; \vee, \wedge, ^*, 0, 1)$ is a distributive p-algebra and $(A; \vee, \wedge, ^+, 0, 1)$ is a dual distributive p-algebra.

For $m, n \geqslant 1$, let $\underline{B}_m$ and $\underline{B}_n$ be respectively the m-atom Boolean lattice and the n-atom Boolean lattice. Define $\underline{P}_{m,n}$ to be the distributive double p-algebra given by the ordinal sum $B_m \oplus B_n$. The unary operations * and $^+$ are defined as follows,

$$x^* = \begin{cases} 1 & \text{if } x = 0, \\ x'_{B_m} & \text{if } x \in B_m,\ x \neq 0, \\ 0 & \text{if } x \in B_n \end{cases} \quad \text{and} \quad x^+ = \begin{cases} 1 & \text{if } x \in B_m, \\ x'_{B_n} & \text{if } x \in B_n,\ x \neq 1, \\ 0 & \text{if } x = 1, \end{cases}$$

where x'_{B_i} is the complement of $x \in B_i$, with $i \in \{m, n\}$.

We denote by d_1 and d_2 respectively the maximum of $\underline{B}_m$ and the minimum of $\underline{B}_n$. Let π_i, $i \in \{1, 2\}$, be the natural projection maps from $\underline{P}^2_{m,n}$ to $\underline{P}_{m,n}$. For a given subalgebra $\underline{r}$ we denote $\pi_i{\restriction}_{\underline{r}}$ by ρ_i.

Let $m, n > 1$ and let $\mathcal{A} = \mathbb{ISP}(\underline{P}_{m,n})$ be the quasivariety generated by $\underline{P}_{m,n}$. Let $\Omega = \mathbb{S}(\underline{P}^2_{m,n})$ be the set of all binary algebraic relations on $\underline{P}_{m,n}$. For a given subalgebra $\underline{r}$ of $\underline{P}^2_{m,n}$, we write $r \subseteq P^2_{m,n}$ when we wish to think of r as a binary relation and $\underline{r} \leqslant \underline{P}^2_{m,n}$ when it is regarded as a subalgebra of $\underline{P}^2_{m,n}$.

In Section 3 we will need some elementary facts on the algebras $\underline{P}_{m,n}$ which we next collect together for future reference.

Proposition 2.1. *Let $\underline{r} \leqslant \underline{P}^2_{m,n}$. Then one of the following cases must occur:*

(a) $(0,1), (1,0) \in r$ *and then $\underline{r}$ is a product of subalgebras of $\underline{P}_{m,n}$.*

(b) $(0,1), (1,0) \notin r$ *and then $r \backslash \{(d_1, d_2), (d_2, d_1)\}$ is the graph of some one-to-one homomorphism $h\colon \underline{N} \leqslant \underline{P}_{m,n} \to \underline{P}_{m,n}$.*

Proof. Note that $(0,1) \in r$ if and only if $(1,0) \in r$. So either $(1,0), (0,1) \in r$ or $(1,0), (0,1) \notin r$.

Suppose that $(0,1), (1,0) \in r$. For every $a \in \rho_1(\underline{r})$ and $b \in \rho_2(\underline{r})$, let $c, d \in P_{m,n}$ be such that $(a, d), (c, b) \in r$. Then we have that $(a, 1) = (a, d) \vee (0, 1) \in r$ and $(1, b) = (c, b) \vee (1, 0) \in r$, and consequently $(a, b) = (a, 1) \wedge (1, b) \in r$. Thus (a) holds.

Now consider (b). We claim that we only have $(a, 0) \in r$ for $a = 0$. Suppose that $(a, 0) \in r$, with $a \neq 0$. Since $(a^*, 1) \in r$ and $(0, 1) \notin r$, we must have $a < d_1$. But then $(d_1, 1) = (a, 0) \vee (a^*, 1) \in r$ and so $(1, 0) = (d_1^+, 1^+) \in r$, contrary to hypothesis. By using the same argument we also prove that $(0, a) \in r$ if and only if $a = 0$. Analogously, we prove that $(a, 1) \in r \Leftrightarrow a = 1$ and $(1, a) \in r \Leftrightarrow a = 1$. Let $(a, b), (a, c) \in r$. We have that $(0, b^* \wedge c), (0, b \wedge c^*) \in r$ which implies that $b^* \wedge c = 0 = b \wedge c^*$ and consequently $b^* = c^*$. But then $b = c$ or $d_1 \leqslant b, c$. We also have that $(1, b^+ \vee c), (1, b \vee c^+) \in r$ which implies $b^+ = c^+$. Thus $b = c$ or $b, c \leqslant d_2$. If b or c is in $\{d_1, d_2\}$ then $(a^*, 0), (a^+, 1) \in r$, and so $a \in \{d_1, d_2\}$ because $a^* = 0$ and $a^+ = 1$. We prove in a similar way that $(b, a), (c, a) \in r$ implies that ($b = c$ or $a, b, c \in \{d_1, d_2\}$). At last observe that if $(a, b) \in r$ then

$(d_1, d_1) \in r$ whenever $0 < a < d_1$, and $(d_2, d_2) \in r$ whenever $d_2 < a < 1$. Now it follows that $r \backslash \{(d_1, d_2), (d_2, d_1)\}$ is a subalgebra of $\underline{P}^2_{m,n}$ and finally we have that $r \backslash \{(d_1, d_2), (d_2, d_1)\}$ is the graph of a one-to-one (partial) endomorphism of $\underline{P}_{m,n}$. $\square$

From this proposition it follows immediately the following result.

Corollary 2.2. *The endomorphisms of $\underline{P}_{m,n}$ are exactly the automorphisms of $\underline{P}_{m,n}$.*

Proposition 2.3.

(a) *There exists a group isomorphism between* $\operatorname{Aut} \underline{P}_{m,n}$ *and $S_m \times S_n$, the direct product of the symmetric groups S_m and S_n.*
(b) *If $f \in \operatorname{Aut} \underline{P}_{m,n} \backslash \{\mathrm{id}\}$ then there exists a maximal subgroup H of* $\operatorname{Aut} \underline{P}_{m,n}$ *such that $f \notin H$.*

Proof. First observe that the group of automorphisms of a k-atom Boolean lattice is isomorphic to the symmetric group S_k. Since the restrictions of the automorphisms of $\underline{P}_{m,n}$ to B_m and to B_n are respectively automorphisms of $\underline{B}_m$ and $\underline{B}_n$, every automorphim of $\underline{P}_{m,n}$ is uniquely determined by a permutation on its atoms and by a permutation on its coatoms. Also, each automorphism of B_m and each automorphism of B_n define an automorphism g of $\underline{P}_{m,n}$. Now the claim regarding (a) follows easily.

For (b) take the subgroup $H = \{\, g \in \operatorname{Aut} \underline{P}_{m,n} \mid g(a) = a \,\}$ of $\operatorname{Aut} \underline{P}_{m,n}$, where $a \in \underline{P}_{m,n}$ is such that $f(a) \neq a$ and a is either an atom or a coatom of $\underline{P}_{m,n}$. Suppose without less of generality that a is an atom of $\underline{P}_{m,n}$ and take $a_1, \dots, a_m$ to be the atoms of $\underline{P}_{m,n}$ such that $a = a_1$. Then $\{\, \sigma \in S_m \mid \sigma(1) = 1 \,\} \times S_n$ is the subgroup of $S_m \times S_n$ that corresponds to H. Since it is a maximal proper subgroup of $S_m \times S_n$, we have that H is a maximal proper subgroup of $\operatorname{Aut} \underline{P}_{m,n}$. $\square$

Let $a \in \underline{P}_{m,n}$. A subalgebra $\underline{N}$ of $\underline{P}_{m,n}$ is called a **value** of $\underline{P}_{m,n}$ at a if $\underline{N}$ is maximal with respect to not containing a. Denote by $\underline{N}_{d_1}$ and $\underline{N}_{d_2}$ the values of $\underline{P}_{m,n}$ at d_1 and d_2 respectively. Observe that N_{d_1} is $\{0\} \oplus B_n$ and N_{d_2} is $B_m \oplus \{1\}$.

Given a partial endomorphism h of $\underline{P}_{m,n}$, we have that $h(d_i) \in \{d_1, d_2\}$, for $d_i \in \operatorname{dom} h$, and since h is one-to-one on $\operatorname{dom} h \backslash \{d_1, d_2\}$, by Proposition 2.1, we also have that $h(\operatorname{dom} h \cap N_{d_i}) \subseteq N_{d_i}$ whenever $\operatorname{dom} h \cap N_{d_i} \nsubseteq \{0, d_j, 1\}$, with $j \in \{1, 2\}$, $j \neq i$. Then we may observe easily that if the codomain of h is N_{d_i}, for $i \in \{1, 2\}$, its domain is either N_{d_i} or $N_{d_i} \cup \{d_1, d_2\}$. Now the following result follows immediately.

Proposition 2.4. *Let $i \in \{1, 2\}$. Then $D(\underline{N}_{d_i})$ is the set of automorphisms of $\underline{N}_{d_i}$.*

For every subalgebra $\underline{N}$ of $\underline{P}_{m,n}$, we denote the set of atoms of $\underline{N}$ and the set of coatoms of $\underline{N}$ by $\operatorname{At} \underline{N}$ and $\operatorname{Coat} \underline{N}$ respectively.

Lemma 2.5. *Let h be a partial endomorphism of $\underline{P}_{m,n}$ with $d_1, d_2 \in \operatorname{dom} h$ and $h(d_1) = d_1$, $h(d_2) = d_2$. Then h is extendable if and only if*
there is an atom a of $\operatorname{dom} h$ such that neither a nor $h(a)$ is an atom of $\underline{P}_{m,n}$,
or
there is a coatom c of $\operatorname{dom} h$ such that neither c nor $h(c)$ is a coatom of $\underline{P}_{m,n}$.

Proof. Let $\underline{N} = \operatorname{dom} h$. If h is extendable then take $h' \colon \underline{N}' \leqslant \underline{P}_{m,n} \to \underline{P}_{m,n}$ to be an extension of h. As $\underline{N}$ is a proper subalgebra of $\underline{N}'$ there exists $a' \in \operatorname{At} \underline{N}'$ such that $a' < a$, for some $a \in \operatorname{At} \underline{N}$, or there exists $c' \in \operatorname{Coat} \underline{N}'$ such that $c < c'$, for some $c \in \operatorname{Coat} \underline{N}$. Suppose without loss of generality that there is such an atom a'. Since h' is one-to-one, by Proposition 2.1, $0 < h'(a') < h(a)$. But then $a \in \operatorname{At} \underline{N}$ and $a, h(a) \notin \operatorname{At} \underline{P}_{m,n}$. Conversely suppose without loss of generality that $a \in \operatorname{At} \underline{N}$ and that $a, h(a) \notin \operatorname{At} \underline{P}_{m,n}$. Then there are $a_1, a_2 \in \operatorname{At} \underline{P}_{m,n}$ such that $a_1 < a$ and $a_2 < h(a)$. For every $x \in N$, we have $a_1 \leqslant x \Leftrightarrow a \leqslant x \Leftrightarrow a_2 \leqslant h(x)$ because $a \in \operatorname{At} \underline{N}$ and h is one-to-one. Take

$$s = \operatorname{graph}(h) \cup \{(a_1, a_2) \vee (x, h(x)) \mid x \in N\} \cup \{({a_1}^*, {a_2}^*) \wedge (x, h(x)) \mid x \in N\}.$$

It is not difficult to verify that s is the universe of a subalgebra of $\underline{P}^2_{m,n}$. Then $s = \operatorname{graph}(\overline{h})$ for some one-to-one partial endomorphism $\overline{h}$ of $\underline{P}_{m,n}$. But then $\overline{h}$ extends h. □

Lemma 2.6. *Let $\underline{N}$ and $\underline{Q}$ be two maximal proper subalgebras of $\underline{P}_{m,n}$. If $\underline{N}$ and $\underline{Q}$ are isomorphic then there exists an automorphism g of $\underline{P}_{m,n}$ such that $g(N) = Q$.*

Proof. Since $\underline{N}$ is a maximal proper subalgebra of $\underline{P}_{m,n}$, N must contain either N_{d_1} or N_{d_2}. Suppose without loss of generality that $N_{d_1} \subseteq N$. Then both $\underline{N}$ and $\underline{Q}$ are isomorphic to $\underline{P}_{m-1,n}$. Let $a_1, \dots, a_m$ be the atoms of $\underline{P}_{m,n}$. If $m = 2$ then $N = N_{d_1} \cup \{d_1\} = Q$. If $m > 2$ then there exist $a_{i_1}, a_{i_2}, a_{j_1}, a_{j_2} \in \operatorname{At} \underline{P}_{m,n}$ such that $a_{i_1} \vee a_{i_2}$ and $a_{j_1} \vee a_{j_2}$ are respectively the atoms of $\underline{N}$ and $\underline{Q}$ that are not atoms of $\underline{P}_{m,n}$. Take $((i_1\, j_1) \circ (i_2\, j_2), \mathrm{id}) \in S_m \times S_n$ and let g be the corresponding automorphism of $\underline{P}_{m,n}$. Then we have that $g(\underline{N}) = \underline{Q}$. □

Proposition 2.7. *Let $h \colon \underline{N} \to \underline{P}_{m,n}$ be a non-extendable partial endomorphism. If either $N_{d_1} \subseteq N$ and $\underline{N}$ has $m-1$ atoms, with $m > 2$, or $N_{d_2} \subseteq N$ and $\underline{N}$ has $n-1$ coatoms, with $n > 2$, then*

(a) *there exists $g \in \operatorname{Aut} \underline{P}_{m,n}$ such that $h \circ g \circ h$ is extendable;*
(b) *for every non-extendable partial endomorphism $f \colon \underline{Q} \to \underline{P}_{m,n}$ of $\underline{P}_{m,n}$, where $\underline{Q}$ is isomorphic to $\underline{N}$, there exist $g_1, g_2 \in \operatorname{Aut} \underline{P}_{m,n}$ such that $f = g_1 \circ h \circ g_2{\restriction}_N$.*

Proof. Suppose that $N_{d_1} \subseteq N$ and $\underline{N}$ has $m-1$ atoms, with $m > 2$. We may assume that $\operatorname{At} \underline{N} = \{a_1, \dots, a_{m-2}, a_{m-1} \vee a_m\}$, where $a_1, \dots, a_m$ are the atoms of $\underline{P}_{m,n}$. Since h is one-to-one, by Proposition 2.1, and h is non-extendable, there are $a_i, a_j, a_{i_1}, a_{i_2} \in \operatorname{At}\underline{P}_{m,n}$ such that $h(a_i) = a_{i_1} \vee a_{i_2}$ and $h(a_{m-1} \vee a_m) = a_j$, by applying Lemma 2.5. There exists $\sigma \in S_m$ such that $\sigma(i_1) = m-1$, $\sigma(i_2) = m$ and $\sigma(j) = i$. Now (a) holds by taking the automorphism g of $\operatorname{Aut}\underline{P}_{m,n}$ determined by $(\sigma, \mathrm{id}) \in S_m \times S_n$.

By Lemma 2.6, we only need to prove (b) for the case $\underline{Q} = \underline{N}$. Let $f\colon \underline{N} \to \underline{P}_{m,n}$ be a non-extendable partial endomorphism of $\underline{P}_{m,n}$. By Lemma 2.5, there are $a_i, a_j \in N$ such that $a_i, a_j \in \operatorname{At}\underline{P}_{m,n}$ and, for some $a_{i_1}, a_{i_2}, a_{j_1}, a_{j_2} \in \operatorname{At}\underline{P}_{m,n}$, $f(a_j) = a_{j_1} \vee a_{j_2}$ and $h(a_i) = a_{i_1} \vee a_{i_2}$. Take g_2 to be the automorphism of $\underline{P}_{m,n}$ determined by $((i\ j), \mathrm{id}) \in S_m \times S_n$. Let $f' = f \circ {g_2}^{-1} \circ h^{-1}$, where h^{-1} denotes the inverse of the isomorphism from $\underline{N}$ to $h(\underline{N})$ given by h. Since $h(a_i)$ is the only atom of $h(N)$ which is not an atom of $\underline{P}_{m,n}$ and $f'(h(a_i)) = f(a_j)$, f' is extendable. Let $g_1 \in \operatorname{Aut}\underline{P}_{m,n}$ be an extension of f'. We have $g_1 \circ h \circ g_2{\restriction}_N = f$.

In case $N_{d_2} \subseteq N$ and $\underline{N}$ has $n-1$ coatoms, with $n > 2$, the proof of the claim is dual to this we have just done. $\square$

Recall that a subset R of Ω **entails** a relation r (in symbols, $R \vdash r$) if, for every $\underline{A} \in \mathcal{A}$, every continuos map $\varphi\colon D(\underline{A}) \to P_{m,n}$ which preserves every relation in R also preserves r. The map $R \mapsto \overline{R} := \{\, r \in \Omega \mid R \vdash r \,\}$ is a closure operator, referred as **entailment closure** (see [**4**] and [**5**] for further details). There are different ways of building relations from a subset R of Ω in order that those new relations are entailed by R. In [**4**] the authors present a list of constructs sufficient to describe entailment. Next we present some of those constructs which we will need in Section 3.

Permutation. From a binary relation r, construct $r^{\smile} = \{\, (c,d) \in P^2_{m,n} \mid (d,c) \in r \,\}$.

Intersection. From binary relations r and s, construct $r \cap s$.

Domains. From a partial endomorphism $e\colon \underline{N} \to \underline{P}_{m,n}$, construct the domain, $\operatorname{dom} e$, of e.

Joint kernels. From (partial) endomorphisms $e_1\colon \underline{N}_1 \to \underline{P}_{m,n}$ and $e_2\colon \underline{N}_2 \to \underline{P}_{m,n}$, with $\underline{N}_1, \underline{N}_2 \leq \underline{P}_{m,n}$, construct $\ker(e_1, e_2) := \{\, (c_1, c_2) \in N_1 \times N_2 \mid e_1(c_1) = e_2(c_2) \,\}$.

Composition. From (partial) endomorphisms $e_1\colon \underline{N}_1 \to \underline{P}_{m,n}$ and $e_2\colon \underline{N}_2 \to \underline{P}_{m,n}$, with $\underline{N}_1, \underline{N}_2 \leq \underline{P}_{m,n}$, construct the composite (partial) endomorphism $e_2 \circ e_1$ with domain $\{\, c \in \operatorname{dom} e_1 \mid e_1(c) \in \operatorname{dom} e_2 \,\}$.

Action by (partial) endomorphisms. From $r \subseteq \underline{P}^2_{m,n}$ and a (partial) endomorphism $e\colon \underline{N} \to \underline{P}_{m,n}$ of $\underline{P}_{m,n}$, construct $e \cdot r := \{\, (c,d) \in P^2_{m,n} \mid c \in N$ and $(e(c), d) \in r \,\}$.

Let $r \in \Omega$ and let $u\colon D(\underline{r}) \to P_{m,n}$ be any map. Define

$$U = \operatorname{Fail}_{\underline{r}}(u) := \{\, s \in \Omega \mid u \text{ fails to preserve } s \,\};$$

U is called a **failset of** r (within Ω) if it contains r. We say that U is a **failset** whenever it is a failset of some $r \in U$. Let $u\colon D(\underline{r}) \to P_{m,n}$ is a map and $x, y \in D(\underline{r})$. Then we say that (x, y) **witnesses** $s \in \mathrm{Fail}_{\underline{r}}(u)$ if $(x, y) \in s_{D(\underline{r})}$ but $(u(x), u(y)) \notin s$.

Proposition 2.8. *For any map $u\colon D(\underline{r}) \to P_{m,n}$, the complement of* $\mathrm{Fail}_{\underline{r}}(u)$ *in Ω is a closed set for entailment closure.*

Proof. This follows from the definitions. □

A failset U is said to be a **minimal failset of** r if U is a minimal element of the set of all failsets of r ordered by inclusion. If U is minimal in the set of all failsets ordered by inclusion then U is called a **globally minimal failset**.

The following two results are Corollary 3.6 and part of Theorem 3.14 of [**5**].

Proposition 2.9. *Let $s \in \Omega$. Let $U = \mathrm{Fail}_{\underline{r}}(u)$ be a failset containing s. Then there is a minimal failset U_s of s with $U_s \subseteq U$.*

Theorem 2.10. *Let $\emptyset \neq U \subseteq \Omega$. Then the following are equivalent:*

(a) *U is a globally minimal failset;*
(b) *U is a minimal failset of r for all $r \in U$.*

3. Globally Minimal Failsets in $\mathbb{S}(\underline{P}^2_{m,n})$

As in the preceding section, let Ω be $\mathbb{S}(\underline{P}^2_{m,n})$. It is known that Ω yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P}_{m,n})$, once $\underline{P}_{m,n}$ has a lattice reduct (see [**3**] and [**6**]). For a given (partial) endomorphism h of $\underline{P}_{m,n}$, we refer to $r = \mathrm{graph}(h)$ as h whenever this causes no confusion. Note that each $x \in D(\underline{r})$ may be identified with the homomorphism $x \circ f$, where $f\colon \mathrm{dom}\, h \to \underline{r}$ is the isomorphism defined by $f(a) = (a, h(a))$, and hence we may identify $D(\underline{r})$ with $D(\mathrm{dom}\, h)$. In case $\mathrm{dom}\, h = \underline{P}_{m,n}$, $D(\underline{r})$ is identified with $\mathrm{Aut}\, \underline{P}_{m,n}$.

Observe that any map $u\colon D(\underline{N}) \to P_{m,n}$ that fails to preserve a homomorphism $h\colon \underline{N} \to \underline{P}_{m,n}$ also fails to preserve either id_N or every extension of h. Consequently a globally minimal failset U in Ω which contains a partial endomorphism of $\underline{P}_{m,n}$ either contains every non-extendable extension of it or contains the identity map on some value of $\underline{P}_{m,n}$. This is the case for the two globally minimal failsets in Ω which contain the identity maps $\mathrm{id}_{N_{d_1}}$ and $\mathrm{id}_{N_{d_2}}$ respectively, as we show next.

Lemma 3.1. *For $i \in \{1, 2\}$, let $u_{d_i}\colon D(\underline{N}_{d_i}) \to P_{m,n}$ be the constant map with value d_i. Then $\mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i})$ is the unique minimal failset of $\mathrm{id}_{N_{d_i}}$ and its elements are the following (and no others):*

(i) the partial endomorphisms of $\underline{P}_{m,n}$ with codomain N_{d_i} and their converses;

(ii) the products $Q \times N_{d_i}$ *and* $N_{d_i} \times Q$, *where* $\underline{N}_{d_i} \leqslant \underline{Q} \leqslant \underline{P}_{m,n}$.
Moreover $\mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i}) = \Phi_{N_{d_i}}$, *where* $\Phi_{N_{d_i}} := \{ r \in \Omega \mid r \vdash \mathrm{id}_{N_{d_i}} \}$.

Proof. Firstly, note that $\mathrm{id}_{N_{d_i}} \in \mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i})$ is witnessed by $(\mathrm{id}_{N_{d_i}}, \mathrm{id}_{N_{d_i}})$. Suppose that $\mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(v)$ is a failset of $\mathrm{id}_{N_{d_i}}$ and that (h,h) witnesses $\mathrm{id}_{N_{d_i}} \in \mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(v)$. We claim that $\mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i}) \subseteq \mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(v)$. Let $r \in \mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i})$. If r is a product of two subalgebras of $\underline{P}_{m,n}$, then the identity map on one of them must belong to $\mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i})$. Since $\mathrm{id}_{N_{d_i}}$ is the only identity map in $\mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i})$, r must be either $Q \times N_{d_i}$ or $N_{d_i} \times Q$, for some $\underline{Q} \leqslant \underline{P}_{m,n}$. But then $r \in \mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(v)$. Thus, by Lemma 2.1, we only need to consider r when $r\backslash\{(d_1,d_2),(d_2,d_1)\}$ is the graph of a one-to-one partial endomorphism of $\underline{P}_{m,n}$. Let h' be such a partial endomorphism. Since r contains the graph of an automorphism of $\underline{N}_{d_i}$ and it does not contain (d_i,d_i), h' must be an automorphism of $\underline{N}_{d_i}$ and there is $j \in \{1,2\}$ such that $d_i \notin \rho_j(r)$. Consequently r is a partial endomorphism of $\underline{P}_{m,n}$ with codomain $\underline{N}_{d_i}$ or the converse of its graph. Suppose without loss of generality that $j = 1$. Then $(h, h' \circ h) \in r$ and $v(h) \notin N_{d_i} = \rho_1(r)$. Thus $(h, h' \circ h)$ witnesses $r \in \mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(v)$. We have just proved that $\mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i})$ is the unique minimal failset of $\mathrm{id}_{N_{d_i}}$ and that its elements are between those of (i) and (ii), which belong to $\Phi_{N_{d_i}}$. Finally it is immediate that $\Phi_{N_{d_i}} \subseteq \mathrm{Fail}_{\mathrm{id}_{N_{d_i}}}(u_{d_i})$. □

Proposition 3.2. *The minimal failsets* $\Phi_{N_{d_1}}$ *and* $\Phi_{N_{d_2}}$ *are globally minimal failsets.*

Proof. Let $\underline{r} \leqslant \underline{P}^2_{m,n}$ and let $v \colon D(\underline{r}) \to P_{m,n}$ be such that $r \in \mathrm{Fail}_{\underline{r}}(v) \subseteq \Phi_{N_{d_i}}$, with $i \in \{1,2\}$. Suppose that $\mathrm{id}_{N_{d_i}} \notin \mathrm{Fail}_{\underline{r}}(v)$ or otherwise $\Phi_{N_{d_i}} \subseteq \mathrm{Fail}_{\underline{r}}(v)$. Since $\mathrm{id}_{N_{d_i}}$ is the only identity map in $\Phi_{N_{d_i}}$ we have that $Q \times N_{d_i}$, $N_{d_i} \times Q \notin \mathrm{Fail}_{\underline{r}}(v)$, for every $\underline{N}_{d_i} \leqslant \underline{Q} \leqslant \underline{P}_{m,n}$. But then, by Lemma 3.1, we may assume that r is the graph of some partial endomorphism h of $\underline{P}_{m,n}$ with codomain N_{d_i}. Let $x, y \in D(\underline{r})$ such that $(x,y) \in r$ but $(v(x), v(y)) \notin r$. We have that $(v(x), v(y)) \in \mathrm{dom}\, h \times N_{d_i}$. Take $r' = r \cup \{(d_i, d_i)\}$. It is easy to verify that $r' \leqslant \underline{P}^2_{m,n}$. Since $v(y) \neq d_i$ we have $(v(x), v(y)) \notin r'$ and then $r' \in \mathrm{Fail}_{\underline{r}}(v)$. Thus $r' \in \Phi_{N_{d_i}}$ and so $(d_i, d_i) \notin r'$. □

Take U to be a failset such that U contains an automorphism of $\underline{P}_{m,n}$. The set of the automorphisms of $\underline{P}_{m,n}$ which are not in U forms a subgroup of $\mathrm{Aut}\,\underline{P}_{m,n}$. The smaller U is, the bigger this subgroup will be. So we may ask if globally minimal failsets in Ω containing automorphisms are "associated" with maximal subgroups of $\mathrm{Aut}\,\underline{P}_{m,n}$. The following results give us the answer.

Let g be a (partial) endomorphism of $\underline{P}_{m,n}$ and let r be a binary algebraic relation on $\underline{P}_{m,n}$, with $(d_1,d_1),(d_2,d_2) \in r$. We denote by $g^{\leqslant}$ and $\overline{r}$ the binary relations $\mathrm{graph}(g) \cup \{(d_1,d_2)\}$ and $r \cup \{(d_1,d_2),(d_2,d_1)\}$ respectively. Observe that both these relations are algebraic.

Lemma 3.3. *Let H be a maximal proper subgroup of* $\mathrm{Aut}\,\underline{P}_{m,n}$ *and* $K = \mathrm{Aut}\,\underline{P}_{m,n}\backslash H$. *Then*

$$W_H := K \cup \{\, g^{\leqslant} \;\mid\; g \in K\,\} \cup \{\, (g^{\leqslant})^{\smile} \;\mid\; g \in K\,\}$$

is a minimal failset of each $g \in K$.

Proof. Let $g \in K$ and define a map $u\colon \mathrm{Aut}\,\underline{P}_{m,n} \to P_{m,n}$ as follows:

$$u(x) = \begin{cases} d_1 & \text{if } x \in K, \\ d_2 & \text{otherwise.} \end{cases}$$

We have that $g \in \mathrm{Fail}_g(u)$ is witnessed by (id, g). Let $s \in \mathrm{Fail}_g(u)$. There exist $x, y \in \mathrm{Aut}\,\underline{P}_{m,n}$ such that $(x, y) \in s$ and $(u(x), u(y)) \notin s$. Then $\rho_1(s) = \rho_2(s) = P_{m,n}$ and $s \neq P^2_{m,n}$. Hence, by Lemma 2.1, $s = \mathrm{graph}(f)$ or $s = f^{\leqslant}$ or $(f^{\leqslant})^{\smile}$, for some $f \in K$. Conversely suppose $f \in K$. Then (id, f) witnesses $f, f^{\leqslant} \in \mathrm{Fail}_g(u)$. Thus $\mathrm{Fail}_g(u) = W_H$ is a failset of g. Now let $v\colon \mathrm{Aut}\,\underline{P}_{m,n} \to P_{m,n}$ be a map such that $g \in \mathrm{Fail}_g(v) \subseteq \mathrm{Fail}_g(u)$. There exists $x \in \mathrm{Aut}\,\underline{P}_{m,n}$ such that $v(g \circ x) \neq g(v(x))$. We claim that $\forall y \in \mathrm{Aut}\,\underline{P}_{m,n}$, $v(y) \in \{d_1, d_2\}$. If $v(x)$ or $v(g \circ x)$ is not in $\{d_1, d_2\}$, then $(x, g \circ x)$ witnesses $\overline{r} \in \mathrm{Fail}_g(v)$, where $r = \mathrm{graph}(g)$, and then $\overline{r} \in W_H$. Hence $v(x), v(g \circ x) \in \{d_1, d_2\}$. Let $y \in \mathrm{Aut}\,\underline{P}_{m,n}$. If $v(y) \neq d_1, d_2$ then (x, y) witnesses $\overline{\mathrm{graph}(y \circ x^{-1})} \in \mathrm{Fail}_g(v)$. Thus $\overline{\mathrm{graph}(y \circ x^{-1})} \in \mathrm{Fail}_g(u)$ and this is false. Let $K' = \{\, f \in \mathrm{Aut}\,\underline{P}_{m,n} \mid f \in \mathrm{Fail}_g(v)\,\}$ and let $H' = \mathrm{Aut}\,\underline{P}_{m,n}\backslash K'$. Then H' is a proper subgroup of $\mathrm{Aut}\,\underline{P}_{m,n}$. By the maximality of H, $H' = H$ and hence $K' = K$. Now it suffices to prove that $f \in \mathrm{Fail}_g(v)$ implies that $f^{\leqslant} \in \mathrm{Fail}_g(v)$ in order to conclude that $W_H \subseteq \mathrm{Fail}_g(v)$. Suppose that $f \in \mathrm{Fail}_g(v)$. Let $x \in \mathrm{Aut}\,\underline{P}_{m,n}$ such that $v(f \circ x) \neq f(v(x))$. If $f^{\leqslant} \notin \mathrm{Fail}_g(v)$ then $(v(x), v(f \circ x)) = (d_1, d_2)$ and $v(f^k \circ x) = v(f \circ x) = d_2$, for every $k \geqslant 1$ (note that $(f^{k-1} \circ x, f^k \circ x) \in f^{\leqslant}$). But then $v(x) = v(f^{-1} \circ f \circ x) = d_2$. Thus $f^{\leqslant} \in \mathrm{Fail}_g(v)$. $\square$

Proposition 3.4. *Let H be a maximal proper subgroup of* $\mathrm{Aut}\,\underline{P}_{m,n}$. *Then W_H is a globally minimal failset.*

Proof. By Theorem 2.10 and Lemma 3.3, it suffices to prove that W_H is a minimal failset of each $g^{\leqslant}$, with $g \in K$. Fix $g \in K$ and let $s = g^{\leqslant}$. Suppose $s \in \mathrm{Fail}_{\underline{s}}(v) \subseteq W_H$, for some map $v\colon D(\underline{s}) \to P_{m,n}$. Since W_H is a minimal failset of g and, by Proposition 2.9, every failset that contains g must contain a minimal failset of g, we only need to prove that $g \in \mathrm{Fail}_{\underline{s}}(v)$. Observe that $s = g \cdot \mathrm{id}^{\leqslant}$. Since $s \in \mathrm{Fail}_{\underline{s}}(v)$ and $id^{\leqslant} \notin \mathrm{Fail}_{\underline{s}}(v)$ because $id^{\leqslant} \notin W_H$, we must have $g \in \mathrm{Fail}_{\underline{s}}(v)$. $\square$

Proposition 3.5. *Let U be a globally minimal failset. If U intersects* $\mathrm{Aut}\,\underline{P}_{m,n}$ *then U is W_H for some maximal proper subgroup H of* $\mathrm{Aut}\,\underline{P}_{m,n}$.

Proof. Let $K = U \cap \mathrm{Aut}\,\underline{P}_{m,n}$ and let $H = \mathrm{Aut}\,\underline{P}_{m,n}\backslash K$. Then H is a proper subgroup of $\mathrm{Aut}\,\underline{P}_{m,n}$. Take H' to be a maximal subgroup of $\mathrm{Aut}\,\underline{P}_{m,n}$ containing H. Let $K' = \mathrm{Aut}\,\underline{P}_{m,n}\backslash H' \subseteq K$ and let $g \in K'$. Since $g \in U$ we may take $x \in \mathrm{Aut}\,\underline{P}_{m,n}$ and a map $u\colon \mathrm{Aut}\,\underline{P}_{m,n} \to P_{m,n}$ such that $(x, g \circ x)$ witnesses $g \in U = \mathrm{Fail}_g(u)$. If $g^{\leqslant} \notin U$ then $u(x) = d_1$, $u(g \circ x) = d_2$ and $u(g^2 \circ x) = u(g \circ x) = d_2$ (note that $(g \circ x, g^2 \circ x) \in g^{\leqslant}$). Moreover $u(g^k \circ x) = d_2$, for $k \geqslant 1$. But then $u(g^{-1} \circ g \circ x) = d_2$, i.e. $u(x) = d_2$. Thus $W_{H'} \subseteq U$. $\square$

It follows from Lemma 2.1 that for every $g \in \mathrm{Aut}\,\underline{P}_{m,n}$, the subalgebra $\underline{r}$, with $r = g^{\leqslant}$, is maximal with respect to not containing (d_2, d_1), that is what we call a value of $\underline{P}^2_{m,n}$ at (d_2, d_1). The next result gives us a globally minimal failset whose elements are exactly the relations $g^{\leqslant}$ and their converses, with $g \in \mathrm{Aut}\,\underline{P}_{m,n}$.

Proposition 3.6. *The set* $\{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \mathrm{Aut}\,\underline{P}_{m,n}\}$ *is a globally minimal failset.*

Proof. Let $f \in \mathrm{Aut}\,\underline{P}_{m,n}$ and $r = f^{\leqslant}$. Let $u\colon D(\underline{r}) \to P_{m,n}$ be defined as follows:

$$u(x) = \begin{cases} d_2 & \text{if } x(d_1, d_2) = d_1, \\ d_1 & \text{if } x(d_1, d_2) = d_2. \end{cases}$$

Observe that

(i) for every $g \in \mathrm{Aut}\,\underline{P}_{m,n}$ we have

$$u(g \circ \rho_i) = u(\rho_i) \quad \text{and} \quad g^{\leqslant} = \{\, (a, g \circ f^{-1}(b)) \mid (a,b) \in f^{\leqslant} \,\},$$

which implies that

$$(\rho_1, g \circ f^{-1} \circ \rho_2) \in g^{\leqslant} \text{ and } (u(\rho_1), u(g \circ f^{-1} \circ \rho_2)) = (d_2, d_1) \notin g^{\leqslant}\ ;$$

(ii) for every $x \in D(\underline{r})$, $x{\restriction}_{\mathrm{graph}(f)}$ is identified with an automorphism of $\underline{P}_{m,n}$ and then $x(r) = P_{m,n}$. Therefore if $s \in \mathrm{Fail}_{\underline{r}}(u)$ then $\rho_1(s) = \rho_2(s) = P_{m,n}$ and $(d_1, d_1), (d_2, d_2) \in s$. Hence, by Lemma 2.1, s must be one of the relations $\mathrm{graph}(g)$, $g^{\leqslant}$ or $(g^{\leqslant})^{\smile}$, for some $g \in \mathrm{Aut}\,\underline{P}_{m,n}$;

(iii) for every $g \in \mathrm{Aut}\,\underline{P}_{m,n}$ and $x \in D(\underline{r})$, $u(g \circ x) = g(u(x))$ and so $g \notin \mathrm{Fail}_{\underline{r}}(u)$.

Thus $\mathrm{Fail}_{\underline{r}}(u) = \{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \mathrm{Aut}\,\underline{P}_{m,n}\}$.

We claim that $\{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \mathrm{Aut}\,\underline{P}_{m,n}\}$ is a minimal failset of r. Suppose that $r \in \mathrm{Fail}_{\underline{r}}(v) \subseteq \mathrm{Fail}_{\underline{r}}(u)$, for some map $v\colon D(\underline{r}) \to P_{m,n}$. Let $x, y \in D(\underline{r})$ such that $(x,y) \in r$ and $(v(x), v(y)) \notin r$. If $(v(x), v(y)) \neq (d_2, d_1)$ then (x,y) witnesses $\overline{r} \in \mathrm{Fail}_{\underline{r}}(v)$. But then $\overline{r} \in \mathrm{Fail}_{\underline{r}}(u)$ and this is false. Thus $(v(x), v(y)) = (d_2, d_1)$. Now take $g \in \mathrm{Aut}\,\underline{P}_{m,n}$. Since $g \circ f^{-1} \notin \mathrm{Fail}_{\underline{r}}(v)$ we must have $v(g \circ f^{-1} \circ y) = g \circ f^{-1}(v(y)) = v(y)$. But then $(x, g \circ f^{-1} \circ y)$ witnesses $g^{\leqslant} \in \mathrm{Fail}_{\underline{r}}(v)$. Thus $\mathrm{Fail}_{\underline{r}}(v) = \mathrm{Fail}_{\underline{r}}(u)$. Finally we apply Theorem 2.10 and we get that $\{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \mathrm{Aut}\,\underline{P}_{m,n}\}$ is a globally minimal failset because it is a minimal failset of each one of its elements. $\square$

Proposition 3.7. *Let U be a globally minimal failset. If U intersects the globally minimal failset $\{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \operatorname{Aut} \underline{P}_{m,n} \}$ but U does not intersect* $\operatorname{Aut} \underline{P}_{m,n}$ *then $U = \{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \operatorname{Aut} \underline{P}_{m,n} \}$.*

Proof. Let $g \in \operatorname{Aut} \underline{P}_{m,n}$ and suppose that $g^{\leqslant} \in U$. Take $g' \in \operatorname{Aut} \underline{P}_{m,n}$ and let $f = {g'}^{-1} \circ g$. Note that $g^{\leqslant} = f \cdot {g'}^{\leqslant}$. Since $f \notin U$ and $g^{\leqslant} \in U$ we must have ${g'}^{\leqslant} \in U$. □

Denote by $\overline{g}$ the algebraic relation $\overline{\operatorname{graph}(g)}$; the subalgebra $\underline{r}$, where $r = \overline{g}$, is a value of $\underline{P}^2_{m,n}$ at $(0,1)$ and then $\underline{r}$ is uniquely covered by $\underline{P}^2_{m,n}$. Also now we get a globally minimal failset whose elements are the relations $\overline{g}$, with $g \in \operatorname{Aut} \underline{P}_{m,n}$.

Proposition 3.8. *The set $\{\, \overline{g} \mid g \in \operatorname{Aut} \underline{P}_{m,n} \}$ is a globally minimal failset.*

Proof. Let $f \in \operatorname{Aut} \underline{P}_{m,n}$ and $r = \overline{f}$. Let $u\colon D(\underline{r}) \to P_{m,n}$ be defined by

$$u(x) = \begin{cases} 0 & \text{if } x(d_1, d_2) = d_1, \\ 1 & \text{otherwise.} \end{cases}$$

Note that

(i) $\forall g \in \operatorname{Aut} \underline{P}_{m,n}$, $(\rho_1, g \circ f^{-1} \circ \rho_2) \in \overline{g}$ and $(u(\rho_1), u(g \circ f^{-1} \circ \rho_2)) = (0,1) \notin \overline{g}$;

(ii) if $s \in \operatorname{Fail}_{\underline{r}}(u)$ then s must be one of the relations $\operatorname{graph}(g)$, $g^{\leqslant}$, $(g^{\leqslant})^{\smile}$ or $\overline{g}$, for some $g \in \operatorname{Aut} \underline{P}_{m,n}$, by applying Lemma 2.1;

(iii) $\forall x \in D(\underline{r})$, $x(d_1, d_2) \neq x(d_2, d_1)$ since $d_1 = x(d_1, d_1) = x(d_1, d_2) \wedge x(d_2, d_1)$ and $d_2 = x(d_2, d_2) = x(d_1, d_2) \vee x(d_2, d_1)$. If $x, y \in D(\underline{r})$ such that $(x, y) \in s$ and $u(x) \neq u(y)$, then $(d_1, d_2), (d_2, d_1) \in s$.

Thus $\{\, \overline{g} \mid g \in \operatorname{Aut} \underline{P}_{m,n} \} = \operatorname{Fail}_{\underline{r}}(u)$ is a failset of r. Let U be a failset of r contained in $\operatorname{Fail}_{\underline{r}}(u)$. For every $g \in \operatorname{Aut} \underline{P}_{m,n}$, we have $r = (g^{-1} \circ f) \cdot \overline{g}$. Since $g^{-1} \circ f \notin \operatorname{Fail}_{\underline{r}}(u)$, and consequently $g^{-1} \circ f \notin U$, we must have $\overline{g} \in U$ or otherwise $\overline{r} \notin U$. Thus $U = \operatorname{Fail}_{\underline{r}}(u)$ and $\operatorname{Fail}_{\underline{r}}(u)$ is a minimal failset of r. Finally we apply Theorem 2.10 and we have that $\{\, \overline{g} \mid g \in \operatorname{Aut} \underline{P}_{m,n} \}$ is a globally minimal failset.□

Proposition 3.9. *Let U be a globally minimal failset. If U intersects the globally minimal failset $\{\, \overline{g} \mid g \in \operatorname{Aut} \underline{P}_{m,n} \}$ but U does not intersect* $\operatorname{Aut} \underline{P}_{m,n}$ *then $U = \{\, \overline{g} \mid g \in \operatorname{Aut} \underline{P}_{m,n} \}$.*

Proof. Use a similar argument to that in the proof of Proposition 3.7. □

The following result gives us two globally minimal failsets that contain non-extendable partial endomorphisms of $\underline{P}_{m,n}$. Later on we will see that they are the unique globally minimal failsets, within Ω, containing non-extendable partial endomorphisms of $\underline{P}_{m,n}$ and containing no identity maps.

We denote by $\operatorname{End}^i \underline{P}_{m,n}$ the set of non-extendable partial endomorphisms of $\underline{P}_{m,n}$ having as its domain a maximal proper subalgebra of $\underline{P}_{m,n}$ containing N_{d_i}.

Proposition 3.10.

(a) *If* $m > 2$ *then the set* $W_{d_1} = \{ h, h^{\leqslant}, (h^{\leqslant})^{\smile} : h \in \mathrm{End}^1 \underline{P}_{m,n} \}$ *is a globally minimal failset.*

(b) *If* $n > 2$ *then the set* $W_{d_2} = \{ h, h^{\leqslant}, (h^{\leqslant})^{\smile} : h \in \mathrm{End}^2 \underline{P}_{m,n} \}$ *is a globally minimal failset.*

Proof. Next we prove the claim regarding (a). By Theorem 2.10, it is enough to prove that W_{d_1} is a minimal failset of each of its elements. Take $h\colon \underline{N} \to \underline{P}_{m,n}$ to be a homomorphism in $\mathrm{End}^1 \underline{P}_{m,n}$. Hence $\underline{N}$ is isomorphic to $\underline{P}_{m-1,n}$. Define a map $u : D(\underline{N}) \to P_{m,n}$ by

$$u(x) = \begin{cases} d_1 & \text{if } x \text{ is non-extendable,} \\ d_2 & \text{otherwise.} \end{cases}$$

We claim that $W_{d_1} = \mathrm{Fail}_h(u)$. Let $f\colon \underline{Q} \to \underline{P}_{m,n}$ be a homomorphism in $\mathrm{End}^1 \underline{P}_{m,n}$. By Lemma 2.6, there is $g \in \mathrm{Aut}\,\underline{P}_{m,n}$ such that $g(\underline{N}) = \underline{Q}$. Consequently $(g{\restriction_N}, f \circ g{\restriction_N})$ witnesses $f, f^{\leqslant} \in \mathrm{Fail}_h(u)$. Conversely let $s \in \mathrm{Fail}_h(u)$. Then one of the pairs (d_1, d_2),(d_2, d_1) is not in s. By applying Lemma 2.1, $s = \mathrm{graph}(f)$ or $s = f^{\leqslant}$ or $s = (f^{\leqslant})^{\smile}$ for some one-to-one partial endomorphism f of $\underline{P}_{m,n}$. Let $x, y \in D(\underline{N})$ be such that (x, y) witnesses $s \in \mathrm{Fail}_h(u)$. Then either x is non-extendable or y is non-extendable. Since $(x, y) \in s \backslash \{ (d_1, d_2), (d_2, d_1) \}$ which is either graph f or (graph $f)^{\smile}$, and $y \circ x^{-1}$ is non-extendable, we must have that f is either $y \circ x^{-1}$ or $x \circ y^{-1}$. Hence f is non-extendable and its domain is isomorphic to $\underline{P}_{m-1,n}$.

Next we prove that $\mathrm{Fail}_h(u)$ is a minimal failset of h. Let $v\colon D(\underline{N}) \to P_{m,n}$ be a map such that $h \in \mathrm{Fail}_h(v) \subseteq \mathrm{Fail}_h(u)$. For every $f \in \mathrm{Fail}_h(u)$, there are $g_1, g_2 \in \mathrm{Aut}\,\underline{P}_{m,n}$ such that $h = g_1 \circ f \circ g_2{\restriction_N}$, by Lemma 2.7. Consequently $\mathrm{Fail}_h(v)$ contains every $f \in \mathrm{Fail}_h(u)$ because it contains no automorphisms of $\underline{P}_{m,n}$. Suppose that $\mathrm{Fail}_h(v) \neq \mathrm{Fail}_h(u)$. Then there exists $f \in \mathrm{Fail}_h(u)$ such that $f^{\leqslant} \notin \mathrm{Fail}_h(v)$. We claim that $h'^{\leqslant} \notin \mathrm{Fail}_h(v)$, for every $h' \in \mathrm{Fail}_h(v)$. Let $h' \in \mathrm{Fail}_h(v)$ and suppose that (x, y) witnesses $h'^{\leqslant} \in \mathrm{Fail}_h(v)$, for some $x, y \in D(\underline{N})$. Then (x, y) also witnesses $h' \in \mathrm{Fail}_h(v)$. We apply Lemma 2.7 again and we have that $h' = g_1' \circ f \circ g_2'{\restriction_{x(N)}}$, for some $g_1', g_2' \in \mathrm{Aut}\,\underline{P}_{m,n}$. Since

$$g_1'(v(f \circ g_2' \circ x) = v(h' \circ x) \neq h'(v(x)) = g_1' \circ f(v(g_2' \circ x))$$

and $(g_2' \circ x, f \circ g_2' \circ x) \in f^{\leqslant}$ we must have $v(g_2' \circ x) = d_1$ and $v(f \circ g_2' \circ x) = d_2$. But then $(d_1, d_2) = ((v(x), v(h' \circ x)) = (v(x), v(y)) \notin h'^{\leqslant}$ and this is absurd. Denote by h^{-1} the partial endomorphism from $h(\underline{N})$ into $\underline{P}_{m,n}$ given by the inverse of the isomorphism from $\underline{N}$ to $h(\underline{N})$ given by h. As $h, h^{-1} \in \mathrm{Fail}_h(v)$ we have that $h^{\leqslant}, (h^{-1})^{\leqslant} \notin \mathrm{Fail}_h(v)$. Therefore $h = h^{\leqslant} \cap ((h^{-1})^{\leqslant})^{\smile} \notin \mathrm{Fail}_h(v)$. Hence W_{d_1} is a minimal failset of h.

In order to finish our proof we only need to see that W_{d_1} is a minimal failset of $s = h^{\leqslant}$. Suppose that $s \in \mathrm{Fail}_{\underline{s}}(v) \subseteq \mathrm{Fail}_h(u)$, for some map $v\colon D(\underline{s}) \to P_{m,n}$. Observe that $s = h \cdot \mathrm{id}^{\leqslant}$. Since $\mathrm{id}^{\leqslant} \notin \mathrm{Fail}_{\underline{s}}(v)$, h must be in $\mathrm{Fail}_{\underline{s}}(v)$. By the minimality of $\mathrm{Fail}_h(u)$ as a failset of h we have that $\mathrm{Fail}_{\underline{s}}(v) = W_{d_1}$.

The proof of (b) is done by using the same kind of arguments as those used to prove (a). $\square$

Proposition 3.11. *Let U be a globally minimal failset. If there is a non-extendable partial endomorphism h of $\underline{P}_{m,n}$ such that* $\mathrm{dom}\, h$ *is a maximal proper subalgebra of $\underline{P}_{m,n}$ and $h \in U$, then either $U = \Phi_{N_{d_i}}$ or $U = W_{d_i}$, for some $i \in \{1,2\}$.*

Proof. Let $h\colon \underline{N} \to \underline{P}_{m,n}$ be a non-extendable partial endomorphism of $\underline{P}_{m,n}$, where $\underline{N}$ is a maximal proper subalgebra of $\underline{P}_{m,n}$. Then N must contain either N_{d_1} or N_{d_2}, so that either $h \in \mathrm{End}^1 \underline{P}_{m,n}$ or $h \in \mathrm{End}^2 \underline{P}_{m,n}$. Suppose that $h \in U$ and suppose without loss of generality that $h \in \mathrm{End}^1 \underline{P}_{m,n}$. By Proposition 3.5, U does not intersect $\mathrm{Aut}\, \underline{P}_{m,n}$.

Firstly consider the case $m = 2$. Then $N = N_{d_1} \cup \{d_1\}$ and $h(d_1) = d_2$. Take $f \in \mathrm{Aut}\, \underline{N}_{d_1}$ to be $h{\restriction}_{N_{d_1}}$. Observe that $\mathrm{graph}\, h = (f^{-1} \cdot (\mathrm{id}^{\leqslant})^{\smile})^{\smile}$ and so $f^{-1} \cdot (\mathrm{id}^{\leqslant})^{\smile} \in U$. Consequently either $f^{-1} \in U$ or $\mathrm{id}^{\leqslant} \in U$. Since $U \cap \{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \mathrm{Aut}\, \underline{P}_{m,n}\} = \emptyset$, by Proposition 3.7, we have that $\mathrm{id}^{\leqslant} \notin U$. Hence we must have that $f^{-1} \in U$. Let $g \in \mathrm{Aut}\, \underline{P}_{m,n}$ be given by $g{\restriction}_{N_{d_1}} = f^{-1}$ and $g{\restriction}_{N_{d_2}} = \mathrm{id}_{N_{d_2}}$. We have that $g \notin U$ and consequently $\mathrm{id}_{N_{d_1}} \in U$ because $\mathrm{graph}\, f^{-1} = g \cdot \Delta_{N_{d_1}}$. Finally $U = \Phi_{N_{d_1}}$ by Theorem 2.10 and Lemma 3.1.

Now consider the case $m > 2$. We claim that $U = W_{d_1}$. Let $u\colon D(\underline{N}) \to P_{m,n}$ be a map and $x, y \in D(\underline{N})$ such that (x,y) witnesses $h \in U = \mathrm{Fail}_h(u)$. Let $f \in W_{d_1}$ be a non-extendable partial endomorphism of $\underline{P}_{m,n}$. We apply Lemma 2.7 and we get that there exist $g_1, g_2 \in \mathrm{Aut}\, \underline{P}_{m,n}$ such that $h = g_1 \circ f \circ g_2{\restriction}_N$. Since $g_1, g_2 \notin U$ and $h \in U$, f must be in U. We still need to prove that $f^{\leqslant} \in U$. Observe that $h^{\leqslant} = (g_1^{-1} \cdot (g_2 \cdot f^{\leqslant})^{\smile})^{\smile}$. Therefore if $h^{\leqslant} \in U$ then $f^{\leqslant} \in U$ because $g_1, g_2 \notin U$. Hence it only remains to prove that $h^{\leqslant} \in U$. Suppose that $(u(x), u(y)) = (d_1, d_2)$. Let $h^{-1} \in W_{d_1}$ be the partial endomorphism of $\underline{P}_{m,n}$ corresponding to the inverse of the isomorphism $\underline{N} \to h(\underline{N})$ given by h. Once again there are $g_1', g_2' \in \mathrm{Aut}\, \underline{P}_{m,n}$ such that $h = g_1' \circ h^{-1} \circ g_2'$. Since $y = h \circ x$ we have that $u(h^{-1} \circ g_2' \circ x) = {g_1'}^{-1}(u(y)) = d_2$ and $u(g_2' \circ x) = g_2'(u(x)) = d_1$. But then $(h^{-1} \circ g_2' \circ x, g_2' \circ x)$ witnesses $h^{\leqslant} \in U$. $\square$

Our next step is to find out if there are any other globally minimal failsets containing proper partial endomorphisms of $\underline{P}_{m,n}$.

Lemma 3.12. *Let U be a failset and suppose that* $\mathrm{id}_{\{0,1\}} \in U$. *Then $\Phi_{N_{d_1}} \subseteq U$ or $\Phi_{N_{d_2}} \subseteq U$.*

Proof. This is a consequence of the equality $\Delta_{N_{d_1}} \cap \Delta_{N_{d_2}} = \Delta_{\{0,1\}}$ and of Proposition 2.9 and Lemma 3.1. $\square$

Lemma 3.13. *Let U be a failset. For every subalgebra $\underline{Q}$ of $\underline{P}_{m,n}$ such that $N_{d_1} \subseteq Q$, $N_{d_2} \cap Q \nsubseteq \{0, d_1, 1\}$ and $\mathrm{id}_Q \in U$, there exists a maximal proper subalgebra $\underline{N}$ of $\underline{P}_{m,n}$ such that $Q \subseteq N$ and $\mathrm{id}_N \in U$.*

Proof. We prove the result by induction on $m - k$, where k is the number of atoms of $\underline{Q}$.

For $m - k = 1$, $\underline{Q}$ has $m - 1$ atoms and therefore $\underline{Q}$ is a maximal proper subalgebra of $\underline{P}_{m,n}$ since $N_{d_1} \subseteq Q$. Now suppose that the result is valid for subalgebras of $\underline{P}_{m,n}$ containing N_{d_1} and having k atoms. Let $\underline{Q}$ be a subalgebra of $\underline{P}_{m,n}$ containing N_{d_1} and let $a_1, \ldots, a_{k-1}$ be the atoms of $\underline{Q}$. There exists $i \in \{1, \ldots, k-1\}$ such that $a_i \notin \mathrm{At}\,\underline{P}_{m,n}$. We may assume that $i = k-1$. Then a_{k-1} is of the form $a_{i_1} \vee a_{i_2} \vee a$ for some $a_{i_1}, a_{i_2} \in \mathrm{At}\,\underline{P}_{m,n}$ and some $a \in P_{m,n}$ such that $a_{i_1} \nleqslant a \vee a_{i_2}$ and $a_{i_2} \nleqslant a \vee a_{i_1}$. Now we take $\underline{Q}_1, \underline{Q}_2 \leqslant \underline{P}_{m,n}$ such that $N_{d_1} \subseteq Q_1, Q_2$ and $\mathrm{At}\,\underline{Q}_1 = \{a_1, \ldots, a_{k-2}, a \vee a_{i_1}, a_{i_2}\}$ and $\mathrm{At}\,\underline{Q}_2 = \{a_1, \ldots, a_{k-2}, a \vee a_{i_2}, a_{i_1}\}$. Note that $Q = Q_1 \cap Q_2$. Since $\mathrm{id}_Q \in U$ one of the two identity maps $\mathrm{id}_{Q_1}, \mathrm{id}_{Q_2}$ must be in U. Then by our inductive hypothesis there exists a maximal proper subalgebra $\underline{N}$ of $\underline{P}_{m,n}$ such that $Q \subseteq N$ and $\mathrm{id}_N \in U$. $\square$

By using a similar argument we also have the following result:

Lemma 3.14. *Let U be a failset. For every subalgebra $\underline{Q}$ of $\underline{P}_{m,n}$ such that $N_{d_2} \subseteq Q$, $N_{d_1} \cap Q \nsubseteq \{0, d_2, 1\}$ and $\mathrm{id}_Q \in U$, there exists a maximal proper subalgebra $\underline{N}$ of $\underline{P}_{m,n}$ such that $Q \subseteq N$ and $\mathrm{id}_N \in U$.*

Lemma 3.15. *Let U be a failset and suppose that U contains neither $\mathrm{id}_{N_{d_1}}$ nor $\mathrm{id}_{N_{d_2}}$ and that U does not intersect $\mathrm{Aut}\,\underline{P}_{m,n}$. Let $\underline{Q}$ be a subalgebra of $\underline{P}_{m,n}$ such that $Q \neq \{0, 1\}$. If $\mathrm{id}_Q \in U$ then U contains the identity map on some maximal proper subalgebra of $\underline{P}_{m,n}$.*

Proof. Let $Q_1 = N_{d_1} \cup Q$ and $Q_2 = N_{d_2} \cup Q$. Observe that Q_1 and Q_2 are universes of two subalgebras $\underline{Q}_1$ and $\underline{Q}_2$ of $\underline{P}_{m,n}$. Since $\mathrm{id}_Q \in U$ and $\underline{Q} = \underline{Q}_1 \cap \underline{Q}_2$ one of the identity maps id_{Q_1}, id_{Q_2} must be in U. Suppose without loss of generality that $\mathrm{id}_{Q_1} \in U$. By hypothesis, $\mathrm{id}_{N_{d_1}} \notin U$ and therefore $Q_1 \neq N_{d_1}$. If $Q_1 \cap N_{d_2} \nsubseteq \{0, d_1, 1\}$ then, by applying Lemma 3.13, there exists a maximal proper subalgebra $\underline{N}$ of $\underline{P}_{m,n}$ satifying $Q_1 \subseteq N$ and $\mathrm{id}_N \in U$. Now suppose that $Q_1 \cap N_{d_2} \subseteq, \{0, d_1, 1\}$ and so $Q_1 = N_{d_1} \cup \{d_1\}$. There exists a map $u \colon D(\underline{Q}_1) \to P_{m,n}$ such that $\mathrm{id}_{Q_1} \in \mathrm{Fail}_{\mathrm{id}_{Q_1}}(u) \subseteq U$. Let $x \in D(\underline{Q}_1)$ such that $x \in Q_1$ and $u(x) \notin Q_1$. Hence $0 < u(x) < d_1$ and then there exist distinct atoms a, b of $\underline{P}_{m,n}$ such that $a \leqslant u(x)$ and $b \nleqslant u(x)$. Now we take $\underline{N}$ to be the maximal proper subalgebra of $\underline{P}_{m,n}$, containing N_{d_1}, whose atoms are $a \vee b$ and all the atoms of $\underline{P}_{m,n}$ different from a and b. Observe that $x \in N$ and $u(x) \notin N$. Thus $\mathrm{id}_N \in U$. $\square$

Proposition 3.16. *Let U be a globally minimal failset. Suppose that U contains neither* $\mathrm{id}_{N_{d_1}}$ *nor* $\mathrm{id}_{N_{d_2}}$ *and suppose that U does not intersect* $\mathrm{Aut}\,\underline{P}_{m,n}$. *For every subalgebra $\underline{Q}$ of $\underline{P}_{m,n}$, $\mathrm{id}_Q \notin U$.*

Proof. Let $\underline{Q}$ be a subalgebra of $\underline{P}_{m,n}$. If $Q = \{0,1\}$ then $\mathrm{id}_{N_{d_1}}$, $\mathrm{id}_{N_{d_2}} \notin U$ implies $\mathrm{id}_Q \notin U$ by Lemma 3.12. Now consider $Q \neq \{0,1\}$. Suppose that $\mathrm{id}_Q \in U$. By Proposition 3.15, there exists a maximal proper subalgebra $\underline{N}$ of $\underline{P}_{m,n}$ such that $\mathrm{id}_N \in U$. Let h be a non-extendable partial endomorphism of $\underline{P}_{m,n}$ having $\underline{N}$ as its domain. We have that $\mathrm{id}_N \in U$ implies $h \in U$ and from Proposition 3.10 and Proposition 3.11 we get $U = W_{d_i}$, for some $i \in \{1,2\}$, and $\mathrm{id}_N \notin W_{d_i}$. Thus $\mathrm{id}_Q \notin U$. □

Lemma 3.17. *Let U be a failset and suppose that U contains no identity maps. For every one-to-one non-extendable partial endomorphism $h\colon \underline{N} \to \underline{P}_{m,n}$ of $\underline{P}_{m,n}$, if $N_{d_i} \subseteq N$, for some $i \in \{1,2\}$, and $h \in U$ then U intersects* $\mathrm{End}^i\,\underline{P}_{m,n}$.

Proof. Let $h\colon \underline{N} \to \underline{P}_{m,n}$ be a one-to-one non-extendable partial endomorphism such that $N_{d_i} \subseteq N$, for some $i \in \{1,2\}$, and $h \in U$. Suppose without loss of generality that $i = 1$. Note that $d_1 \in \mathrm{dom}\,h$ because h is non-extendable. Since h is one-to-one and $h(d_1), h(d_2) \in \{d_1, d_2\}$ we must have $h(d_1) = d_1$ and $h(d_2) = d_2$. Thus $N_{d_2} \cap N \neq \{0, d_1, 1\}$ or otherwise h would be extendable. Let k be the number of atoms of $\underline{N}$. We will prove the result by induction on $m - k$.

If $m - k = 1$ then $\underline{N}$ has m-1 atoms. Hence $\underline{N}$ is a maximal proper subalgebra of $\underline{P}_{m,n}$ and there is nothing more to prove.

Now suppose the result is valid for partial endomorphisms whose domains have at least k atoms. Let $h\colon \underline{N} \to \underline{P}_{m,n}$ be a one-to-one non-extendable partial endomorphism of $\underline{P}_{m,n}$ such that $h \in U$, $N_{d_1} \subseteq N$ and $\underline{N}$ has $k-1$ atoms. Let $a_1, \dots, a_{k-1}$ be the atoms of $\underline{N}$. Since h is non-extendable and $k - 1 < m$, there exists $a_j \in \mathrm{At}\,\underline{N}$ such that $a_j \notin \mathrm{At}\,\underline{P}_{m,n}$ but $h(a_j) \in \mathrm{At}\,\underline{P}_{m,n}$. Suppose without loss of generality that $j = k-1$ and let b be the atom $h(a_j)$ of $\underline{P}_{m,n}$. Then a_{k-1} must be of the form $a_{k_1} \vee a_{k_2} \vee a$ for some distinct atoms a_{k_1}, a_{k_2} of $\underline{P}_{m,n}$ and for some $a \in \underline{P}_{m,n}$ (eventually 0), with $a < d_1$. The one-to-one non-extendability of h also implies the existence of some l in $\{1, \dots, k-2\}$ such that $a_l \in \mathrm{At}\,\underline{N} \cap \mathrm{At}\,\underline{P}_{m,n}$, but $h(a_l) \notin \mathrm{At}\,\underline{P}_{m,n}$, and so $h(a_l)$ is of the form $a_{l_1} \vee a_{l_2} \vee a'$ for some distinct atoms a_{l_1}, a_{l_2} of $\underline{P}_{m,n}$ and some $a' \in \underline{P}_{m,n}$ with $a' < d_1$. Now two cases may occur:

Case 1: there exists $a_l \in \mathrm{At}\,\underline{P}_{m,n}$ for which $h(a_l) = a_{l_1} \vee a_{l_2} \vee a'$, for some distinct atoms a_{l_1}, a_{l_2} of $\underline{P}_{m,n}$ and some $0 < a' < d_1$ such that $a_{l_1}, a_{l_2} \nleq a'$;

Case 2: for every $a_l \in \mathrm{At}\,\underline{N} \cap \mathrm{At}\,\underline{P}_{m,n}$, $h(a_l)$ is either an atom of $\underline{P}_{m,n}$ or is the join of two atoms of $\underline{P}_{m,n}$.

Observe that the subalgebras of $\underline{P}_{m,n}$ are completely determined by their atoms and their coatoms, and every one-to-one partial endomorphism of $\underline{P}_{m,n}$ is completely determined by the following conditions:

(i) The bijection from the set of the atoms of the domain, if non empty, into the set of the atoms of the codomain;
(ii) The bijection from the set of the coatoms of the domain, if non empty, into the set of the coatoms of the codomain.

We begin with case 1. Let $\underline{N}_1$ and $\underline{N}_2$ be the subalgebras of $\underline{P}_{m,n}$ such that $N_{d_1} \subseteq N_1, N_2$ and

$$\mathrm{At}\,\underline{N}_1 = \{a_1, \ldots, a_{k-2}, a_{k_1}, a_{k_2} \vee a\}$$

and

$$\mathrm{At}\,\underline{N}_2 = \{a_{l_1}, a_{l_2} \vee a'\} \cup \mathrm{At}\,h(\underline{N})\backslash\{h(a_l)\}.$$

Let $h_1 \colon \underline{N}_1 \to \underline{P}_{m,n}$ be the one-to-one partial endomorphism determined by $h_1{\restriction_{N_{d_1}}} = h{\restriction_{N_{d_1}}}$ and

$$h_1(x) = \begin{cases} a_{l_2} \vee a' & \text{if } x = a_l, \\ b & \text{if } x = a_{k_1}, \\ a_{l_1} & \text{if } x = a_{k_2} \vee a, \\ h(x) & \text{otherwise} \end{cases}$$

for every $x \in \mathrm{At}\,\underline{N}_1$. Observe that the non-extendability of h implies that h_1 is also non-extendable. Let $h_2 \colon \underline{N}_2 \to \underline{P}_{m,n}$ be the one-to-one partial endomorphism determined by $h_2{\restriction_{N_{d_1}}} = id{\restriction_{N_{d_1}}}$ and

$$h_2(x) = \begin{cases} a_{l_2} & \text{if } x = a_{l_1}, \\ a' & \text{if } x = a_{l_2} \vee a', \\ a_{l_1} \vee b & \text{if } x = b, \\ x & \text{otherwise} \end{cases}$$

for every $x \in \mathrm{At}\,\underline{N}_2$. We claim that $ker(h_1, h_2) = \mathrm{graph}(h)$. Note that $ker(h_1, h_2)$ $= (N_{d_1}^2 \cap ker(h_1, h_2)) \cup (N_{d_2}^2 \cap ker(h_1, h_2))$. Since $h_1{\restriction_{N_{d_1}}} = h{\restriction_{N_{d_1}}}$ and $h_2{\restriction_{N_{d_1}}} = \mathrm{id}{\restriction_{N_{d_1}}}$ we have $N_{d_1}^2 \cap ker(h_1, h_2) = N_{d_1}^2 \cap \mathrm{graph}\, h$. Let $x, y \in N_{d_2}$. We are going to prove that $(x, y) \in ker(h_1, h_2)$ only for $x \in N$. Suppose $(x, y) \in ker(h_1, h_2)$ with $x \in N_1 \backslash N$. Then one and only one of the atoms $a_{k_1}, a_{k_2} \vee a$ of $\underline{N}_1$ is less or equal to x.

If $a_{k_1} \leqslant x$ then $b = h_1(a_{k_1}) \leqslant h_1(x) = h_2(y)$. But then $a_{l_1} \vee b \leqslant h_2(y)$. Thus $h_1(a_{k_2} \vee a) = a_{l_1} \leqslant h_1(x)$ and this is false because $h_1(x) \wedge h_1(a_{k_2} \vee a) =$ $h_1(x \wedge (a_{k_2} \vee a)) = 0$.

If $a_{k_2} \vee a \leqslant x$ then $a_{l_1} = h_1(a_{k_2} \vee a) \leqslant h_1(x) = h_2(y)$. But then $a_{l_1} \vee b \leqslant h_2(y)$. Thus $h_1(a_{k_1}) = b \leqslant h_1(x)$ and this is false because $h_1(x \wedge a_{k_1}) = 0$.

Now let $x \in N$. We are going to consider three different possibilities in order to prove that $(x, y) \in ker(h_1, h_2)$ if and only if $(x, y) \in \mathrm{graph}(h)$.

If $a_{k-1}, a_l \nleqslant x$ then $h_1(x) = h(x) = h_2(h(x))$. Thus $h_1(x) = h_2(y)$ if and only if $y = h(x)$.

If $a_{k-1} \nleqslant x$ and $a_l \leqslant x$ then $x = a_l \vee x'$ for some $x' \ngeqslant a_l$ in $\underline{N}$. Then we have $h_1(x) = a_{l_2} \vee a' \vee h(x')$. Since $a_{l_2} \vee a' \leqslant h_2(y)$ if and only if $h(a_l) \leqslant y$ we have $h_1(x) = h_2(y)$ if and only if

$$h_2(y) = h_2(h(a_l)) \vee h_1(x) = h_2(h(a_l)) \vee h(x') = h_2(h(a_l)) \vee h_2(h(x')) = h_2(h(x))$$

if and only if $(x, y) \in \operatorname{graph} h$.

If $a_{k-1} \leqslant x$ then $x = a_{k-1} \vee x'$ for some $x' \ngeqslant a_{k_1}, a_{k_2} \vee a$ in $\underline{N}_1$. Then we have $h_1(x) = b \vee a_{l_1} \vee h_1(x')$ and $h_1(x') \ngeqslant b, a_{l_1}$. Observe that

$$y = h(x) \Rightarrow b \vee a_{l_1} \leqslant h_2(y) \Leftrightarrow b \leqslant y \Leftrightarrow y = b \vee y',$$

for some $y' \ngeqslant b$ in $\underline{N}_2$. Also note that $y' \ngeqslant b \Rightarrow h_2(y') \ngeqslant a_{l_1} \vee b \Rightarrow h_2(y') \ngeqslant a_{l_1}, b$ because $a_{l_1} \vee b$ is an atom of $h_2(\underline{N}_2)$. Hence we have

$$\begin{aligned} h_1(x) = h_2(y) \Leftrightarrow h_1(x') \vee b \vee a_{l_1} = h_2(y) \Leftrightarrow h_1(x') \vee b \vee a_{l_1} = h_2(y') \vee h_2(b) \\ \Leftrightarrow h_1(x') \vee b \vee a_{l_1} = h_2(y') \vee b \vee a_{l_1} \Leftrightarrow h_1(x') = h_2(y'). \end{aligned}$$

Since $a_{k-1} \nleqslant x'$ and $x' \in N$ we already know that $h_1(x') = h_2(y') \Leftrightarrow y' = h(x') \Leftrightarrow y = b \vee h(x') = h(x)$.

Thus $h \in U$ implies there is $i' \in \{1, 2\}$ such that $h_{i'} \in U$.

If $i' = 1$ then $h_{i'}$ satisfies the inductive conditions and the result comes immediately.

If $i' = 2$ and h_2 is non-extendable then $h_{i'}$ satisfies the inductive conditions and the result comes immediately.

If $i' = 2$ and h_2 is extendable then take $h_3 \colon \underline{N}_3 \to \underline{P}_{m,n}$ to be a non-extendable partial endomorphism of $\underline{P}_{m,n}$ such that h_3 extends h_2. We only need to prove that $h_3 \in U$ in order that we can apply the inductive hypothesis. Since $h_2 \in U$ we may take $u : D(\underline{N}_2) \to P_{m,n}$ such that $h_2 \in \operatorname{Fail}_{h_2}(u) \subseteq U$. Let (x, y) witness $h_2 \in \operatorname{Fail}_{h_2}(u)$. Since $\mathrm{id}_{N_2} \notin U$ we must have $u(x) \in N_2$ and $u(y) \neq h_2(u(x)) = h_3(u(x))$. Hence $h_3 \in \operatorname{Fail}_{h_2}(u) \subseteq U$.

Finally we consider case 2. Recall that we only need to consider $k - 1 < m - 1$. Hence $\underline{N}$ and consequently $h(\underline{N})$ are not maximal proper subalgebras of $\underline{P}_{m,n}$. But then there are distinct atoms a_j, a_l of $\underline{N}$ in $\operatorname{At}\underline{P}_{m,n}$ such that $h(a_j), h(a_l) \notin \operatorname{At}\underline{P}_{m,n}$. Hence $h(a_j) = a_{j_1} \vee a_{j_2}$ and $h(a_l) = a_{l_1} \vee a_{l_2}$, for some $a_{j_1}, a_{j_2}, a_{l_1}, a_{l_2} \in \operatorname{At}\underline{P}_{m,n}$ (recall that h is non-extendable). Observe that $h(a_j)$ and $h(a_l)$ are atoms of $h(\underline{N})$ and then $a_{j_1} \neq a_{l_1}$. Let $h_1 \colon \underline{N} \to \underline{P}_{m,n}$ be the one-to-one partial endomorphism determined by $h_1{\restriction}_{N_{d_1}} = h{\restriction}_{N_{d_1}}$ and

$$h_1(x) = \begin{cases} a_{j_1} \vee a_{j_2} \vee a_{l_1} & \text{if } x = a_j, \\ a_{l_2} & \text{if } x = a_l, \\ h(x) & \text{if } x \neq a_j, a_l, \end{cases}$$

for every $x \in \mathrm{At}\,\underline{N}$. Let $\underline{N}'$ be the maximal proper subalgebra of $\underline{P}_{m,n}$ containing N_{d_1} whose atoms are $a_{j_1} \vee a_{l_1}$ and all the atoms of $\underline{P}_{m,n}$ different from a_{j_1} and from a_{l_1}. Now let $h_2 \colon \underline{N}' \to \underline{P}_{m,n}$ be the one-to-one partial endomorphism determined by $h_2{\restriction}_{N_{d_1}} = \mathrm{id}_{N_{d_1}}$ and

$$h_2(x) = \begin{cases} a_{j_1} & \text{if } x = a_{j_1} \vee a_{l_1}, \\ a_{l_1} \vee a_{l_2} & \text{if } x = a_{l_2}, \\ x & \text{for } x \neq a_{j_1} \vee a_{l_1}, a_{l_2}, \end{cases}$$

for every $x \in \mathrm{At}\,\underline{N}'$. Now we claim that $h = h_2 \circ h_1$. Let $x \in N$.

If $a_j, a_l \nleqslant x$ then $h_1(x) = h(x)$ is the join of some atoms of $\underline{P}_{m,n}$ different from $a_{j_1}, a_{j_2}, a_{l_1}$ and a_{l_2}. But then $h(x) = h_2(h(x)) = h_2 \circ h_1(x)$.

If $a_j \nleqslant x$ and $a_l \leqslant x$ then $x = x' \vee a_l$ for some $x' \in N$ with $a_l \nleqslant x'$. But now we have

$$h_2 \circ h_1(x) = h_2(h(x')) \vee h_2 \circ h_1(a_l) = h(x') \vee h(a_l) = h(x).$$

If $a_j \leqslant x$ then $x = x' \vee a_j$ for some $x' \in N$ with $a_j \nleqslant x'$. But now we have

$$h_2 \circ h_1(x) = h_2 \circ h_1(x') \vee h_2 \circ h_1(a_j) = h(x') \vee h(a_j) = h(x).$$

Since $h \in U$ we must have $h_1 \in U$ or $h_2 \in U$.

If $h_1 \in U$ then the result comes by applying case 1.

If $h_2 \in U$ then there is nothing more to prove since h_2 is already a non-extendable partial endomorphism of $\underline{P}_{m,n}$ having as its domain a maximal proper subalgebra of $\underline{P}_{m,n}$. □

Proposition 3.18. *Let U be a globally minimal failset and suppose that U contains a partial endomorphism of $\underline{P}_{m,n}$. Then either $U = \Phi_{N_{d_i}}$ or $U = W_{d_i}$, for some $i \in \{1, 2\}$.*

Proof. Observe that U must not intersect $\mathrm{Aut}\,\underline{P}_{m,n}$, by Proposition 3.5. If U contains an identity map then, by Proposition 3.16, U must contain $\mathrm{id}_{N_{d_i}}$, for some $i \in \{1, 2\}$, and therefore $U = \Phi_{N_{d_i}}$, by Lemma 3.1.

Now suppose that U does not contain any identity map. Let $h \colon \underline{N} \to \underline{P}_{m,n}$ be a partial endomorphism of $\underline{P}_{m,n}$ in U. There exists a map $u \colon D(\underline{N}) \to P_{m,n}$ such that $\mathrm{Fail}_h(u) = U$. Since $\mathrm{id}_N \notin U$ there is $x \in D(\underline{N})$ such that $u(x) \in x(N) \subseteq N$ and $u(h \circ x) \neq h(u(x))$. Thus every non-extendable extension of h is also in U. So we may assume that h is non-extendable. Note that $d_1, d_2 \in N$. We claim that U contains a one-to-one non-extendable partial endomorphism of $\underline{P}_{m,n}$ whose domain contains N_{d_i}, for some $i \in \{1, 2\}$.

If h is not one-to-one then $h(d_1) = h(d_2)$. Suppose without loss of generality that $h(d_1) = h(d_2) = d_2$. We must have $N \cap N_{d_2} = \{0, d_1, 1\}$ and, by Proposition 2.1, $h{\restriction}_{N_{d_1} \cap N}$ is one-to-one. Let $\underline{N}'$ be the subalgebra of $\underline{P}_{m,n}$ that contains

N_{d_2} and whose coatoms are the coatoms of $\underline{N}$, so that the universe of $\underline{N}'$ is $N \cup N_{d_2}$. Let $h' : \underline{N}' \to \underline{P}_{m,n}$ be the one-to-one partial endomorphism of $\underline{P}_{m,n}$ that satisfies $h'\restriction_{N_{d_2}} = \mathrm{id}_{N_{d_2}}$ and $h'\restriction_{N_{d_1} \cap N} = h\restriction_{N_{d_1} \cap N}$. Since h is non-extendable h' must also be non-extendable. Suppose that $h' \notin U$. We have $x(d_1) = d_1$ or otherwise

$$x, u(x) \in N_{d_1},\ (x, h \circ x) \in h\restriction_{N_{d_1} \cap N} = h'\restriction_{N_{d_1} \cap N}$$

and

$$u(h' \circ x) = u(h \circ x) \neq h(u(x)) = h'(u(x)).$$

Take $y \in D(\underline{N})$ to be defined by $y\restriction_{N_{d_1} \cap N} = x\restriction_{N_{d_1} \cap N}$ and $y(d_1) = d_2$. Now we have $(x, y) \in \mathrm{id}^{\leqslant}$. By Proposition 3.7, $\mathrm{id}^{\leqslant} \notin U$. Hence $u(x) = u(y)$ or $(u(x), u(y)) = (d_1, d_2)$. If $u(x) = u(y)$ then $u(x) \in y(N) \subseteq N_{d_1}$ and

$$u(h \circ x) = u(h' \circ y) = h'(u(y)) = h(u(y)) = h(u(x)).$$

If $(u(x), u(y)) = (d_1, d_2)$ then

$$u(h \circ x) = u(h' \circ y) = h'(u(y)) = h'(d_2) = d_2 = h(d_1) = h(u(x)).$$

Thus $h' \in U$.

If h is one-to-one then take $\underline{Q}$ to be the subalgebra of $\underline{P}_{m,n}$ whose atoms are the atoms of $\underline{N}$ and whose coatoms are the coatoms of $h(\underline{N})$. Let $h_1 : \underline{N} \to \underline{P}_{m,n}$ be the one-to-one partial endomorphism determined by $h_1\restriction_{\mathrm{At}\,\underline{N}} = \mathrm{id}\restriction_{\mathrm{At}\,\underline{N}}$ and $h_1\restriction_{\mathrm{Coat}\,\underline{N}} = h\restriction_{\mathrm{Coat}\,\underline{N}}$, and let $h_2 : \underline{Q} \to \underline{P}_{m,n}$ be the one-to-one partial endomorphism determined by $h_2\restriction_{\mathrm{At}\,\underline{N}} = h\restriction_{\mathrm{At}\,\underline{N}}$ and $h_2\restriction_{\mathrm{Coat}\,h(\underline{N})} = \mathrm{id}\restriction_{\mathrm{Coat}\,h(\underline{N})}$. Then $h_1 \in U$ or $h_2 \in U$ because $h = h_2 \circ h_1 \in U$. Finally observe that the one-to-one non-extendability of h implies that h_1 and h_2 are extended by one-to-one non-extendable partial endomorphisms whose domains contain N_{d_2} and N_{d_1} respectively.

Now the result follows from Proposition 3.11 and Lemma 3.17. □

Finally we are going to prove that the globally minimal failsets within Ω are exactly those which have already been described.

Theorem 3.19. *The globally minimal failsets, within Ω, are the following:*

(a) $\Phi_{N_{d_i}}$*, where* $i \in \{1, 2\}$*;*
(b) W_H*, for every maximal proper subgroup* H *of* $\mathrm{Aut}\,\underline{P}_{m,n}$*;*
(c) $\{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \mathrm{Aut}\,\underline{P}_{m,n} \}$*;*
(d) $\{\, \overline{g} \mid g \in \mathrm{Aut}\,\underline{P}_{m,n} \}$*;*
(e) W_{d_1} *if* $m > 2$*;*
(f) W_{d_2} *if* $n > 2$.

Proof. Let U be a globally minimal failset and suppose that U is not one of the globally minimal failsets described before. By Lemma 3.1, $\mathrm{id}_{N_{d_1}}, \mathrm{id}_{N_{d_2}} \notin U$ and, by Proposition 3.5, U does not intersect $\mathrm{Aut}\,\underline{P}_{m,n}$. Consequently U does not contain any id_Q, with $\underline{Q} \leqslant \underline{P}_{m,n}$, by Proposition 3.16. We also have that there is no partial endomorphisms of $\underline{P}_{m,n}$ in U, by applying Proposition 3.18. Then for every $r \in U$, we use Lemma 2.1 and Propositions 3.7 and 3.9 and we have that r must be one of the relations $h^{\leqslant}$, $(h^{\leqslant})^{\smile}$ and $\overline{h}$, for some one-to-one partial endomorphism h of $\underline{P}_{m,n}$ such that $h(d_i) = d_i$ when $i \in \{1,2\}$ and $d_i \in \mathrm{dom}\, h$.

Let $h\colon \underline{N} \to \underline{P}_{m,n}$ be a one-to-one partial endomorphism and suppose that $r \in U$, with $r = h^{\leqslant}$ or $r = \overline{h}$. If h is extendable then there exists an extension f of h such that either f is an automorphism of $\underline{P}_{m,n}$ or f is a one-to-one non-extendable partial endomorphism of $\underline{P}_{m,n}$ with $d_i \in \mathrm{dom}\, f$ and $f(d_i) = d_i$, for $i \in \{1,2\}$. Observe that $r = r' \cap (\rho_1(r) \times \rho_2(r))$, where $r' = f^{\leqslant}$ if $r = h^{\leqslant}$ and $r' = \overline{f}$ otherwise. Since $\rho_1(r) \times \rho_2(r) \notin U$ we have that $r' \in U$. But then f must be a non-extendable partial endomorphism, by Proposition 3.7 and Proposition 3.9. Thus we may assume that h is non-extendable. Hence $d_1, d_2 \in N$. Take $u\colon D(\underline{r}) \to P_{m,n}$ to be a map such that $U = \mathrm{Fail}_{\underline{r}}(u)$ and let (x,y) witness $r \in \mathrm{Fail}_{\underline{r}}(u)$. Since $h \notin U$ we must have $(x,y) \notin h$ and so $(x(a,b), y(a,b)) \in r \cap \{(d_1,d_2),(d_2,d_1)\}$ for some $(a,b) \in r \cap \{d_1,d_2\}^2$. Denote by r' the set $r \backslash \{d_1,d_2\}^2$ Let $z \in D(\underline{r})$ be defined by $z{\restriction_{r'}} = y{\restriction_{r'}}$ and $z(a,b) = x(a,b)$, for $(a,b) \in r \cap \{d_1,d_2\}^2$. We have that $(x,z) \in h$ and $(z,y) \in \mathrm{id}'$, where $\mathrm{id}' = \Delta_{P_{m,n}} \cup (r \cap \{(d_1,d_2),(d_2,d_1)\})$. Since h, $\mathrm{id}' \notin U$ we must have $u(z) = h(u(x))$ and $(h(u(x), u(y)) = (u(z), u(y)) \in \mathrm{id}'$. Then either $(u(x), u(y)) \in h$ or $(u(x), u(y)) \in r \cap \{(d_1,d_2),(d_2,d_1)\}$, so that $(u(x), u(y)) \in r$.□

4. Optimal Natural Dualities on $\mathbb{ISP}(\underline{P}_{m,n})$

A subset R of Ω yields an optimal duality on $\mathcal{A} = \mathbb{ISP}(\underline{P}_{m,n})$ if R yields a duality on $\mathcal{A}$ but no proper subset of R does so. Let $\mathcal{G}$ be a family of globally minimal failsets within Ω and let T be a subset of Ω. We say T is a **transversal** of $\mathcal{G}$ if T intersects each $U \in \mathcal{G}$ but no proper subset of T does.

The following result is part of Theorem 4.4 (The Optimal Duality Theorem) of [**5**].

Theorem 4.1. *Assume that $\Omega' \subseteq \bigcup_{n \geqslant 1} \mathbb{S}(\underline{M}^n)$ is finite and yields a duality on $\mathbb{ISP}(\underline{M})$, where $\underline{M}$ is a finite algebra. Then the following are equivalent:*

(a) *$R \subseteq \Omega'$ yields an optimal duality on $\mathcal{A}$;*
(b) *R is a transversal of the globally minimal failsets in Ω'.*

Recall that $\Omega = \mathbb{S}(\underline{P}^2_{m,n})$ yields a duality on $\mathcal{A} = \mathbb{ISP}(\underline{P}_{m,n})$. Thus the following result is obtained by applying Theorem 3.19 and Theorem 4.1.

Theorem 4.2. *The optimal dualities on $\mathcal{A} = \mathbb{ISP}(\underline{P}_{m,n})$ given by binary relations are the dualities yielded by the transversals of the family $\mathcal{G}$ whose elements are the following:*

(a) $\Phi_{N_{d_i}}$ *for* $i \in \{1,2\}$*;*
(b) W_H*, for every maximal proper subgroup* H *of* $\operatorname{Aut}\underline{P}_{m,n}$*;*
(c) $\{\, g^{\leqslant}, (g^{\leqslant})^{\smile} \mid g \in \operatorname{Aut}\underline{P}_{m,n}\}$*;*
(d) $\{\, \overline{g} \mid g \in \operatorname{Aut}\underline{P}_{m,n}\}$*;*
(e) W_{d_1} *if* $m > 2$*;*
(f) W_{d_2} *if* $n > 2$.

We may take transversals of $\mathcal{G}$ whose intersections with the globally minimal failsets W_H only contain automorphisms of $\operatorname{Aut}\underline{P}_{m,n}$. This implies that the intersections of these transversals with $\operatorname{Aut}\underline{P}_{m,n}$ are transversals of the family of the globally minimal failsets W_H, where H is a maximal proper subgroup of $\operatorname{Aut}\underline{P}_{m,n}$, because these are the unique globally minimal failsets that contain automorphisms of $\operatorname{Aut}\underline{P}_{m,n}$. The next result tells us that these transversals of the family $\{\, W_H \mid H$ is a maximal proper subgroup of $\operatorname{Aut}\underline{P}_{m,n}\,\}$ are precisely the minimal generating sets of $\operatorname{Aut}\underline{P}_{m,n}$.

Proposition 4.3. *A set T_A of automorphisms of $\underline{P}_{m,n}$ is a transversal of the family $\mathcal{G}_A = \{\, W_H \mid H$ is a maximal proper subgroup of $\operatorname{Aut}\underline{P}_{m,n}\,\}$ if and only if T_A is a minimal generating set of $\operatorname{Aut}\underline{P}_{m,n}$.*

Proof. Let T_A be a set of automorphisms of $\underline{P}_{m,n}$. We have that T_A is a transversal of $\{\, W_H \mid H$ is a maximal proper subgroup of $\operatorname{Aut}\underline{P}_{m,n}\,\}$ if and only if

$T_A \cap W_H \neq \emptyset$, for every W_H in $\mathcal{G}_A$, and for every $S \subsetneq T_A$, $S \cap W_H = \emptyset$ for some W_H in $\mathcal{G}_A$,

if and only if

$T_A \nsubseteq H$, for every maximal proper subgroup H of $\operatorname{Aut}\underline{P}_{m,n}$, and for every $S \subsetneq T_A$, there is a maximal proper subgroup H of $\operatorname{Aut}\underline{P}_{m,n}$ that contains S

if and only if

T_A generates $\operatorname{Aut}\underline{P}_{m,n}$ but no proper subset of T_A does so

if and only if

T_A is a minimal generating set of $\operatorname{Aut}\underline{P}_{m,n}$. □

Hence we may take as transversals of $\mathcal{G}$ the unions of minimal generating sets of $\operatorname{Aut}\underline{P}_{m,n}$ with transversals of the family of globally minimal failsets containing no automorphisms of $\underline{P}_{m,n}$.

Let σ, τ be the following elements of $S_m \times S_n$:

$$\sigma = \begin{cases} ((1\,2), \mathrm{id}) & \text{if } n = 2, \\ ((1\,2), (1\,3 \ldots n)) & \text{if } n \text{ is even and } n > 2, \\ ((1\,2), (1\,2 \ldots n)) & \text{if } n \text{ is odd} \end{cases}$$

and

$$\tau = \begin{cases} (\mathrm{id}, (1\,2)) & \text{if } m = 2, \\ ((1\,3\ldots m), (1\,2)) & \text{if } m \text{ is even and } m > 2, \\ ((1\,2\ldots m), (1\,2)) & \text{if } m \text{ is odd.} \end{cases}$$

Proposition 4.4. *The set* $\{\sigma, \tau\}$ *is a minimal generating set of* $S_m \times S_n$.

Proof. If n is even and $n > 2$ then $\sigma^n = (\mathrm{id}, (1\,3\ldots n))$ and $\sigma^{n-1} = ((1\,2), \mathrm{id})$. If n is odd then $\sigma^{n+1} = (\mathrm{id}, (1\,2\ldots n))$ and $\sigma^n = ((1\,2), \mathrm{id})$. Then σ generates either $\{((1\,2), \mathrm{id}), (\mathrm{id}, (1\,3\ldots n))\}$ or $\{((1\,2), \mathrm{id}), (\mathrm{id}, (1\,2\ldots n))\}$. Similarly τ generates either $\{(\mathrm{id}, (1\,2)), ((1\,3\ldots m), \mathrm{id})\}$ or $\{(\mathrm{id}, (1\,2)), ((1\,2\ldots m), \mathrm{id})\}$. Since $(1\,2\ldots k) = (1\,3\ldots k)(1\,2)$, for $k > 1$, the set $\{\sigma, \tau\}$ generates $(\mathrm{id}, (1\,2))$, $((1\,2), \mathrm{id})$, $(\mathrm{id}, (1\,2\ldots n))$ and $((1\,2\ldots m), \mathrm{id})$. Consequently $\{\sigma, \tau\}$ generates $S_m \times S_n$. $\square$

Corollary 4.5. *The minimum size of the generating sets of* $S_m \times S_n$ *is* 2.

Now take g_1 and g_2 to be the automorphisms of $\underline{P}_{m,n}$ determined by σ and τ respectively. Then the set $\{g_1, g_2\}$ is a minimal generating set of $\mathrm{Aut}\,\underline{P}_{m,n}$ of minimum size. The set $\{h_1{\restriction}_{N_{d_1}}, h_2{\restriction}_{N_{d_2}}, h_1, h_2, \mathrm{id}^{\leqslant}, \overline{\mathrm{id}}\}$, where $h_1 \in \mathrm{End}^1\,\underline{P}_{m,n}$ and $h_2 \in \mathrm{End}^2\,\underline{P}_{m,n}$, is a minimum size transversal of the family of globally minimal failsets containing no automorphisms. Thus the set

$$T = \{h_1{\restriction}_{N_{d_1}}, h_2{\restriction}_{N_{d_2}}, h_1, h_2, g_1, g_2, \mathrm{id}^{\leqslant}, \overline{\mathrm{id}}\}$$

is a minimum size transversal of $\mathcal{G}$ containing a generating set of $\mathrm{Aut}\,\underline{P}_{m,n}$. The set

$$T' = \{h_1{\restriction}_{N_{d_1}}, h_2{\restriction}_{N_{d_2}}, h_1, h_2, g_1, g_2^{\leqslant}, \overline{\mathrm{id}}\}$$

is also a transversal of $\mathcal{G}$ and it has minimum size. Both sets T and T' yield optimal dualities on $\mathcal{A}$ and T' is one of the smallest sets of binary relations yielding an optimal duality on $\mathcal{A}$.

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