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CONGRUENCE CLASSES IN REGULAR VARIETIES

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Abstract. A characterization of congruence classes of algebras of regular varieties is presented. The problem of deciding whether a given subset of an algebra of regular variety is a congruence class is shown to be solvable in polynomial time.

It has been proved by A. I. Mal'cev [**6**] that a nonempty subset $C \subseteq A$ of the support of an algebra $\mathcal{A} = (A, F)$ is a class of some congruence relation on $\mathcal{A}$ if and only if

$$\text{either} \quad \tau(C) \cap C = \emptyset \quad \text{or} \quad \tau(C) \subseteq C$$

for any unary polynomial τ of $\mathcal{A}$. This characterization, whatever useful, is not much efficient. In [**1**], the authors found a simple characterization of congruence classes of algebras from varieties which are both regular and permutable. They also showed that the decision problem of being a congruence class for algebras from a given regular and permutable variety is solvable in polynomial time. In this paper we give a characterization of congruence classes of algebras from regular varieties.

Recall that an algebra $\mathcal{A} = (A, F)$ is **regular** if $\theta = \Phi$ for $\theta, \Phi \in \operatorname{Con} \mathcal{A}$ whenever they have a congruence class in common. $\mathcal{A}$ is n-permutable if $\theta \circ \phi \circ \theta \circ \cdots = \phi \circ \theta \circ \phi \circ \cdots$ (n factors in both relational products) for every $\theta, \phi \in \operatorname{Con} \mathcal{A}$. A variety $\mathcal{V}$ is regular or n-permutable if each $\mathcal{A} \in \mathcal{V}$ has this property.

Regular varieties have been characterized independently by B. Csákány, G. Grätzer and R. Wille in 1970s. For our purposes we present a Mal'cev condition which is rather similar to that one of R. Wille (cf. Theorem 6.11 in [**8**]).

Theorem 1. *A variety $\mathcal{V}$ is regular if and only if there exist a positive integer n, ternary terms $t_1, \dots, t_n$, and 5-ary terms $p_1, \dots, p_n$ such that*

$$\begin{aligned} t_i(x, x, z) &= z \quad \textit{for } i = 1, \dots, n \\ x &= p_1(t_1(x, y, z), z, x, y, z) \\ p_i(z, t_i(x, y, z), x, y, z) &= p_{i+1}(t_{i+1}(x, y, z), z, x, y, z), \quad i = 1, \dots, n-1 \\ y &= p_n(t_n(x, y, z), z, x, y, z). \end{aligned}$$

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Proof. Let $\mathcal{V}$ be a regular variety, $F_{\mathcal{V}}(x,y,z) \in \mathcal{V}$ be a free algebra generated by x, y and z, let further $\theta = \theta(x,y)$, $C = [z]_\theta$. For $\theta(x,y)$ and $\theta(C \times \{z\})$ have the class C in common, it follows from regularity that $\theta(x,y) = \theta(C \times \{z\})$. We have therefore $\langle x,y\rangle \in \theta(C \times \{z\})$. The compactness of congruence lattice implies that there is a finite subset $\{d_1,\dots,d_k\} \subseteq C$ such that $\langle x,y\rangle \in \theta(\{d_1,\dots,d_k\}\times\{z\})$. By Mal'cev lemma, there are $e_1,\dots,e_m \in F_{\mathcal{V}}(x,y,z)$ and $(2+m)$-ary terms $q_1,\dots,q_n$ such that $x = q_1(d_{j_1}, z, \vec{e})$, $q_i(z, d_{j_i}, \vec{e}) = q_{i+1}(d_{j_{i+1}}, z, \vec{e})$ for $i = 1,\dots,n-1$, and $y = q_n(d_{j_n}, z, \vec{e})$ where $j_i \in \{1,\dots,k\}$. Clearly, $q_i(u,v,\vec{e}) = p_i(u,v,x,y,z)$ and $d_{j_i} = t_i(x,y,z)$, $i = 1,\dots,n$, which are the required terms.

Conversely, let $\mathcal{V}$ satisfy the listed identities, let $\mathcal{A} \in \mathcal{V}$. To prove regularity of $\mathcal{A}$ it is enough to prove that each $\theta \in \operatorname{Con}\mathcal{A}$ with some singleton class $\{c\}$ is the identity relation ω. Let then $\theta \in \operatorname{Con}\mathcal{A}$, $\{c\}$ be a class of θ, $\langle a,b\rangle \in \theta$. Thus $\langle t_i(a,b,c), c\rangle = \langle t_i(a,b,c), t_i(a,a,c)\rangle \in \theta$, i.e. $t_i(a,b,c) \in \{c\}$, i.e. $t_i(a,b,c) = c$. We conclude

$$\begin{aligned} a &= p_1(t_1(a,b,c),c,a,b,c) = p_1(c,c,a,b,c) = \dots = p_n(c,c,a,b,c) \\ &= p_n(c, t_n(a,b,c), a,b,c) = b, \end{aligned}$$

hence $\theta = \omega$. □

Theorem 2. *Let the variety $\mathcal{V}$ be regular and $p_1,\dots,p_n$ be terms of Theorem 1. Then $\mathcal{V}$ is $(n+1)$-permutable.*

Proof. Put $q_i(x,y,z) = p_i(t_i(x,y,z), t_i(y,z,z), x, z, z)$. The identities

$$\begin{aligned} x &= q_1(x,y,y) \\ q_i(x,x,y) &= q_{i+1}(x,y,y), \quad i = 1,\dots,n-1 \\ y &= q_n(x,x,y) \end{aligned}$$

are easy to verify. Hence, by [**5**], $\mathcal{V}$ is $(n+1)$-permutable. □

Theorem 3. *Let $\mathcal{V}$ be a regular variety, and $t_1,\dots,t_n$ be the terms of Theorem 1. Let $\mathcal{A} = (A,F) \in \mathcal{V}$ and $\emptyset \neq C \subseteq A$. The following conditions are equivalent:*

(1) *C is a class of some $\theta \in \operatorname{Con}\mathcal{A}$.*

(2) *(i) for each m-ary $f \in F$, $a_j, b_j \in A$, $j = 1,\dots,m$, $c \in C$, it holds*

$$\&_{i=1}^n\, t_i(a_j,b_j,c) \in C \;\Rightarrow\; \&_{i=1}^n\, t_i(f(a_1,\dots,a_m), f(b_1,\dots,b_m), c) \in C;$$

(ii) if $a,b,d \in A$ then

$$\&_{i=1}^n \Big(t_i(a,b,c) \in C \,\&\, t_i(b,d,c) \in C\Big) \;\Rightarrow\; \&_{i=1}^n\, t_i(a,d,c) \in C;$$

(iii) if $a \in A$, $c,d \in C$, then $t_i(d,c,c) \in C$ for $i = 1,\dots,n$, and

$$\&_{i=1}^n\, t_i(a,c,c) \in C \;\Rightarrow\; a \in C.$$

Proof. Let $\mathcal{A} \in \mathcal{V}$, $\emptyset \neq C \subseteq A$, $c \in C$ and let (i), (ii) and (iii) hold. Let θ_C be a binary relation on A defined by

$$(*) \qquad \langle x, y\rangle \in \theta_C \quad \text{iff} \quad t_1(x,y,c) \in C, \ldots, t_n(x,y,c) \in C.$$

Since $t_i(x,x,c) = c \in C$, the relation θ_C is reflexive. Compatibility and transitivity of θ_C follow from the conditions (i) and (ii), respectively. Applying Theorem 2 we conclude that $\mathcal{V}$ is $(n+1)$-permutable. By [**3**], each reflexive, transitive and compatible relation in a $(n+1)$-permutable variety is a congruence relation, hence $\theta_C \in \operatorname{Con} \mathcal{A}$.

Let $x \in [c]_{\theta_C}$. Then $\langle x, c\rangle \in \theta_C$ and, by $(*)$, $t_i(x,c,c) \in C$ for $i = 1, \ldots, k$. From (iii) it follows $x \in C$. Conversely, let $x \in C$. Then by (iii) we get $t_i(x,c,c) \in C$, $i = 1, \ldots, k$. By $(*)$ this implies $\langle x, c\rangle \in \theta_C$, i.e. $x \in [c]_\theta$. Hence, $C = [c]_\theta$.

Conversely, let $C \subseteq A$ be a class of some $\theta \in \operatorname{Con} \mathcal{A}$ and $c \in C$. If $a_j, b_j \in A$ and $t_i(a_j, b_j, c) \in C$ ($j = 1, \ldots, m$, $i = 1, \ldots, n$) and if $f \in F$ is m-ary then then $\langle t_i(a_j, b_j, c), c\rangle \in \theta$ and, by Theorem 1, we have

$$\begin{aligned} a_j &= p_1(t_1(a_j,b_j,c), c, a_j, b_j, c)\,\theta\, p_1(c, t_1(a_j,b_j,c), a_j, b_j, c) \\ &= p_2(t_2(a_j,b_j,c), c, a_j, b_j, c)\,\theta\, p_2(c, t_2(a_j,b_j,c), a_j, b_j, c) \\ &\ \ \vdots \\ &= p_n(t_n(a_j,b_j,c), c, a_j, b_j, c)\,\theta\, p_n(c, t_n(a_j,b_j,c), a_j, b_j, c) = b_j, \end{aligned}$$

hence $\langle a_j, b_j\rangle \in \theta$. From compatibility of θ it follows

$$\begin{aligned} &\langle t_i(f(a_1,\ldots,a_m), f(b_1,\ldots,b_m), c), c\rangle \\ &\quad = \langle t_i(f(a_1,\ldots,a_m), f(b_1,\ldots,b_m), c), t_i(f(b_1,\ldots,b_m), f(b_1,\ldots,b_m), c), c\rangle \in \theta, \end{aligned}$$

i.e. $t_i(f(a_1,\ldots,a_m), f(b_1,\ldots,b_m), c) \in [c]_\theta = C$. Hence, (i) holds.

If $t_i(x,y,c) \in C$, $t_i(y,z,c) \in C$ ($i = 1, \ldots, n$), then as in the previous case, $\langle x,y\rangle \in \theta$, $\langle y,z\rangle \in \theta$, hence, $\langle x,z\rangle \in \theta$. Therefore, $\langle t_i(x,z,c), c\rangle = \langle t_i(x,z,c), t_i(z,z,c)\rangle \in \theta$, i.e. $t_i(x,z,c) \in [c]_\theta = C$, proving (ii).

If $t_i(a,c,c) \in C$ ($i = 1, \ldots, n$), then again $\langle a, c\rangle \in \theta$, i.e. $a \in C$. If $c, d \in C$ then $\langle c, d\rangle \in \theta$, and thus $\langle t_i(d,c,c), c\rangle = \langle t_i(d,c,c), t_i(d,d,c)\rangle \in \theta$, i.e. $t_i(d,c,c) \in C$. We have proved (iii). $\square$

Let us turn to computational aspects of our problem. Computational properties of universal algebra are of recent interest, see e.g. [**2**]. Recall first that by a time complexity of an algorithm it is meant a function $f\colon N \mapsto N$ such that every problem of size n will be solved after at most $f(n)$ number of (computational) steps (see [**4**]). The class of all problems for which there is an deterministic algorithm (nondeterministic algorithm) of polynomial time complexity ($f(n)$ is a polynomial) is denoted by P (NP). Algorithms of polynomial time complexities are considered as practically usable, algorithms of greater complexities (e.g. exponential) are considered as unusable. The class P is therefore the class of tractable problems. In our case, we are given a class $\mathcal{K}$ of algebras. We face the following decision problem: For an algebra $\mathcal{A} \in \mathcal{K}$ and a subset $C \subseteq A$, decide whether C is a congruence class. Suppose that one evaluation step consists in the evaluation of one term. Denote the problem $p_{\mathcal{K}}$. It has been shown in [**1**] that in general $p_{\mathcal{K}} \in NP$ but for a regular and permutable variety $\mathcal{V}$ of a finite type, $p_{\mathcal{V}} \in P$. The following theorem shows that being a regular variety for $\mathcal{K}$ is sufficient for the problem to belong to P.

Theorem 4. *Let $\mathcal{V}$ be a regular variety of a finite type, for which the terms $t_1, \dots, t_n$ of Theorem 1 are known. Then $p_{\mathcal{V}} \in P$.*

Proof. Let $\emptyset \neq C \subseteq A$, $\mathcal{A} = \langle A, F\rangle \in \mathcal{V}$. Denote further $F = \{f_1, \dots, f_k\}$, $l = \text{card } C$, $m = \text{card } A$, and let $\sigma(f)$ denote the arity of $f \in F$. To check whether C is a class of some congruence relation on $\mathcal{A}$ we can use Theorem 3, i.e. we have to test the conditions (i), (ii) and (iii) of (2). Consider first condition (i). We choose $f \in F$ (k choices) and $a_j, b_j \in A$ (m^2 choices). For this choice we have to test the implication. The test of the antecedent consists of n steps. The test of the consequent part consists of $n\, m^{2\sigma(f)}$ steps (there are $m^{2\sigma(f)}$ possible substitutions for the arguments of f). Since the choices are independent we have $\sum_{i=1}^{k} m^2\,(n + n\, m^{2(\sigma(f_i)-1)})$ computational steps altogether. Similarly, to test the conditions (ii) and (iii) we have to perform $m^2\, l\,(2n+n)$ and $m\, l^2\,(n+(n+1))$ steps, respectively. For a given variety, the derived expressions are polynomials. Since the overall number of steps is given by the sum of the expressions the assertion is proved. □

Remark. The proof of the foregoing theorem gives a polynomial algorithm solving our problem for $\mathcal{K}$ being a regular variety. The time complexity of the algorithm is

$$n \sum_{i=1}^{k} m^{2\sigma(f_i)} + m^2\, n(3l + k) + ml^2(2n + 1).$$

Note that this algorithm is of the same asymptotic complexity as that one for regular and permutable varieties based on Theorem 1 of [**1**].

Recall the following concept, see e.g. [**1**]. If $p(x_1, \dots, x_n, y_1, \dots, y_m)$ is an $(m + n)$-ary term of an algebra $\mathcal{A} = (A, F)$ and $C \subseteq A$ we say that C is $y-$**closed under** p if $p(a_1, \dots, a_n, c_1, \dots, c_m) \in C$ for every $a_1, \dots, a_n \in A$ and $c_1, \dots, c_m \in C$. The following theorem presents a characterization of special congruence classes of regular varieties by means of y-closeness.

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