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INVARIANT CONE OF SLOWLY OSCILLATING SOLUTION IN TWO DELAYS DIFFERENTIAL EQUATIONS

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ABSTRACT. Scalar delay differential equations with two delays are considered in this paper. Some monotonicity results permit to establish existence of non trivial slowly oscillating solutions.

1. Introduction

Oscillations of delay differential equations have been considered recently by many authors (see [**1**], [**2**], [**3**], [**6**], [**7**]). In this work, we investigate the monotony properties to establish existence of slowly oscillating solution (s.o.) of retarded differential equations with two delays.

This paper is organized in three sections. In the introduction, we present our model and give the definition of a s.o. solution. In the second section, we construct an invariant cone $K \subseteq C\left([-\sigma, 0], \mathbb{R}\right)$ of s.o. solutions. Section 3 is devoted to an example.

We consider the equation

$$\dot{x}\left(t\right) = -f\left(x\left(t-\tau\right)\right) - g\left(x\left(t-\sigma\right)\right) \tag{1}$$

under the following general assumptions:
f and g are smooth functions such that:

1. $0 \leq \dfrac{f(x)}{x} \leq a$ for $x \neq 0$,
2. $0 \leq \dfrac{g(x)}{x}$ for $x \neq 0$,
3. $\tau < \sigma$.

Definition 1. A solution of (1) is slowly oscillating if the distance between two successive zeros of $x(t)$ is not less than $\max(\tau, \sigma)$.

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2. Slowly Oscillating Solution

In this section, we start with the study of the retarded differential equation

$$\dot{x}(t) = -a\,x(t-\tau) - g(x(t-\sigma)). \tag{2}$$

We assume that:
(H) a satisfies $0 \le a\tau < \dfrac{1}{e}$,
and we indicate how this hypothesis may be used to derive existence of slowly oscillating solutions.

The advantage of using hypothesis (H) lies in the fact that the fundamental solution of the retarded differential equation

$$\dot{x}(t) = -a\,x(t-\tau) \tag{3}$$

is positive. This implies the same conclusion for $x(t)$ solution of the retarded differential equation

$$\dot{x}(t) = -a\,x(t-\tau) + f(t)$$

with initial condition $x_0 \ge 0$, when f is positive.

From hypothesis (H), we know that there are two negative roots $\lambda_1 < \lambda_2$ of the characteristic equation $\lambda = -a\,e^{-\lambda\tau}$, associated with the linear equation (3).

We define the cone ([**5**], [**9**]):

$$\Gamma = \left\{\varphi \in C : \varphi \ge 0,\ \varphi(\theta) \le e^{\lambda_2\theta}\varphi(0)\right\}$$

where $C = C\left([-\tau, 0], \mathbb{R}\right)$.

Proposition 2. *Under hypothesis (H), if $\varphi \in \Gamma$, then, the solution $x(t)$ of the retarded differential equation*

$$\begin{cases} \dot{x}(t) = -a\,x(t-\tau), \\ x_0 = \varphi, \end{cases} \tag{4}$$

satisfies $x_t \in \Gamma$, for all $t > 0$, where x_t denotes as usual, the function defined by $x_t(\theta) = x(t+\theta)$ for all $-\tau \le \theta \le 0$.

Proof. Let $y(t) = e^{-\lambda_2 t}x(t)$. In terms of y, equation (4) reads:

$$\begin{cases} \dot{y}(t) = -\lambda_2(y(t) - y(t-\tau)), \\ y_0 = \tilde{\varphi}, \end{cases}$$

where $\tilde{\varphi} = e^{-\lambda_2 \cdot}\varphi$.

The cone Γ becomes:

$$\Gamma_0 = \left\{\tilde{\varphi} \in C : \tilde{\varphi} \geq 0,\ \tilde{\varphi}(\theta) \leq \tilde{\varphi}(0)\right\}.$$

So, we are lead to prove that $y_t \in \Gamma_0$. To do this, we demonstrate that $y(t)$ is positive and increasing for all $t \in [0, +\infty)$.
$\dot{y}(0) \geq 0$ gives $\dot{y}(t) \geq 0$ for all t in a right neighborhood of 0. Let us prove that $\dot{y}(t) \geq 0$ for all $t > 0$.

In case $\tilde{\varphi}(\theta) < \tilde{\varphi}(0)$, we assert that $\dot{y}\,(t) > 0$ for all $t > 0$. Indeed, on the contrary, suppose there exists $\underline{t} > 0$ such that $\dot{y}(\underline{t}) = 0$ and $\dot{y}(t) > 0$ for all $0 \leq t < \underline{t}$, this implies:

If $0 \leq \underline{t} - \tau < \underline{t}$ then $y(\underline{t}) - y(\underline{t} - \tau) > 0$, if $\underline{t} - \tau \leq 0 < \underline{t}$ then $y(\underline{t} - \tau) = \varphi(\underline{t} - \tau) \leq \varphi\,(0) < y(\underline{t})$. In either case, we have, $\dot{y}(\underline{t}) = -\lambda_2(y(\underline{t}) - y(\underline{t} - \tau) > 0$, which is impossible.

In the other case, it suffices to let $\tilde{\varphi}_\varepsilon = \max(\tilde{\varphi}(\theta) + \varepsilon\theta, 0)$, $\varepsilon > 0$.

For $\theta < 0$, we have

$$\begin{aligned} \tilde{\varphi}_\varepsilon(\theta) &= \tilde{\varphi}(t) + \varepsilon\theta < \tilde{\varphi}\,(\theta) \leq \tilde{\varphi}\,(0) = \tilde{\varphi}_\varepsilon\,(0) && \text{if } \tilde{\varphi}(\theta) + \varepsilon\theta \geq 0, \\ \text{and } \tilde{\varphi}_\varepsilon(\theta) &= 0 < \tilde{\varphi}\,(0) = \tilde{\varphi}_\varepsilon\,(0) && \text{otherwise.} \end{aligned}$$

So, for all θ, we have $\tilde{\varphi}_\varepsilon(\theta) < \tilde{\varphi}_\varepsilon\,(0)$. By letting $\varepsilon \to 0$, the same conclusion can be drawn. ☐

Remark 1. In the proof of Proposition 2, $\varphi \in \Gamma$ is taken non constant since for φ constant everything is clear.

Proposition 3. *Let $X\,(t)$ be the fundamental solution associated with the initial condition X_0 of the retarded differential equation* (3)

$$X_0\,(\theta) = \begin{cases} 0 & si\ -\tau \leq \theta < 0, \\ 1 & si\ \theta = 0\,. \end{cases}$$

Then X_t satisfies

$$X_t \geq 0,\ \textit{and}\ X_t\,(\theta) \leq e^{\lambda_2\theta} X_t\,(0) \quad \textit{for} \quad t > 0. \tag{5}$$

Proof. Let $V\,(t) = e^{-\lambda_2 t} X\,(t)$. So, we have to prove that V_t solution of the differential equation

$$\begin{cases} \dot{V}\,(t) = -\lambda_2\,(V\,(t) - V\,(t - \tau))\,, \\ \quad V_0 = X_0, \end{cases} \tag{6}$$

satisfies

$$V_t \geq 0 \ \text{ and } \ V_t\,(\theta) \leq V_t\,(0) \quad \text{for} \quad t > 0.$$

Indeed, for $t \in [0, \tau[$ we have: $\dot{V}\,(t) = -\lambda_2 V\,(t)$. This implies $V\,(t) = e^{-\lambda_2 t} V\,(0) = e^{-\lambda_2 t}$. So, $V\,(t)$ is positive and increasing and finally, $V_t\,(\theta) \leq V_t\,(0)$ for all θ : $-\tau \leq \theta \leq 0$.

If $t = \tau$, then V_τ is in Γ_0 which concludes that V_t is positive increasing for all $t \geq \tau$. ☐

Proposition 4. *Under hypothesis (H), the solution $x(t)$ of the retarded differential equation*

$$\begin{cases} \dot{x}(t) = -a\,x(t-\tau) + f(t), \\ \quad x_0 = 0, \end{cases}$$

is positive, if f is locally integrable positive.

Proof. Because that X and f are positive and from the constant variation formula, we conclude that $x(t) = \int_0^t X(t-s)f(s)\,ds$ is positive. □

Remark 2. It is clear that if in the last proposition f is negative, then the solution $x(t)$ is also negative.

Proposition 5. *If $\varphi \in \Gamma$ and f is non negative, locally integrable, then the solution $x(t)$ of retarded differential equation*

$$\begin{cases} \dot{x}(t) = -a\,x(t-\tau) + f(t), \\ \quad x_0 = \varphi, \end{cases}$$

verifies $x_t \in \Gamma$ for all $t \geq 0$.

Proof. From [**6**], the solution $x(t)$ satisfies:

$$x_t = T(t)\varphi + \int_0^t X_{t-s} f(s)\,ds$$

where $T(t)$ is the semi-group associated to the linear retarded differential equation (4) and $X(\cdot)$ is the fundamental solution.

It is clear, from Proposition 2 that $T(t)\varphi \in \Gamma$. Moreover, since X_t verifies (5) and f is positive, then $\int_0^t X_{t-s} f(s)\,ds$ verifies (5). We finally, conclude that $x_t \in \Gamma$ for all $t \geq 0$. □

Having accomplished this preliminary step, we can now return to equation (2).

Proposition 6. *We assume that $\varphi \in \Gamma$. Let $x(t)$ be the solution of (2) with initial condition φ. If t_1 is the first zero of $x(t)$, then $x_t \in -\Gamma$ for all $t_1 + \tau \leq t \leq t_1 + \sigma$.*

Proof. We can easily see that $\dot{x}(t) \leq 0$ for all $t_1 \leq t \leq t_1 + \tau$ and deduce the proposition for $t = t_1 + \tau$.

To complete the proof, we let

$$y(t) = -x(t + t_1 + \tau) \quad \text{and} \quad f(t) = g(x(t_1 + \tau + t - \sigma)).$$

This yields

$$\begin{cases} \dot{y}(t) = -ay(t-\tau) + f(t), & t \geq 0, \\ y_0(\theta) = -x_{t_1+\tau}(\theta), & -\tau \leq \theta \leq 0, \end{cases}$$

It is clear that $f(t) \geq 0$ for all $0 \leq t \leq \sigma - \tau$, so we conclude from Proposition 5, that $y_t \in \Gamma$ for all $0 \leq t \leq \sigma - \tau$ which implies:

$$x_t \in -\Gamma \quad \text{for all} \quad t_1 + \tau \leq t \leq t_1 + \sigma. \qquad \square$$

In the sequel, we generalize this result to the retarded differential equation (1). We consider the cone

$$K = \left\{ \begin{array}{ll} \varphi \in C\left([-\sigma, 0], \mathbb{R}\right) & : \varphi \geq 0, \\ & \theta \longmapsto \varphi(\theta)\, e^{-\lambda_2 \theta} \text{ is nondecreasing on } [-\sigma, 0], \\ & \varphi(-\sigma) = 0 \end{array} \right\}$$

where $C([-\sigma, 0], \mathbb{R})$ is the set of continuous functions defined on $[-\sigma, 0]$.

We can now state the analogue of Proposition 6.

Proposition 7. *Let $x(t)$ be the solution of (1) with initial condition x_0 i.e. $x(t) = x_0(t)$ for $-\sigma \leq t \leq 0$. If $x_0 \in K$ and t_1 is the first value $t_1 > 0$ such that $x(t_1) = 0$, then x is decreasing on $[t_1, t_1 + \tau]$ and $x_t \in -K$ for all $t_1 + \tau \leq t \leq t_1 + \sigma$.*

Proof. We denote

$$a(t) \equiv \begin{cases} \dfrac{f(x(t-\tau))}{x(t-\tau)}, & \text{if } x(t-\tau) \neq 0, \\ f'(0), & \text{if } x(t-\tau) = 0\,. \end{cases}$$

Then, equation (1) reads

$$\dot{x}(t) = -a(t)\, x(t-\tau) - g(x(t-\sigma)).$$

Let $x_0 \in K$ and $t_1 > 0$ be such that $x(t_1) = 0$ and $x(t) > 0 \ \ \forall t : 0 < t < t_1$. For $t_1 \leq t \leq t_1 + \tau$, we have $x(t-\tau) \geq 0$ and $x(t-\sigma) \geq 0$, so, $f\left(x\left(t-\tau\right)\right) \geq 0$ and $g\left(x\left(t-\sigma\right)\right) \geq 0$. This implies

$$\begin{aligned} \dot{x}\left(t\right) &= -f\left(x\left(t-\tau\right)\right) - g\left(x\left(t-\sigma\right)\right), \\ &\leq 0\,. \end{aligned}$$

Thus, $x\left(t\right)$ is decreasing on $[t_1, t_1 + \tau]$.

For $t_1 + \tau \leq t \leq t_1 + \sigma$, let $y(t) = e^{-\lambda_2 t} x(t)$.

So

$$\dot{y}(t) = -\lambda_2 y(t) - e^{-\lambda_2 \tau} a(t) y(t-\tau) - e^{-\lambda_2 t} g\left(e^{\lambda_2 (t-\sigma)} y(t-\sigma)\right) \tag{7}$$

it is clear that $\dot{y}(t) < 0$ for all $t \in [t_1, t_1 + \tau]$. Let us prove that $\dot{y}(t) \leq 0$ for all $t \in [t_1 + \tau, t_1 + \sigma]$.

We have

$$\left.\begin{array}{l} 0 < a(t) \leq a \\ \lambda_2 = -ae^{-\lambda_2 \tau} \text{ and } y \leq 0 \end{array}\right\} \Rightarrow$$
$$a(t)y(t-\tau)e^{-\lambda_2 \tau} \geq ae^{-\lambda_2 \tau} y(t-\tau) = -\lambda_2 y(t-\tau),$$
$$t_1 + \tau \leq t \leq t_1 + 2\tau$$

so,

$$\begin{aligned} \dot{y}(t) &\leq -\lambda_2 \left(y\left(t\right) - y\left(t-\tau\right)\right) - e^{-\lambda_2 t} g\left(e^{\lambda_2 (t-\sigma)} y\left(t-\sigma\right)\right), \\ &\leq \left|\lambda_2\right| \left(y\left(t\right) - y\left(t-\tau\right)\right), \quad t_1 + \tau \leq t \leq t_1 + \min\left(\sigma, 2\tau\right). \end{aligned}$$

Let $z\left(t\right)$ be the solution of the retarded differential equation

$$\left\{\begin{array}{l} \dot{z}\left(t\right) = -\lambda_2 \left(z\left(t\right) - z\left(t-\tau\right)\right) \\ z_{t_1+\tau} = y_{t_1+\tau} . \end{array}\right.$$

From the comparison result in [**8**], we have

$$y\left(t\right) \leq z\left(t\right).$$

Since $z_{t_1+\tau} \in -\Gamma_0$, we deduce from Proposition 2 that $z\left(t\right)$ is negative, decreasing. On the other hand, because $\lambda_2 < 0$ and $t - \tau < t_1 + \tau$, we have

$$\begin{aligned} -\lambda_2 y\left(t\right) &\leq -\lambda_2 z\left(t\right) \\ \text{and} \ \ y\left(t-\tau\right) &= z\left(t-\tau\right). \end{aligned}$$

This implies

$$-\lambda_2 \left(y\left(t\right) - y\left(t-\tau\right)\right) \leq -\lambda_2 \left(z\left(t\right) - z\left(t-\tau\right)\right),$$

and finally

$$\dot{y}(t) \leq \dot{z}\left(t\right) \leq 0\,.$$

Thus $y\left(t\right)$ is also negative, decreasing and finally $y_t \backslash_{[-\tau,0]} \in -\Gamma_0$ for all $t \in [t_1 + \tau, t_1 + \min\left(\sigma, 2\tau\right)[$ which concludes $x_t \left|_{[-\tau,0]}\right. \in -\Gamma$.

We can repeat this derivation for $t_1 + 2\tau \leq t \leq t_1 + 3\tau, \ldots$, as long as $k\tau < \sigma$, and deduce that $y\left(t\right)$ is negative decreasing for all $t \in [t_1 + \tau, t_1 + \sigma]$. In conclusion, we proved that: x is decreasing on $[t_1, t_1 + \tau]$ while the function $e^{-\lambda_2 t} x\left(t\right)$ is decreasing on $[t_1 + \tau, t_1 + \sigma]$, that is: $x_t \in -K$, for all $t_1 + \tau \leq t \leq t_1 + \sigma$. Which is the desired conclusion. □

Let us mention the following important consequence of this proposition.

Theorem 8. *Let $x_0 \in K$ and let x be the solution of (1) associated with x_0. Then, we have either:*

1. *$x(t)$ has a finite number of zeros $t_1 < t_2 < \cdots < t_N$ such that: $t_{j+1} - t_j \geq \sigma$; $x_{t_j+\sigma} \in \pm K$ for j even, j odd respectively and moreover: $x(t) \longrightarrow 0$ when $t \longmapsto \infty$. Or,*
2. *$x(t)$ has infinitely many zeros $t_1 < t_2 < \cdots < t_j < \cdots$. Any two consecutive zeros t_j, t_{j+1} verify:*

$$t_{j+1} - t_j \geq \sigma.$$

Finally $x_{t_j+\sigma} \in \pm K$ for j even, j odd respectively.

3. Example

We have encountered in [**4**], in a second attempt to understand the behavior of subjects trying to perform a "simple" motor control task, the non linear differential equation

$$\dot{x}(t) = -A_1 \tanh(x(t-\tau)) - A_2 \tanh(x(t-\sigma)) \tag{8}$$

where A_1, A_2 are positive constants. The linearized equation of (8) is

$$\dot{x}(t) = -A_1 x(t-\tau) - A_2 x(t-\sigma),$$

which can be rewriten

$$\dot{x}(t) = -x(t-\tilde{\tau}) - \alpha x(t-\tilde{\sigma}) \tag{9}$$

where $\tilde{\tau} = \tau A_1$, $\tilde{\sigma} = \sigma A_1$ and $\alpha = \dfrac{A_2}{A_1}$.

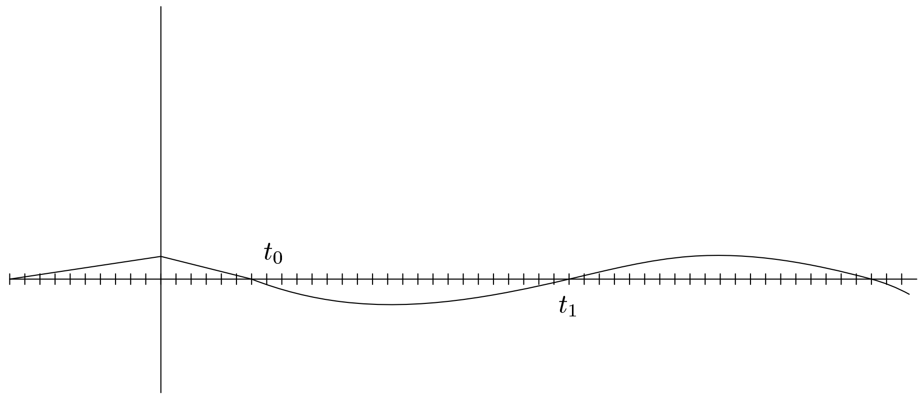

Figure 1. Solution $x(t)$ of equation (9) with initial condition $x(t) = t + \frac{4}{15}$ for $-\frac{4}{15} \leq t \leq 0$.

We have chosen values $\tilde{\tau} = \dfrac{1}{4} < \dfrac{1}{e}$, $\sigma = \dfrac{4}{15}$ and $\alpha = 5.59$. We compute, by Euler's method, the solution of the linear equation (9) with initial condition $x(t) = t - \sigma$ for $-\sigma \le t \le 0$. Figure 1 shows that the solution is slowly oscillating.

References

1. Arino O. and Györi I., *Necessary and sufficient condition for oscillation of a neutral differential system with delays*, J. Diff. Eq. **81** (1989), 98–105.
2. Arino O., Györi I. and Jawhari A., *Oscillation criteria in delay equation*, J. Diff. Eq. **53** (1984), 115–123.
3. Arino O. and Niri K., *Oscillations in vector: A comparison result for monotone delay differential system*, J. Math. Anal. Appl. **160** (1991), 267–283.
4. Bélair J. and Campbell S. A., *Stability and bifurcations of equilibria in a multiple-delayed differential equation*, SIAM J. Appl. Math. **54** (1994), 1402–1424.
5. Györi I., *Invariant cones of positive initial functions for delay differential equations*, Appl. Anal. **35** (1990), 21–42.
6. Hale J., *Functional differential equations*, Springer, Berlin, 1977.
7. Nussbaum R., *Differential delay equations with two time lags*, Mem. Amer. Math. Soc. **16** (1978).
8. Smith H., *Monotone semiflows generated by functional differential equations*, J. Diff. Eq. **66** (1987), 420–442.
9. Stroinski U., *Oscillatory semigroups*, Dissertation der Mathematischen Fakultät der Eberhard-Karls-Univesität zu Tübingen,.

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