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SKEW PRODUCTS IN THE CENTRALIZER OF COMPACT ABELIAN GROUP EXTENSIONS

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Abstract. If T_ψ is an ergodic group extension of a weakly mixing transformation T having minimal self-joinings, then it is shown that isomorphism between T_ψ and its inverse implies isomorphism between T and T^{-1}. However, if T satisfies the weaker condition of being simple it is shown that that isomorphism between T_ψ and its inverse does not imply the isomorphism between T and T^{-1}. This answers a question asked by D. Rudolph.

0. Introduction

Let T be an ergodic automorphism defined on a standard Borel probability space $(X, \mathcal{F}, \mu)$. Given a compact abelian group G with Haar measure m, and a cocycle $\phi\colon X \to G$, the corresponding **group extension** $T_\phi\colon X \times G \to X \times G$ is defined by

$$T_\phi(x, g) = (Tx, \phi(x) + g).$$

Let $S \in C(T_\phi)$, where $C(T)$ denotes the **centralizer** of T. The question of when S can be represented as a **skew product**: $S(x, g) = (S_0 x, \psi(x, g))$ is often of importance. In this situation (when the first coordinate is independent of g), we call S a G-**map**. We examine some new sufficient conditions for the existence of G–maps, and give some examples where they do not occur. It is important to know when G-maps exist since if they do, the form of members of the centralizer can be completely determined.

The following was shown in [**2**]: Suppose that T is a simple map (in the sense of del Junco and Rudolph [**3**]), which is isomorphic to its inverse, and suppose that T_ϕ is also isomorphic to its inverse, then every conjugation between T_ϕ and its inverse is a G-map. D. Rudolph asked the question whether this result is still true if we do not assume that T is isomorphic to its inverse. We construct a family of simple maps T_ψ, each of which is an abelian group extension of a simple map T which is not isomorphic to its inverse, even though T_ψ is isomorphic to its inverse. Consequently, the conjugations between T_ψ and its inverse cannot be G-maps. Our last theorem shows that this phenomenon cannot happen if T is assumed to

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have minimal self joinings. I would like to thank Dan Rudolph for his interest in this paper.

1. Preliminaries

Let $T\colon (X,\mathcal{F},\mu) \to (X,\mathcal{F},\mu)$ be an ergodic automorphism defined on a non-atomic standard Borel probability space. We denote the identity automorphism and the identity group automorphism by I. The group of all automorphisms $\mathrm{Aut}(X)$ of $(X,\mathcal{F},\mu)$ becomes a completely metrizable topological group when endowed with the weak convergence of transformations ($T_n \to T$ if for all $A \in \mathcal{F}$, $\mu(T_n^{-1}(A)\triangle T^{-1}(A)) + \mu(T_n(A)\triangle T(A)) \to 0$ as $n \to \infty$). Denote by $C(T)$ the centralizer (or commutant) of T, i.e., those automorphisms of $(X,\mathcal{F},\mu)$ which commute with T. Since we are assuming the members of $C(T)$ are invertible, $C(T)$ is a group.

Much of the discussion concerns the set

$$\mathcal{B}(T) = \{S \in \mathrm{Aut}\,(X) : TS = ST^{-1}\},$$

whose basic properties are discussed in [**2**]. In particular we note that $\{S^2 : S \in \mathcal{B}(T)\} \subseteq C(T)$.

Let G be a compact abelian group equipped with Haar measure m and denote by $(X\times G,\mathcal{F}_G,\tilde{\mu})$ the product measure space, where $\tilde{\mu} = \mu\times m$. Let $\phi\colon X \to G$ be measurable (i.e., a G-cocycle), then the corresponding G-extension: $T_\phi\colon X\times G \to X\times G$ preserves the measure $\tilde{\mu}$.

Recall that for an automorphism $T\colon (X,\mathcal{F},\mu) \to (X,\mathcal{F},\mu)$, a T-invariant sub σ-algebra $\mathcal{C}$ (i.e., $T^{-1}\mathcal{C} = \mathcal{C}$) is said to be a **factor** of T (or rather the map $T\colon (X,\mathcal{C},\mu) \to (X,\mathcal{C},\mu)$ is a factor of $T\colon (X,\mathcal{F},\mu) \to (X,\mathcal{F},\mu)$). $J(T,S)$ will be used to denote the space of all 2-**joinings** of T with another such map S i.e., $\lambda \in J(T,S)$ if λ is a $T\times S$-invariant probability measure on $\mathcal{F}\otimes\mathcal{F}$ whose marginals on each coordinate are μ. In this paper we will have either $S = T$, or $S = T^{-1}$. The general theory of joinings is developed in [**3**] (see also [**6**]).

T is said to be 2-**simple**, if the the only ergodic 2-joinings $\lambda \in J(T,T)$ are product measure $\mu\times\mu$, and measures of the form μ_S, $S \in C(T)$, (**graph-joinings**) defined by $\mu_S(A\times B) = \mu(A\cap S^{-1}B)$. T is said to have **minimal self–joinings** (MSJ) (of order two), if in addition the only graph joinings arise from powers of T.

If T and S have a common factor $\mathcal{C}$, and λ is a self-joining of the factor map, we can lift it to a joining of T and S by

$$\widehat{\lambda}(A\times B) = \int_{X\times X} E(A|\mathcal{C})E(B|\mathcal{C})\,d\lambda, \qquad A,B \in \mathcal{B},$$

called the **relatively independent extension** of λ.

We say that a measurable transformation $S\colon X \times G \to X \times G$ is a G-map if it factors as a skew product, of the form $S(x,g) = (kx, \psi(x,g))$, for some measurable $k\colon X \to X$, and $\psi\colon X \times G \to G$. We make the distinction between skew products and G-maps because a transformation may be a G-map for one particular G, but not for a different one (and of course is a skew product in each case). If $\tilde{S} \in C(T_\phi)$ is a G–map (respectively $\tilde{K} \in \mathcal{B}(T_\phi)$ is a G-map), then it is known ([**2**], [**5**]) that there exists $S \in C(T)$, (respectively $k \in \mathcal{B}(T)$), $f\colon X \to G$ measurable and a continuous epimorphism $v\colon G \to G$ for which

$$\tilde{S}(x,g) = (Sx, f(x) + v(g)); \qquad \phi(Sx) - v(\phi(x)) = f(Tx) - f(x)$$

(respectively

$$\tilde{K}(x,g) = (kx, f(x) + v(g)); \qquad \phi(kTx) + v((\phi(x)) = f(x) - f(Tx)).$$

We say that S can be **lifted** to $C(T_\phi)$ and k can be **lifted** to $\mathcal{B}(T_\phi)$ when the above hold. It is known that if T is simple and T_ϕ is ergodic, then every $\tilde{S} \in C(T_\phi)$ is a G-map (and if we also have T isomorphic to its inverse, then every $\tilde{K} \in \mathcal{B}(T_\phi)$ is a G-map), [**2**], [**5**].

2. New Sufficient Conditions for the Existence of G-maps

Recall that T has the weak closure property if the weak closure of the powers of T equals $C(T)$. In this section we shall see that automorphisms T with the property that the set $\{S^2 : S \in \mathcal{B}(T)\}$ is a singleton set, are of importance. For example, it was shown in [**2**] that transformations having the weak closure property and that are isomorphic to their inverses always satisfy this. Also if T has simple spectrum, $S^2 = I$ for all $S \in \mathcal{B}(T)$. We mention here a simple property of such automorphisms related to Proposition 5 of [**2**].

Proposition 1. *(i) S conjugates $C(T)$ to $C(T)^{-1}$ for every $S \in \mathcal{B}(T)$ if and only if $\{S^2 : S \in \mathcal{B}(T)\}$ is a singleton set.*
(ii) If $\{S^2 : S \in \mathcal{B}(T)\}$ is a singleton set, then $S^4 = I$ for all $S \in \mathcal{B}(T)$.

Proof. (i) If $S \in \mathcal{B}(T)$ conjugates $C(T)$ to $C(T)^{-1}$, then $S \circ R = R^{-1} \circ S$ for all $R \in C(T)$. This implies that $(S \circ R)^2 = S^2$, and the result follows since given $S_1, S_2 \in \mathcal{B}(T)$, $S_1 = S_2 \circ R$ for some $R \in C(T)$.

Conversely suppose that $\{S^2 : S \in \mathcal{B}(T)\}$ is a singleton set, then $(S \circ R)^2 = S^2$ for any $S \in \mathcal{B}(T)$ and $R \in C(T)$. This immediately gives $S \circ R = R^{-1} \circ S$, or that S conjugates $C(T)$ to $C(T)^{-1}$.

(ii) Suppose $\{S^2 : S \in \mathcal{B}(T)\}$ is a singleton set, and let $S \in \mathcal{B}(T)$, then $S^3 \in \mathcal{B}(T)$ and we have $S^6 = S^2$, so $S^4 = I$. □

The following was proved in [**1**].

Theorem 1. *Suppose that T is an ergodic transformation isomorphic to its inverse and having the weak closure property. If there exists a conjugation of order 4, then every conjugation is of order 4, and T can be represented as a $\mathbb{Z}_2$-extension of an ergodic map T_0. Furthermore, every $S \in \mathcal{B}(T)$ can be represented in the form*

$$S(x,j) = (kx, \psi(x) + j), \quad \text{for some} \quad k \in \mathcal{B}(T_0) \quad \text{satisfying} \quad k^2 = I,$$

and every $\widehat{\phi} \in C(T)$ is the lift of some $\phi \in C(T_0)$. As a consequence, $\widehat{\phi}(x,j) = (\phi(x), u(x) + j)$ for some measurable $u\colon X \to \mathbb{Z}_2$.

We prove here some related results, giving conditions under which a map is a skew product, and hence a G-map.

Theorem 2. *Let $T_\phi\colon X \times G \to X \times G$ be a compact group extension which is isomorphic to its inverse. If the set $\{S^2 : S \in \mathcal{B}(T_\phi)\}$ is a singleton set, then for each $S \in \mathcal{B}(T_\phi)$ there exists $k \in \mathcal{B}(T)$, and measurable $\psi\colon X \to G$ with $S(x,g) = (kx, \psi(x) - g)$, a.e. $x \in X$, $g \in G$.*

Proof. We use the fact that $S\colon X \times G \to X \times G$ is essentially one–to–one, so that if $S(x,g) = (k(x,g), \psi(x,g))$, and if

$$k(x_1,g_1) = k(x_2,g_2) \ \text{ and } \ \psi(x_1,g_1) = \psi(x_2,g_2) \ \text{ then } (x_1,g_1) = (x_2,g_2) \ \text{ a.e. } \tilde{\mu}.$$

Now let $S \in \mathcal{B}(T_\phi)$ and $\sigma(x,g) = (x, g+h)$ for some $h \in G$, then $S \circ \sigma_h \in \mathcal{B}(T_\phi)$, so that $S^2 = (S \circ \sigma_h)^2$. This says that

$$k(k(x,g+h), \psi(x,g+h) + h) = k(k(x,g), \psi(x,g)),$$

and

$$\psi(k(x,g+h), \psi(x,g+h) + h) = \psi(k(x,g), \psi(x,g)).$$

These two together imply that

$$k(x,g+h) = k(x,g) \quad \text{and} \quad \psi(x,g+h) + h = \psi(x,g), \quad \text{a.e.} \quad \tilde{\mu},$$

and hence for almost all $h, g \in G$ and $x \in X$. In particular, k is independent of h almost everywhere, so we can write $k(x,g) = k(x)$, say. Also, if $\psi'(x) = \psi(x,g_0)$, where g_0 is chosen so that the above equation holds a.e., then we have $\psi(x,h) = \psi'(x) - h + g_0$ a.e. Now replace ψ' by $\psi(x) = \psi'(x) + g_0$, and we see that $\psi(x,h)$ is of the required form. It can now be seen that since $S \in \mathcal{B}(T_\phi)$ then $k \in \mathcal{B}(T)$.□

Corollary 1. *Suppose T_ϕ is isomorphic to its inverse. If either $\tilde{K}^2 = I$ for all $\tilde{K} \in \mathcal{B}(T_\phi)$, or if T_ϕ has the weak closure property, then all members of $C(T_\phi)$ and $\mathcal{B}(T_\phi)$ are G-maps. In fact, if $\tilde{S} \in C(T_\phi)$, then $\tilde{S}(x,g) = (Sx, u(x) + g)$ for some $S \in C(T)$ and measurable $u\colon X \to G$.*

Proof. We simply use the fact that if $\tilde{K} \in \mathcal{B}(T_\phi)$, then $\tilde{K} \circ \tilde{S} \in \mathcal{B}(T_\phi)$ for any $\tilde{S} \in C(T_\phi)$. □

3. Conjugations Which Are Not G-maps

In [**2**] the following theorem was proved:

Theorem 3. *Suppose that T_ϕ is an ergodic compact group extension of a 2-simple map T. If T_ϕ is isomorphic to its inverse, then every member of $\mathcal{B}(T_\phi)$ is a G-map if and only if T is isomorphic to its inverse.*

The question arose in [**2**] whether for a simple map T, the isomorphism between T_ϕ and its inverse is enough to guarantee the same being true for T. In this section we show that this is not the case, even if T_ϕ is also a simple map. In particular this shows that it is possible for a group extension T_ϕ to be isomorphic to its inverse without the base transformation T being isomorphic to its inverse. Our final theorem shows that if T has minimal self-joinings, then it does follow that T must be isomorphic to its inverse.

We start with some new results which parallel results in [**4**]. We say that a cocycle $\phi\colon X \to G$ is **ergodic** if the corresponding group extension T_ϕ is ergodic. Given $k \in \mathcal{B}(T)$ we obtain a graph joining $\mu_k \in J(T, T^{-1})$ defined by $\mu_k(A \times B) = \mu(A \cap k^{-1}B)$, for $A, B \in \mathcal{F}$.

Theorem 4. *Let $\phi\colon X \to \mathbb{Z}_2$ be an ergodic cocycle and $k \in \mathcal{B}(T)$. If $\widehat{\mu}_k \in J(T_\phi, T_\phi^{-1})$ denotes the relatively independent extension of μ_k, then the following are equivalent:*

(i) $\widehat{\mu}_k$ is ergodic.
(ii) $T_{\phi\times\phi\circ kT}(x,i,j) = (Tx, \phi(x)+i, \phi(kTx)+j)$ is ergodic.
(iii) $k \in \mathcal{B}(T)$ does not lift to $\mathcal{B}(T_\phi)$.
(iv) $S(x,g) = (Tx, \phi(kTx)+\phi(x)+g)$ is ergodic.

Proof. (i) $\Leftrightarrow$ (ii) Let us define $f\colon X\times\mathbb{Z}_2\times\mathbb{Z}_2 \to X\times\mathbb{Z}_2\times X\times\mathbb{Z}_2$ by $f(x,i,j) = (x,i,kx,j)$, then it is easy to check that

$$f \circ T_{\phi\times\phi\circ kT} = (T_\phi \times T_\phi^{-1}) \circ f,$$

(here T_ϕ^{-1} denotes the inverse of the extension and not the extension of the inverse). This implies that the dynamical systems $(T_{\phi\times\phi\circ kT}, \mu\times\nu_2\times\nu_2)$ and $(T_\phi\times T_\phi^{-1}, \lambda)$ are isomorphic (where λ is the image of $\mu\times\nu_2\times\nu_2$ under f and ν_2 is Haar measure on $\mathbb{Z}_2$). We need only check that λ is the relatively independent extension of μ_k, i.e., $\lambda = \widehat{\mu}_k$. Since the support of λ is the set $\{(x,i,kx,j) : x\in X;\ i,j\in\mathbb{Z}_2\}$, we have $\lambda = \mu_k\times\nu_2\times\nu_2$ and this is just $\widehat{\mu}_k$.

(ii) $\Leftrightarrow$ (iii) Suppose that $\phi\times\phi\circ kT$ is ergodic (where $\phi\times\phi\circ kT(x) = (\phi(x), \phi(kTx))$), and k lifts to $\mathcal{B}(T_\phi)$, then there exists measurable f such that

$$\phi(kTx) + \phi(x) = f(x) - f(Tx).$$

Let $\chi \in \widehat{\mathbb{Z}}_2$, $\chi \neq 1$, then

$$\chi(\phi \circ kT(x)) \cdot \chi(\phi x) = \chi \circ f(x)/\chi \circ f(Tx).$$

But $\phi \times \phi \circ kT$ ergodic implies that $\chi \circ f = \text{constant}$, and $\tilde{\chi}(i,j) = \chi(i)\chi(j) = 1$, but $\tilde{\chi}(i,j) = (-1)^{i+j} \neq 1$, so we have a contradiction.

Conversely suppose the product cocycle $\phi \times \phi \circ kT$ is not ergodic. Then there exist measurable $\tilde{f}$ and characters χ_1 and χ_2 (both not 1, for otherwise ϕ would not be ergodic), satisfying

$$\tilde{f}(Tx)/\tilde{f}(x) = \chi_1(\phi(x))\chi_2(\phi(kTx)).$$

Therefore $\tilde{f}^2(Tx) = \tilde{f}^2(x)$, and the ergodicity of T implies that $\tilde{f}^2 = $ a constant. We see that $\tilde{f}$ takes 2 values, which we may assume are ± 1. It follows that there exists f such that

$$\phi(kTx) + \phi(x) = f(x) - f(Tx),$$

and this implies that k lifts to $\mathcal{B}(T_\phi)$, a contradiction.

(ii) $\Rightarrow$ (iv) It is easy to see that S is a factor of $T_{\phi\times\phi\circ kT}$, via the map $F(x,i,j) = (x, i-j)$, i.e., $F \circ T_{\phi\times\phi\circ kT} = S \circ F$. The ergodicity of $T_{\phi\times\phi\circ kT}$ now implies that of S.

(iv) $\Rightarrow$ (iii) This now follows using a similar argument to (ii) $\Rightarrow$ (iii). □

Corollary 2. *$k \in \mathcal{B}(T)$ lifts to $\mathcal{B}(T_\phi)$ if and only if $T_{\phi\times\phi\circ kT}$ is not ergodic.*

Proposition 2. *Define $R\colon X \times \mathbb{Z}_2 \times \mathbb{Z}_2 \to X \times \mathbb{Z}_2 \times \mathbb{Z}_2$ by*

$$R(x,i,j) = (kx, j, i), \quad \text{where} \quad k^2 = I,$$

then $R \circ T_{\phi\times\phi\circ kT} = T^{-1}_{\phi\times\phi\circ kT} \circ R$, i.e., R conjugates $T_{\phi\times\phi\circ kT}$ to its inverse.

Proof. Note that $T^{-1}_{\phi\times\phi\circ kT}(x,i,j) = (T^{-1}x, -\phi(T^{-1}x) + i, -\phi(kx) + j)$, and

$$R \circ T_{\phi\times\phi\circ kT}(x,i,j) = R(Tx, \phi(x) + i, \phi(kTx) + j) = (kTx, \phi(kTx) + j, \phi(x) + i),$$

$$T^{-1}_{\phi\times\phi\circ kT} \circ R(x,i,j) = T^{-1}_{\phi\times\phi\circ kT}(kx, j, i)$$

$$= (T^{-1}kx, -\phi(T^{-1}kx) + j, -\phi(k^2x) + i) = (kTx, \phi(kTx) + j, \phi(x) + i)$$

in $\mathbb{Z}_2$. □

Remark. R is a G-map for T with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, but is not a $\mathbb{Z}_2$-map for T_ϕ. Although R is an involution, there are members of $\mathcal{B}(T_{\phi\times\phi kT})$ which are not involutions. For example

$$\tilde{K}(x,i,j) = (kx, j+1, i).$$

Theorem 5. *There is a weakly mixing automorphism $T\colon X \to X$ having a weakly mixing compact group extension $T_\psi\colon X\times G \to X \times G$ of a compact abelian group G with the following properties:*

(i) T *and* T_ψ *are simple maps.*
(ii) $T_\psi \cong T_\psi^{-1}$.
(iii) T *and* T^{-1} *are not isomorphic.*
(iv) If $K \in \mathcal{B}(T_\psi)$*, then* K *is not a* G*-map for* T*.*

Proof. Start by choosing S to be an automorphism having minimal self–joinings and isomorphic to its inverse (for example, the rank one substitution $\theta(0) = 00100$, $\theta(1) = 1$, see [**3**] or [**6**]). Choose a cocycle $\phi\colon X \to \mathbb{Z}_2$ with the property that $\mathcal{B}(S_\phi) = \emptyset$ and S_ϕ is ergodic. In particular S_ϕ is not isomorphic to its inverse and hence is weakly mixing (see [**2**], Proposition 11(i)).

Let $T = S_\phi$, and put $\psi = \phi \circ kS$, for some $k \in \mathcal{B}(S)$, then by Corollary 2 and Proposition 2 (using $k^2 = I$), T_ψ is ergodic and isomorphic to its inverse. T and T_ψ, being extensions of a map with minimal self-joinings, are both simple and hence weakly mixing (by Proposition 11(iii) of [**2**]). Clearly T_ψ is an extension of T, and the conclusions of the theorem are satisfied. □

Remarks. 1. Every $K \in \mathcal{B}(T_\psi)$ is of the form $K = T_\psi^m \circ R$, where $R(x,i,j) = (kx, j + c_1, i + c_2)$, $c_1, c_2 \in \mathbb{Z}_2$, $k \in \mathcal{B}(T)$ and $m \in \mathbb{Z}$.

2. It follows from results in [**1**] that T_ψ has an even multiplicity function on the subspace

$$\{f \in L^2(X \times \mathbb{Z}_2 \times \mathbb{Z}_2) : f \circ K^2 = f\}^\perp = \{f : f(x,i,j) = f(x, i+1, j+1)\}^\perp,$$

and also that the spectra of T and that of T_ψ (on $L^2(X \times \mathbb{Z}_2)^\perp$) cannot be mutually singular.

Finally we show that if T has minimal self-joinings, then the phenomena of Theorem 5 cannot happen.

Theorem 6. *Suppose that* $T\colon X \to X$ *is weakly mixing and has minimal self–joinings. Let* $T_\phi\colon X \times G \to X \times G$ *be a weakly mixing group extension for which* $T_\phi \cong T_\phi^{-1}$*, then* $T \cong T^{-1}$.

Proof. Suppose S satisfies $ST_\phi = T_\phi^{-1}S$, and let $\tilde{\mu}_S \in J(T_\phi, T_\phi^{-1})$ be the corresponding graph joining (where $\tilde{\mu} = \mu \times m$).

Now $\tilde{\mu}_S|\mathcal{F} \times \mathcal{F}$ is a joining of T and T^{-1}, i.e.,

$$\tilde{\mu}_S|\mathcal{F} \times \mathcal{F} \in J(T, T^{-1}).$$

However, since T and T^{-1} have minimal self–joinings, T and T^{-1} are either disjoint or conjugate (see [**3**]).

If they are disjoint, then $J(T,T^{-1}) = \{\mu \times \mu\}$, i.e., $\tilde{\mu}_S|\mathcal{F} \times \mathcal{F} = \mu \times \mu$. This, together with the fact that $T_\phi \times T_\phi^{-1}$ is weakly mixing implies that $\tilde{\mu}_S = \tilde{\mu} \times \tilde{\mu}$, which is clearly impossible for a graph joining. Hence T and T^{-1} must be conjugate, i.e., there exists a measure preserving conjugation $k\colon X \to X$ for which $Tk = kT^{-1}$, and the result follows. $\square$

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