Acta Math. Univ. Comenianae
Vol. LXVII, 2(1998), pp. 373-376

ON METRIC PRESERVING FUNCTIONS AND INFINITE DERIVATIVES

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Abstract. We give two examples to answer a question of J. Doboš and Z. Piotrowski concerning the points at which a metric preserving function has an infinite derivative.

Metric preserving functions appear in the literature as far back as 1935 (see [**8**]). Many papers have recently been written concerning properties of these functions (see for example [**7**], [**1**], [**3**], [**6**], [**5**] and [**4**]).

Definition. We call a function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ a **metric preserving function** if and only if $f(\rho)\colon M \times M \to \mathbb{R}^+$ is a metric on M for every metric $\rho\colon M \times M \to \mathbb{R}^+$, where (M, ρ) is an arbitrary metric space and $\mathbb{R}^+$ denotes the nonnegative reals.

An analytic way to describe the functions is as follows: a function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ is metric preserving if and only if $f^{-1}(\{0\}) = \{0\}$ and for all $a, b, c > 0$ such that $|a - b| \le c \le a + b$ we have

$$|f(a) - f(b)| \le f(c)\,.$$

In [**3**] an example was given of a continuous metric preserving function f which had $f'(2^{-n}) = \infty$. Also in [**3**] we find the following question about metric preserving functions:

$$\text{Is it possible to characterize the set } f'^{-1}(\infty)?$$

It is this question we address in this paper, concentrating on functions f which are everywhere differentiable (including $f'(x) = \infty$). We know (see [**1**]) that if f is a differentiable metric preserving function then

$$|f'(x)| \le f'(0)$$

where at $x = 0$ it's a one-sided derivative. So if the set in question is nonempty, it must contain zero. In this paper we will look at this set in terms of the following lemma that is a modification of Lemma 1.2 of [**2**, Chapter IX].

Received October 2, 1996; revised June 9, 1997.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 54E40; Secondary 26A24.

Lemma 1. *Let $Z \subseteq \mathbb{R}^+$ be a measure zero $\mathcal{G}_\delta$ set. There exists a differentiable (in the extended sense) absolutely continuous function $G\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $G'(x) = \infty$ for all $x \in Z$ and $\infty > G'(x) > 0$ for all $x \in \mathbb{R}^+ \setminus Z$.*

First we note that by using the proposition below (from [**1**]), it is trivial to show that the set in question can be an arbitrary $\mathcal{G}_\delta$ and measure zero set containing the origin if we do not care about the function being everywhere continuous.

Proposition. *Let a function $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ have the properties that $f(0) = 0$ and there exists a positive value a such that for all positive x we have*

$$a \leq f(x) \leq 2a.$$

Then f is metric preserving.

Example 1. Let $Z \subseteq \mathbb{R}^+$ be the set from Lemma 1. There exists an $f\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that

1. f is metric preserving,
2. f' exists (in the extended sense) at all x, and
3. $\{x : f'(x) = \infty\} = Z \cup \{0\}$.

Proof. Given Z use Lemma 1 to define G. Then define f by

$$f(x) = \begin{cases} \frac{2}{\pi}\arctan\left(G(x)\right) + 1, & x \in (0,\infty), \\ 0, & x = 0. \end{cases}$$

By the Proposition f is metric preserving. □

This function is, however, not continuous at the origin. Continuity at the origin is very important to metric preserving functions. From the analytic description above along with the fact that $f(0) = 0$, continuity at the origin implies continuity everywhere. Thus our next goal is to construct a metric preserving function with infinite derivative on $Z \cup \{0\}$ which is continuous on all of $\mathbb{R}^+$. Requiring continuous and differentiable (in the extended sense) on $\mathbb{R}^+$ makes our result as complete as possible. We begin with a lemma from [**3**].

Lemma 2. *Let g and h be metric preserving functions. Let $t > 0$ be such that $g(t) = h(t)$. Define $w\colon \mathbb{R}^+ \to \mathbb{R}^+$ as follows*

$$w(x) = \begin{cases} g(x), & x \in [0,t), \\ h(x), & x \in [t,\infty). \end{cases}$$

Suppose g is non-decreasing and concave. Then if the condition

$$\forall\, x, y \in [t,\infty) : |x - y| \leq t \implies |h(x) - h(y)| \leq g(|x - y|)$$

is met w is metric preserving.

In applying Lemma 2 we must be careful in constructing the differentiable $g(x)$. Another property of metric preserving functions is that

$$\left|\frac{f(x+h)-f(x)}{h}\right| \le \frac{f(h)}{h}\,.$$

Thus the function $h \to \frac{1}{h}(g(x+h)-g(x))$ must become infinite (when h approaches 0) at $x=0$ faster than it does anywhere else.

Example 2. Let $Z \subseteq \mathbb{R}^+$ be a measure zero $\mathcal{G}_\delta$ set. There exists a continuous metric preserving function whose derivative exists everywhere and is infinite precisely on $Z \cup \{0\}$.

Proof. Let $G(x)$ be the function from Lemma 1 which is absolutely continuous and

$$G'(x) = \infty \text{ for } x \in Z, \text{ while}$$
$$G'(x) \text{ exists and is finite for } x \notin Z.$$

Define $\hat{G}(x) = \frac{2}{\pi}\arctan(G(x)) + 1$. This $\hat{G}$ is clearly uniformly continuous.

Let $s(x)$ be a continuous increasing differentiable concave function that maps $\mathbb{R}^+$ onto $\mathbb{R}^+$ such that $s(0)=0$ and for all $h \in (0,2)$

$$s(h) \ge 2\sup\{|\hat{G}(x+h)-\hat{G}(x)| : x \in \mathbb{R}^+\}.$$

Using this s we can put together a sequence of continuous, differentiable metric preserving functions f_m.

Start with $\{a_m\}$, a sequence of points in $(0,1)\setminus Z$ converging monotonically to zero. For each m find the point b_m such that $s(b_m) = \hat{G}(a_m)$. Since for all but finitely many m the condition $b_m > \frac{1}{2}a_m$ must be true we require that it is true for **all** m in our sequence. Define

$$f_m(x) = \begin{cases} s((2b_m x)/a_m) & \text{on } [0, a_m/2], \\ t(x) & \text{on } [a_m/2, a_m], \\ \hat{G}(x) & \text{on } [a_m, \infty), \end{cases}$$

where $t(x)$

1. is a differentiable function with $1 \le t(x) \le 2$,
2. meets $t(a_m/2) = s(b_m)$, $t(a_m) = \hat{G}(a_m)$ and t connects s and $\hat{G}$ smoothly, and
3. satisfies $|t(x)-t(y)| \le \frac{1}{2}s\left(\frac{2b_m}{a_m}|x-y|\right)$ for $|x-y| \le a_m/2$.

Now $s((2b_m x)/a_m)$ is metric preserving from Proposition 2 in [**1**] and the function

$$y = \begin{cases} 0, & x = 0, \\ t(a_m/2), & (0, a_m/2], \\ t(x), & [a_m/2, a_m], \\ \hat{G}(x), & [a_m, \infty) \end{cases}$$

is metric preserving via the Proposition on page 2 (we know $1 \leq y \leq 2$ for $x > 0$). Since s is constructed to satisfy the requirements of Lemma 2 each f_m is a metric preserving function. Note $f'_m(x) = \infty$ on $([a_m, \infty) \cap Z) \cup \{0\}$ and elsewhere exists and is finite. Lastly define

$$f(x) = \sum_{m=1}^{\infty} 2^{-m} f_m(x). \qquad \square$$

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