Acta Math. Univ. Comenianae
Vol. LXVIII, 1(1999), pp. 1–15

THE VECTOR INDIVIDUAL WEIGHTED ERGODIC THEOREM FOR BOUNDED BESICOVICH SEQUENCES

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Abstract. In this paper we prove maximal ergodic theorem and a pointwise convergence theorem. Our result is to prove the convergence of

$$B_n(T,\alpha,f) = \frac{1}{n}\sum_{j=0}^{n-1}\alpha_j T^j f$$

for all $f \in L^1(\Omega,X) = L^1(X)$, where n tends to infinity, Ω is a σ-finite measure space, X is a reflexive Banach space, α_j is a bounded Besicovich sequence and T is a linear operator on $L^1(X)$ which is contracting in both $L^1(X)$ and $L^\infty(X)$.

Our result has the additional advantage as it is sufficiently general in order to extend the Beck and Schwartz random theorem.

We can also generalize this result to a multidimensional case.

Notations and Definitions

Denote by X a Banach space, (Ω,β,μ) a σ-finite measure space, $\|x\|_X$ the norm of a vector x in X.

- $L^1(\Omega,X) = L^1(X) = \Big\{f\colon \Omega \to X$, measurable and $\int_\Omega \|f(\omega)\|_X \, d\mu(\omega) < \infty\Big\}$ the space of integrable functions in the sense of Bochner which take values in X, and $L^\infty(\Omega,X) = L^\infty(X) = \Big\{f\colon \Omega \to X$, measurable and bounded a.e. (i.e $\sup_{\omega\in\Omega}\|f(\omega)\|_X < \infty)\Big\}$.
- $L^1 = L^1(\Omega,R)$, $L^\infty = L^\infty(\Omega,R)$.
- For all $f \in L^1(X)$, $\|f\|_1 = \int_\Omega \|f(\omega)\|_X \, d\mu(\omega)$ and $\|f\|_\infty = \sup_{\omega\in\Omega}\|f(\omega)\|_X$.
- For an operator T of $L^1(X)$ into itself: T is contracting in $L^1(X)$ iff $\|Tf\|_1 \le \|f\|_1$ for all $f \in L^1(X)$, similarly, T is contracting in $L^\infty(X)$ iff for all $f \in L^\infty(X)$, $\|Tf\|_\infty \le \|f\|_\infty$.
- For $a > 0$ and $f \in L^1(X)$, $f^{a-} = \frac{f}{\|f\|}\min\{\|f\|, a\}$, $f^{a+} = f - f^{a-}$, $f^* = \sup_n \|B_n(\alpha,T,f)\|$, $e^*(a,\alpha) = \{f^* > \alpha a\}$, $e(a) = \{\|f\| > a\}$ and for $A \subset \Omega$ we denote φ_A the indicator function of A.

Received July 7, 1995; revised March 5, 1998.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 27A35; Secondary 28A65.

We first define the term "Bounded Besicovich sequence". Let α_j be a sequence of complex numbers. We say that α_j is a bounded Besicovich sequence if

(i) there exists a positive real α such that $|\alpha_j| < \alpha$ for every $j \in N$,

(ii) for $\varepsilon > 0$ there exists a trigonometric polynomial φ_ε such that

$$\lim_n \frac{1}{n+1} \sum_{j=0}^{n} |\alpha_j - \varphi_\varepsilon(j)| < \varepsilon \,.$$

Introduction

In [**5**] J. Olsen proved an individual weighted ergodic theorem for bounded Besicovich sequences. He proved the a.e. convergence of

$$B_n(T,\alpha,f) = \frac{1}{n+1} \sum_{j=0}^{n} \alpha_j T^j f$$

where T is linear operator on $L^1 = L^1(\Omega, R)$ which is contracting in $L^1 = L^1(\Omega, R)$ and in $L^\infty = L^\infty(\Omega, R)$, and α_j is a bounded Besicovich sequence.

In [**2**] R. V. Chacon proved a maximal ergodic lemma for operators which act in the space of functions taking their values in a Banach space, and he used this result to obtain a vector valued ergodic theorem as a generalization of the Dunford and Shwartz theorem.

In this paper we intend to generalize the individual ergodic theorem of Olsen to operators acting in $L^1(X)$.

In his proof Olsen used the dominated operator of T-linear modulus (see [**5**, Lemma 2.1]) but in our case we have shown in [**4**] that there exists a vector operator without linear modulus contracting in L^1.

We prove that the method of Chacon can be adapted to this situation, then our result is the convergence μ-a.e. of $B_n(T,\alpha,f)$ where T is a linear operator on $L^1(X)$ which is contracting in both $L^1(X)$ and $L^\infty(X)$.

In the second part, we generalize these results to a multidimensional case: If α_j is a bounded Besicovich sequence and $\lambda_d = i_1 s_1 + \cdots + i_d s_d$ where $s_j \in N$ for $j = 1, \ldots, d$ then the limit of

$$B_n(T,d,\alpha,f) = \frac{1}{(n+1)^d} \sum_{i_1=0}^{n} \cdots \sum_{i_d}^{n} \alpha_{\lambda_d} T^{\lambda_d} f$$

exists a.e. for all $f \in L^1(X)$.

I. One Dimensional Case

We now state Chacon's result [**2**]:

Theorem 1.1 (Chacon)**.** *Let T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$.*

(i) If $a > 0$, $e^*(a) = \left\{\omega \in \Omega;\ \sup_n \left\| \frac{1}{n+1} \sum_{j=0}^{n} T^j f(\omega) \right\|_X > a \right\}$

$$\int_{e^*(a)} \left(a - \left\|f^{a-}(\omega)\right\|_X\right) d\mu(\omega) \leq \int_\Omega \left\|f^{a+}(\omega)\right\|_X d\mu(\omega).$$

(ii) For $f \in L^1(X)$, the limit of $A_n(T,f)(\omega) = \frac{1}{n}\sum_{j=0}^{n-1} T^j f(\omega)$ exists strongly for every $\omega \in \Omega$ as n tends to infinity.

(iii) If $1 < p < \infty$, then there exists a function $f^{**} \in L^p(X)$ such that

$$\left\|A_n(T)f\right\|_X \leq \left\|f^{**}\right\|_X \quad (a.e.\ n \geq 0).$$

Main Result

We now state and prove our main result.

Theorem 1.2. *Let X be a reflexive Banach space, T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$, α_j be a bounded Besicovich sequence then:*

(i) For $f \in L^1(X)$, the limit of $B_n(T,\alpha,f)(\omega) = \frac{1}{n}\sum_{j=0}^{n-1} \alpha_j T^j f(\omega)$ exists strongly for every $\omega \in \Omega$ as n tends to infinity.

(ii) If $1 < p < \infty$, $f \in L^p(X)$ the average $B_n(T,\alpha,f)$ converges a.e. and

$$\left\| \sup_n \left\|B_n(T,\alpha,f)\right\|_X \right\|_p \leq \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p\,.$$

Before proving Theorem 1.2, let us remark that if $\alpha_j = 1$ for every j, we have

$$e^*(a,1) = e^*(a) = \left\{ \sup_n \left\| \frac{1}{n+1} \sum_{k=0}^{n} T^k f \right\| > a \right\}.$$

We will have to prove that Chacon's lemma is valid also for the averages $B_n(T,\alpha,f)$.

Lemma 1.3. *If $f \in L^1(X)$ and $a > 0$, we have then*

$$\int_{e^*(a,\alpha)} \left(a - \left\|f^{a-}(\omega)\right\|_X\right) d\mu(\omega) \leq \int_\Omega \left\|f^{a+}(\omega)\right\|_X d\mu(\omega)\,.$$

Proof of Lemma 1.3. We can suppose that $\alpha_j > 0$ for all j in N. As in [**2**] we define:

$$\begin{aligned} f_0 &= f^{a+}\,,\\ f_{i+1} &= Tf_i - \frac{Tf_i}{\left\|Tf_i\right\|_X} \min\left\{\left\|Tf_i\right\|_X, a - \left\|f^{a-}\right\|_X - \sum_{k=0}^{i}\left\|Tf_k - f_{k+1}\right\|_X\right\} \end{aligned}$$

and

$$\begin{aligned} d_0 &= 0\,,\\ d_{i+1} &= \frac{Tf_i}{\left\|Tf_i\right\|_X} \min\left\{\left\|Tf_i\right\|_X, a - \left\|f^{a-}\right\|_X - \sum_{k=0}^{i}\left\|Tf_k - f_{k+1}\right\|_X\right\}\,. \end{aligned}$$

By [**2**] these sequences satisfy the following relations:

(1) $\left\|f^{a-}(\omega)\right\|_X + \sum_{k=0}^{i}\left\|d_k(\omega)\right\|_X \leq a$ for every i in N and for every ω in Ω,

(2) $\left\|Tf_i(\omega)\right\|_X = \left\|f_{i+1}(\omega)\right\|_X + \left\|d_{i+1}(\omega)\right\|_X$ for all ω in Ω,

(3) $T^i f = T^i f^{a-} + f_i + \sum_{k=0}^{i} T^{i-k} d_k$ for every i in N,

(4) $\sum_{i=0}^{n}\left[T^i f^{a-} + \sum_{k=0}^{i} T^{i-k} d_k\right] = \sum_{i=0}^{n} T^i\left(f^{a-} + \sum_{k=0}^{n-i} d_k\right)$ for every n in N,

(5) if for $\omega \in \Omega$ $f_{i+1}(\omega) \neq 0$ then $a = \left\|f^{a-}(\omega)\right\|_X + \sum_{k=0}^{i+1}\left\|d_k(\omega)\right\|_X$.

Multiply equality (3) by α_j to obtain

$$\alpha_j T^i f = \alpha_i T^i f^{a-} + \alpha_i f_i + \sum_{k=0}^{i} \alpha_i T^{i-k} d_k\,.$$

To prove the following equality

$$(*) \qquad \sum_{i=0}^{n}\left[\alpha_i T^i f^{a-} + \sum_{k=0}^{i} \alpha_i T^{i-k} d_k\right] = \sum_{i=0}^{n} T^i\left(\alpha_i f^{a-} + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\right)$$

we shall argue by induction on n. It is true for $n = 0$. Suppose now that it is valid up to n and let us prove it for $n+1$:

$$\begin{aligned}
&\sum_{i=0}^{n+1}\Big[\alpha_i T^i f^{a-} + \sum_{k=0}^{i} \alpha_i T^{i-k} d_k\Big] \\
&\quad = \sum_{i=0}^{n} \alpha_i \Big(T^i f^{a-} + \sum_{k=0}^{i} T^{i-k} d_k\Big) + \alpha_{n+1}\Big(T^{n+1} f^{a-} + \sum_{k=0}^{n+1} T^{n+1-k} d_k\Big) \\
&\quad = \underbrace{\sum_{i=0}^{n} T^i \Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\Big)}_{\mu_n} + \underbrace{\alpha_{n+1}\Big(T^{n+1} f^{a-} + \sum_{k=0}^{n+1} T^{n+1-k} d_k\Big)}_{\lambda_n}.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&\sum_{i=0}^{n+1} T^i \Big(\alpha_i f^{a-} + \sum_{k=0}^{n+1-i} \alpha_i d_k\Big) = \sum_{i=0}^{n} T^i \Big(\alpha_i f + \sum_{k=0}^{n-i} \alpha_{i+k} d_k + \alpha_{n+1} d_{n+1-i}\Big) \\
&\quad = \sum_{i=0}^{n} T^i \Big(\alpha_i f + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\Big) + T^{n+1} f^{a-} + \alpha_{n+1} \sum_{i=0}^{n+1} T^i d_{n+1-i} \\
&\quad = \underbrace{\sum_{i=0}^{n} T^i \left(\alpha_i f^{a-} + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\right)}_{\mu_n} + \underbrace{\alpha_{n+1}\left(T^{n+1} f^{a-} + \sum_{k=0}^{n+1} T^{n+1-k} d_k\right)}_{\lambda_n}.
\end{aligned}$$

It follows that $(*)$ holds for every n in N. Thus

$$\begin{aligned}
\sum_{i=0}^{n+1} \alpha_i T^i f &= \sum_{i=0}^{n} T^i \Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\Big) + \sum_{i=0}^{n+1} \alpha_i f_i \\
&= \Big(\alpha_0 f^{a-} + \sum_{k=0}^{n} \alpha_k d_k\Big) + \sum_{i=1}^{n} T^i \Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\Big) + \sum_{i=0}^{n+1} \alpha_i f_i
\end{aligned}$$

Now, we shall show that for all $\omega \in e^*(\alpha, a) = \Big\{\omega \in \Omega;\ \sup_n \Big\| \frac{1}{n} \sum_{j=0}^{n} \alpha_j T^j f(\omega)\Big\|_X > a\Big\}$ we have

$$a = \Big\|f^{a-}(\omega)\Big\|_X + \sum_{k=0}^{\infty} \Big\|d_k(\omega)\Big\|_X.$$

Let $\omega \in e^*(\alpha, a)$ then there exists $n = n(\omega)$ such that:

$$\begin{aligned}
\alpha a n + \alpha a \le \Big\|\sum_{i=1}^{n-1} \alpha_i T^i f(\omega)\Big\| &\le \Big(\alpha_0 \big\|f^{a-}(\omega)\big\|_X + \sum_{k=0}^{n} \alpha_k \big\|d_k(\omega)\big\|_X\Big) \\
&\quad + \sum_{i=1}^{n} \Big\|T^i \Big(\alpha_i f^{a-} + \sum_{k=0}^{n-i} \alpha_{i+k} d_k\Big)(\omega)\Big\|_X + \sum_{i=0}^{n} \alpha_i \big\|f_i(\omega)\big\|_X.
\end{aligned}$$

By (1) we have

$$\begin{aligned}\alpha_i\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\alpha_{i+k}\big\|d_k(\omega)\big\|_X &\le \alpha_i\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\alpha_i\big\|d_k(\omega)\big\|_X\\ &=\alpha_i\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]\end{aligned}$$

and, as the operator T is contracting in $L^\infty(X)$, then

$$\begin{aligned}\sum_{i=1}^{n}\Big\|T^i(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k(\omega)\Big\|_X &\le \sum_{i=1}^{n}\Big\|T^i(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k)\Big\|_\infty\\ &\le \sum_{i=1}^{n}\Big\|\Big(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k\Big)\Big\|_\infty=\sum_{i=1}^{n}\sup_{\omega\in\Omega}\Big\|\Big(\alpha_i f^{a-}+\sum_{k=0}^{n-i}\alpha_{i+k}d_k\Big)(\omega)\Big\|_X\\ &\le \sum_{i=1}^{n}\sup_{\omega\in\Omega}\Big[\alpha_i\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n-i}\alpha_{i+k}\big\|d_k(\omega)\big\|_X\\ &\le \alpha a\Big]\end{aligned}$$

whence

$$\begin{aligned}(n+a)\alpha a &\le \alpha\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]+na\alpha+\sum_{i=1}^{n-1}\alpha_i\big\|f_i(\omega)\big\|_X\\ &\le \alpha\Big\{\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]+na+\sum_{i=1}^{n-1}\alpha_i\big\|f_i(\omega)\big\|_X\Big\}\end{aligned}$$

and so

$$a\le\Big[\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{n}\big\|d_k(\omega)\big\|_X\Big]+\sum_{i=1}^{n-1}\big\|f_i(\omega)\big\|_X$$

by relation (5) we have

$$a=\big\|f^{a-}(\omega)\big\|_X+\sum_{k=0}^{\infty}\big\|d_k(\omega)\big\|_X$$

for all $\omega\in e^*(\alpha,a)$.

The rest of the proof can be obtained in the same way as in Chacon's method, in fact, using (6) and knowing that $\|T\|_1\le 1$ we deduce

$$\begin{aligned}\int_{e^*(a,\alpha)}\big(a-\big\|f^{a-}(\omega)\big\|_X\big)\,d\mu &\le \sum_{k=0}^{\infty}\big(\big\|Tf_k\big\|_1-\big\|f_{k+1}\big\|_1\big)\le\sum_{k=0}^{\infty}\big(\big\|f_k\big\|_1-\big\|f_{k+1}\big\|_1\big)\\ &\le \big\|f_0\big\|_1=\int_\Omega\big\|f_0(\omega)\big\|\,d\mu(\omega)=\int_\Omega\big\|f^{a+}(\omega)\big\|\,d\mu(\omega).\end{aligned}$$

Proof of Theorem 1.2. Since the Lemma 1.3 gives us maximal weak inequality for averages $B_n(T,\alpha,f)$ it suffices to prove the convergence for f belonging to a set which is dense every where in $L^1(X)$. We know that $L^\infty(X)$ is such a set, so for $f \in L^\infty(X)$ we have:

$$\frac{1}{n+1}\sum_{i=0}^{n}\alpha_i T^i f = \frac{1}{n+1}\sum_{i=0}^{n}\varphi_\varepsilon(i)T^i f + \frac{1}{n+1}\sum_{i=0}^{n}[\alpha_i - \varphi_\varepsilon]T^i f.$$

Let θ be a complex number. The operator $Uf = e^{i\theta}Tf$ is contracting in both $L^1(X)$ and $L^\infty(X)$ and the theorem follows in the case $\alpha_n = e^{in\theta}$ from Chacon's theorem. The linearity of convergence gives that

$$\lim_n \frac{1}{n+1}\sum_{i=0}^{n}\varphi_\varepsilon(i)T^i f$$

exists and is finite a.e. for any trigonometric polynomial φ_ε, and $f \in L^\infty(X)$.

In fact we have for this operators a strong inequality in $L^\infty(X)$:

$$\left\|\sup_n \left\|\frac{1}{n+1}\sum_{i=0}^{n}\varphi_\varepsilon(i)T^i f\right\|_X\right\|_\infty \le k_\varepsilon \|f\|_\infty$$

and by the definition b) we also have

$$\limsup_n \frac{1}{n+1}\sum_{i=0}^{n}|\alpha_i - \varphi_\varepsilon(i)|\left\|T^i f\right\|_X \le \varepsilon\|f\|_\infty.$$

By Lemma 1.3 we have

$$a\mu(e^*(a,\alpha)) \le \int_{e(a)} \|f(\omega)\|_X \, d\mu$$

and so, using the rearrangement formula we get:

$$\begin{aligned}
\left\|f^*\right\|_p^P &= \int_\Omega [f^*]^p \, d\mu = p\alpha^p \int_\Omega \int_0^{f^*/\alpha} \lambda 1_{e^*(\lambda,\alpha)} \, d\lambda \, d\mu(\omega) \\
&= p\alpha^p \int_\Omega \int_0^\infty \lambda^{p-1}\mu\big[e^*(\lambda,\alpha)\big] \, d\lambda \\
&\le p\alpha^p \int_\Omega \int_0^\infty \lambda^{p-2}\left\|f(\omega)\right\|_X d\lambda \, d\mu(\omega) \\
&= p\alpha^p \int_\Omega \int_0^\infty \lambda^{P-2} 1_{e(\lambda)} \, d\lambda \, d\mu(\omega) \\
&= p\alpha^p \int_\Omega \left\|f(\omega)\right\|_X \Big[\int_0^{\|f(\omega)\|} \lambda^{p-2} \, d\lambda\Big] \, d\mu(\omega) \\
&= \alpha^p \frac{p}{p-1} \int_\Omega \left\|f(\omega)\right\|_X^p \, d\mu(\omega).
\end{aligned}$$

$\square$

Remark 1.4. We notice that Lemma 1.3 remains true for any bounded sequence even if it is not a Besicovich one.

Let us consider some examples to which Theorem 1.2 is applied:

Examples 1.5.

1. Let $X = \mathbb{R} \times \mathbb{R}$ (reflexive Banach space) with norm $\|(x, y)\| = \|x| + |y|$. $\Omega = \{1, 2\}$ a probability space, $\mu(1) = \mu(2) = \frac{1}{2}$; $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ being a Banach space of dimension 4. Notice that for

$$T = \begin{pmatrix} a & b & a' & b' \\ c & d & c' & d' \\ e & f & e' & f' \\ g & h & g' & h' \end{pmatrix}$$

T will be contracting on $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ if the sum of the absolute values of the terms in each column is less than 1.

It is easy to show that the operator T is contracting on $L^\infty(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ if the terms of the matrix T satisfy

$$\begin{cases} |a| + |c| + |a'| + |c'| \leq 1 \\ |a| + |c| + |b'| + |d'| \leq 1 \\ |b| + |d| + |a'| + |c'| \leq 1 \\ |b| + |d| + |b'| + |d'| \leq 1 \end{cases} \quad \text{and} \quad \begin{cases} |e| + |g| + |e'| + |g'| \leq 1 \\ |e| + |g| + |f'| + |h'| \leq 1 \\ |f| + |h| + |e'| + |g'| \leq 1 \\ |f| + |h| + |f'| + |h'| \leq 1 \end{cases}$$

Let T be the linear operator on $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ represented by a square matrix of order 4 defined by

$$T = \begin{pmatrix} 2/9 & 0 & 3/8 & 3/7 \\ 1/4 & 1/9 & 2/9 & 2/9 \\ 0 & 1/7 & 2/7 & 1/4 \\ 2/5 & 1/9 & 2/9 & 3/10 \end{pmatrix}$$

T is contracting on both $L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$ and $L^\infty(\{1, 2\}, \mathbb{R} \times \mathbb{R})$.

Consider also the sequence $\alpha_n = e^{i\theta n}$ where θ a complex number, we have by Theorem 1.2 the convergence of the sequence

$$\frac{1}{n+1} \sum_{k=0}^{n} \alpha_k T^k f = \frac{1}{n+1} \sum_{k=0}^{n} e^{i\theta k} \begin{pmatrix} 2/9 & 0 & 3/8 & 3/7 \\ 1/4 & 1/9 & 2/9 & 2/9 \\ 0 & 1/7 & 2/7 & 1/4 \\ 2/5 & 1/9 & 2/9 & 3/10 \end{pmatrix}^k \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \end{pmatrix}$$

for all $f = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \varphi_{\{1\}} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \varphi_{\{2\}} \in L^1(\{1, 2\}, \mathbb{R} \times \mathbb{R})$.

2. Let X be a reflexive Banach space, Ω a probability space, φ is a transformation from Ω to Ω such that for all $f \in L^1(X)$,

$$\int_\Omega \|f \circ \varphi(\omega)\| \, d\mu(\omega) \leq \int_\Omega \|f(\omega)\| \, d\mu(\omega)$$

(or φ is a measure preserving transformation) and let $(\alpha_i)_{i \in N}$ be any Besicovich bounded sequence then

$$\lim_n \frac{1}{n+1} \sum_{i=0}^{n} \alpha_i (f \circ \varphi^i)$$

exists a.e. for every $f \in L^1(X)$.

Applications. The general result of Theorem 1.2 can be applied to give a generalization of the vector valued random ergodic theorem of Beck and Schwartz [**1**].

Theorem 1.6. *Let be defined on Ω a strongly measurable function U_ω with values in the Banach space $B(X)$ of bounded linear operators on a space X. Suppose that $\|U_\omega\| \leq 1$ for all $\omega \in \Omega$. Let φ be a measure preserving transformation in (Ω, β, μ) and $(\alpha_i)_{i \in N}$ be a Besicovich bounded sequence, then for $f \in L^1(X)$, the limit*

$$\lim_n \frac{1}{n+1} \sum_{k=0}^{n} \alpha_k U_\omega U_{\phi(\omega)} \ldots U_{\phi^{k-1}(\omega)} f(\varphi^k(\omega))$$

exists for almost all $\omega \in \Omega$.

Proof. For $f \in L^1(X)$ we define

$$U f(\omega) = U_\omega(f(\varphi(\omega)))\,.$$

Then it can be easily seen that it satisfies the conditions of Theorem 1.2 and hence the condition follows at once from Theorem 1.2. □

II. A Multidimensional Case

Obtaining an extension of Theorem 1.2 to distinct several operators $T_1, \ldots, T_d$ which more general means difficult. But if $T_1 = T^{s_1}, \ldots .\ T_d = T^{s_d}$ where T is an linear operator on $L^1(X)$ and $s_k \in N$ for $k = 1, \ldots, d$, then Theorem 1.2 can be extended to this case. Let

$$B_n(T, d, \alpha, f) = \frac{1}{(n+1)^d} \sum_{i_1=0}^{n} \cdots \sum_{i_d=0}^{n} \alpha_{\lambda_d} T_1^{i_1} \ldots T_d^{i_d} f$$

where $\lambda_d = i_1 s_1 + \ldots i_d s_d$ and α_j be a bounded Besicovich sequence. Let $f_d^* = \sup_n \left\|B_n(T, d, \alpha, f)\right\|_X$ and $e_d^*(a, \alpha) = \left\{f_d^* > \alpha a\right\}$.

Theorem 2.1. *Let X be a reflexive Banach space, T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$, α_j be a bounded Besicovich sequence. Then for $\lambda_d = i_1 s_1 + \cdots + i_d s_d$, $s = s_1 + \ldots, s_d$:*

(i) For $f \in L^1(X)$, et $a > 0$ we have

$$\int_{e_d^*(a,\alpha)} \left(a - \left\|f^{a-}(\omega)\right\|_X\right) d\mu(\omega) \leq s \int_\Omega \left\|f^{a+}(\omega)\right|_X d\mu(\omega).$$

(ii) For $f \in L^1(X)$, the limit of $B_n(T, d, \alpha, f)(\omega)$ exists strongly for every $\omega \in \Omega$ as n tends to infinity.

(iii) If $1 < p < \infty$, $f \in L^p(X)$ and $\alpha = \sup_k |\alpha_k|$, the average $B_n(T, d, \alpha, f)$ converges a.e. and

$$\left\|\sup_n \left\|B_n(T, d, \alpha, f)\right\|_X\right\|_p \leq \alpha \left(\frac{p}{p-1}\right)^{1/p} \|f\|_p .$$

Proof. We will prove the theorem in the case where $d = 2$, and $s_1 = s_2 = 1$ only, for the sake of simplicity. A similar proof to that used in this case gives the general result in the d-dimensional case ($d > 2$).

We now study the following averages:

$$B_n(T, 2, \alpha, f)(\omega) = \frac{1}{(n+1)^2} \sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{i+j} T^{i+j} f.$$

By the relation (3) in the proof of the Theorem 1.2 we can write

$$\sum_{i=0}^{n} \sum_{j=0}^{n} \alpha_{i+j} T^{i+j} f = \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} \Big[T^{j+k} f^{a-} + \sum_{m=0}^{j+k} T^{j+k-m} d_m\Big] + \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} f^{j+k} .$$

We prove an analogous equality to $(*)$ as in the proof of Theorem 1.2

(7)

$$\sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} \Big[T^{j+k} f^{a-} + \sum_{m=0}^{j+k} T^{j+k-m} d_m\Big] = \sum_{j=0}^{n} \underbrace{\sum_{t=j}^{n+j} \alpha_t \Big(T^t f^{a-} + \sum_{m=0}^{t} T^{t-m} d_m\Big)}_{1}$$

$$= \sum_{j=0}^{n} \underbrace{\Big\{ \sum_{t=0}^{n+j} \alpha_t \big(T^t f^{a-} + \sum_{m=0}^{t} T^{t-m} d_m\big) - \sum_{t=0}^{j-1} \alpha_t \Big(T^t f^{a-} + \sum_{m=0}^{t} T^{t-m} d_m\Big)\Big\}}_{1}$$

$$= \sum_{j=0}^{n} \Big\{ \sum_{t=0}^{n+j} T^t \Big(\alpha_t f^{a-} + \underbrace{\sum_{m=0}^{n+j-t} \alpha_{t+m} d_m}_{2} \Big)$$

$$- \sum_{t=0}^{j-1} T^t \Big(\alpha_t f^{a-} + \sum_{m=0}^{j-1-t} \alpha_{t+m} d_m \Big) \Big\} \qquad \text{by } (*)$$

$$= \sum_{j=0}^{n} \Big\{ \sum_{t=0}^{n+j} T^t \Big[\alpha_t f^{a-} + \underbrace{\sum_{m=0}^{j-t-1} \alpha_{t+m} d_m + \sum_{m=j-t}^{n+j-t} \alpha_{t+m} d_m}_{2} \Big]$$

$$- \sum_{t=0}^{j-1} T^t \Big(\alpha_t f^{a-} + \sum_{m=0}^{j-1-t} \alpha_{t+m} d_m \Big) \Big\}$$

$$= \sum_{j=0}^{n} \Big\{ \sum_{t=0}^{n+j} T^t \Big(\alpha_t f^{a-} + \sum_{m=0}^{j-t-1} \alpha_{t+m} d_m \Big) + \underbrace{\sum_{t=0}^{n+j} \Big[\sum_{m=j-t}^{n+j-t} \alpha_{t+m} d_m \Big]}_{3}$$

$$- \sum_{t=0}^{j-1} T^t \Big(\alpha_t f^{a-} + \sum_{m=0}^{j-1-t} \alpha_{t+m} d_m \Big) \Big\}$$

$$= \sum_{j=0}^{n} \sum_{t=j}^{n+j} T^t \Big(\alpha_t f^{a-} + \sum_{m=0}^{j-t-1} \alpha_{t+m} d_m \Big) + \underbrace{\sum_{t=0}^{n+j} T^t \Big(\sum_{m=j-t}^{n+j-t} \alpha_{t+m} d_m \Big) \Big\}}_{3}.$$

Let

$$\chi_n(\omega) = \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} \Big[T^{j+k} f^{a-} + \sum_{m=0}^{j+k} T^{j+k-m} d_m \Big](\omega)$$

and

$$f_C^* = \sup_n \big\| C_n(T, \alpha, f) \big\|_X$$

where

$$C_n(T, \alpha, f) = \frac{1}{(2n+1)^2} \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} T^{j+k} f$$

and $e_C^*(a, \alpha) = \{ f_C^* > a\alpha \}$. Fix $\omega \in e_C^*(a, \alpha)$ there exists $n = n(\omega)$ such that

$$(2n+1)^2 \alpha a \leq \Big\| \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} T^{j+k} f(\omega) \Big\|_X \leq \big\| \chi_n(\omega) \big\|_X + \sum_{j=0}^{n} \sum_{k=0}^{n} \alpha_{j+k} \big\| f_{j+k} \big\|_X.$$

But

$$\begin{aligned}\|\chi_n(\omega)\|_X &\le \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big]\\ &+ \sum_{j=0}^{n}\sum_{t=j}^{n+j}\Big\|T^t\Big(\alpha_t f^{a-} + \sum_{m=0}^{j-t-1}\alpha_{t+m}d_m\Big)(\omega)\Big\|_X + \sum_{j=0}^{n}\sum_{t=0}^{n+j}\Big\|T^t\Big(\sum_{m=j-t}^{n+j-t}\alpha_{t+m}d_m\Big)(\omega)\Big\|_X.\end{aligned}$$

We know that T is contracting in $L^\infty(X)$, using (1) we obtain:

$$\begin{aligned}\|\chi_n(\omega)\|_X &\le \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big] + \sum_{j=0}^{n}\sum_{t=j}^{n+j}a\alpha + \sum_{j=0}^{n}\sum_{t=0}^{n+j}a\alpha\\ &\le \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big] + \sum_{j=0}^{n}(n-j-j)\alpha a + \sum_{j=0}^{n}(n+j)\alpha a\,.\end{aligned}$$

By (3) we can write

$$\begin{aligned}(2n+1)^2\alpha a &\le \Big[\alpha_0\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\alpha_k\|d_k(\omega)\|_X\Big]\\ &\quad + n(n+1)\alpha a + 2n^2\alpha a + \sum_{j=0}^{n}\sum_{k=0}^{n}\alpha_{j+k}\|f_{j+k}\|_X\\ &\le \alpha\Big[\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\|d_k(\omega)\|_X\Big] + (3n^2+n)\alpha a + \alpha\sum_{j=0}^{n}\sum_{k=0}^{n}\|f_{j+k}\|_X.\end{aligned}$$

As in the case $d = 1$ we have by the relations (3), (4) and (5)

$$a \le \Big[\|f^{a-}(\omega)\|_X + \sum_{k=0}^{n}\|d_K(\omega)\|_X\Big] + \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\|f_{j+k}(\omega)\|_X.$$

By the (5) we see for all $\omega \in e^*_C(a,\alpha)$

$$a = \|f^{a-}(\omega)\|_X + \sum_{k=0}^{\infty}\|d_k(\omega)\|_X.$$

This implies

$$\int_{e^*_C(a,\alpha)}\big(a - \|f^{a-}(\omega)\|_X\big)\,d\mu(\omega) \le \sum_{k=0}^{\infty}\|d_k\|_1 \le \|f_0\|_1 = \int_\Omega \|f_0(\omega)\|\,d\mu(\omega).$$

On the other hand we can write

$$B_n(T,\alpha,f) = \frac{(2n+1)^2}{(n+1)^2}C_n(T,\alpha,f)$$

which gives $f_B^* \le 2f_C^*$ hence $e_B^*(a,\alpha) \subseteq e_C^*(a/2,\alpha)$. For $b = a/2$ we have

$$\begin{aligned}\int_{e_B^*(a,\alpha)} \big(a - \|f^{a-}(\omega)\|_X\big)\, d\mu(\omega) &\le 2\int_{e_C^*(a,\alpha)} \big(b - \|f^{b-}(\omega)\|_X\big)\, d\mu(\omega)\\ &\le 2\int_\Omega \|f^{b+}(\omega)\|_X\, d\mu(\omega) \le 2\int_\Omega \|f^{a+}(\omega)\|_X\, d\mu(\omega)\end{aligned}$$

so

$$a\mu\big[e_B^*(a,\alpha)\big] \le a\mu\big[e_C^*(a/2,\alpha)\big] \le 2\int_\Omega \|f(\omega)\|\, d\mu(\omega). \tag{8}$$

Now, we shall prove that averages

$$B(n_1,n_2,T,\alpha,f) = \frac{1}{n_1 n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} \varphi^1(i)\varphi^2(j)T^{i+j}f$$

converge on a dense set in $L^1(X)$. We will need the following lemma:

Lemma 2.2. *Let T be a linear operator on $L^1(X)$ which is contracting in both $L^1(X)$ and $L^\infty(X)$, then for $f \in \mathrm{Inv}\,(T) + \mathrm{Im}\,(I-T) \cap L^\infty(X)$ the limit*

$$\lim \frac{1}{n_1 n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} T^{i+j}f$$

exists as n_1 and n_2 tend to infinity.

Proof. Let $f = g + (h - Th)$ with $Tg = g$, $g \in L^1(X)$ and $h \in L^\infty(X)$. Then

$$\begin{aligned}\frac{1}{n_1 n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2} T^{i+j}f &= g + \frac{1}{n_1 n_2}\sum_{i=0}^{n_1} T^i\Big[\sum_{j=0}^{n_2}(T^j h - T^{j+1}f)\Big]\\ &= g + \frac{1}{n_1 n_2}\sum_{i=0}^{n_1}\big[T^i h - T^{n_2-1}f\big]\\ &= g + \frac{1}{n_1}\Big[\frac{1}{n_2}\sum_{i=0}^{n_1} T^i h\Big] - \frac{1}{n_1}\Big[\frac{1}{n_2}\sum_{i=0}^{n_1} T^{n_2-1+i}h\Big].\end{aligned}$$

But $\|T\|_\infty \le 1$ hence

$$\Big\|\frac{1}{n_2}\Big[\frac{1}{n_1}\sum_{i=0}^{n_1} T^i h\Big]\Big\|_\infty \le \frac{1}{n_2}\|h\|_\infty \xrightarrow{n_2\to\infty} 0$$

and

$$\Big[\frac{1}{n_2}\Big[\frac{1}{n_1}\sum_{i=0}^{n_1} T^{n_2-1+i}h\Big]\Big\|_\infty \le \frac{1}{n_2}\|h\|_\infty \xrightarrow{n_2\to\infty} 0.$$

Let $U_k f = e^{i\theta_k} T_k f$, $k = 1, 2$. U_k is a linear operator satisfying the conditions of Lemma 2.2 and so the theorem holds in the case $\phi^k(n) = e^{\theta_k n}$.

This implies that $\frac{1}{n_1 n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\varphi^1(i)\varphi^2(j)T^{i+j}f$ converges a.e. on $\operatorname{Inv}(T) + \operatorname{Im}(I-T)\cap L^1(X)$ which is dense, by Kakutani-Yoshida theorem in $L^1(X)$ (X a reflexive Banach space). From Theorem 2.1(i) and the linearity of convergence of sequences we obtain that

$$\lim_{n_1, n_2\to\infty}\frac{1}{n_1 n_2}\sum_{i=0}^{n_1}\sum_{j=0}^{n_2}\varphi^1(i)\varphi^2(j)T^{i+j}f$$

exists and is finite a.e. for any trigonometric polynomial ϕ^k, $k = 1, 2$, and $f \in L^1(X)$.

The general case ($d > 2$) is similar to the real case studied by Olsen [**5**].

The second assertion (ii) follows from the maximal equality (3) and the rearrangement formula used in part I.

Let $\alpha_j = 1$, and $s_k = 1$ for all j in N and $k = 1, \dots, d$ the average $B_n(T, d, \alpha, f)$ becomes

$$B_n(T, d, \alpha, f) = \frac{1}{(n+1)^d}\sum_{i_1=0}^{n}\cdots\sum_{i_d=0}^{n}T^{i_1+\cdots+i_d}f = \Big(\frac{1}{n+1}\sum_{j=0}^{n}T^j\Big)^d f.$$

Using Theorem 2.1, we deduce the following:

Corollary 2.3. *Let X be a reflexive Banach space, T be a linear operator on $L^1(X)$ contracting in $L^1(X)$ and in $L^\infty(X)$, then for $d \in N$ and $f \in L^1(X)$*

$$\lim_n\Big(\frac{1}{n+1}\sum_{j=0}^{n}T^j\Big)^d f$$

exists a.e.

Acknowledgement. The author wishes to express his thank to Professors Sylvie Delabriere, Louis Sucheston, Jean Paul Thouvenot and the referee for kind advice.

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