Acta Math. Univ. Comenianae
Vol. LXVIII, 1(1999), pp. 137–151

A PICONE TYPE IDENTITY FOR SECOND ORDER HALF–LINEAR DIFFERENTIAL EQUATIONS

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Abstract. In the paper a Picone-type identity for half-linear differential equations of second order is derived and Sturmian theory for both forced and unforced half-linear and quasilinear equations based on this identity is developed.

1. Introduction

According to the classical Sturm-Picone comparison theorem for linear second order ordinary differential equations of the form

$$l[x] \equiv (p(t)x')' + q(t)x = 0 \tag{1}$$

and

$$L[y] \equiv (P(t)y')' + Q(t)y = 0 \tag{2}$$

where p, q, P and Q are continuous real-valued functions defined on a given interval I, if there exists a nontrivial solution x of (1) with consecutive zeros a and b and if

$$p(t) \geq P(t) > 0 \tag{3}$$

and

$$Q(t) \geq q(t) \tag{4}$$

on $[a, b]$, then every solution y of (2) except a constant multiple of x has a zero in (a, b).

Received December 1, 1998.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 34C10.
Key words and phrases. Half-linear differential equation, Picone's identity, Wirtinger's inequality, Sturm's comparison principle, forced oscillations.
*Research supported by the Matsumae International Foundation, Tokyo, Japan.

The original proof by Picone [**16**] was based on using the identity

$$\frac{d}{dt}\left\{\frac{x}{y}(ypx' - xPy')\right\} = (p-P)x'^2 + (Q-q)x^2 + P(x' - \frac{x}{y}y')^2 + \frac{x}{y}\{yl[x] - xL[y]\} \tag{5}$$

which holds for all real valued functions x and y defined on I such that x, y, px' and Py' are differentiable on I and $y(t) \neq 0$ for $t \in I$.

The identity (5) has proved to be useful tool not only in comparing equations (1) and (2) but also in establishing Wirtinger type inequalities for solutions of the second order linear ordinary differential equations and lower bounds for the eigenvalues of the associated eigenvalue problems, and was generalized to higher-order ordinary differential operators as well as the partial differential operators of the elliptic type (see [**8**]).

The purpose of this paper is to generalize Picone's identity (5) to the case of nonlinear second order differential operators of the form

$$l_\alpha[x] \equiv (p(t)|x'|^{\alpha-1}x')' + q(t)|x|^{\alpha-1}x \tag{6}$$

and

$$L_\alpha[y] \equiv (P(t)|y'|^{\alpha-1}y')' + Q(t)|y|^{\alpha-1}y \tag{7}$$

where $\alpha > 0$ is a constant and p, q, P and Q are real-valued continuous functions defined on a given non-degenerate interval I with $p(t) > 0$ and $P(t) > 0$ on I and to apply it to the study of qualitative properties of the associated differential equations $l_\alpha[x] = 0$ and $L_\alpha[y] = 0$ as well as the equations with forcing terms.

The operators of the form (6) and (7) are sometimes called **half-linear** (or **homogeneous of degree** α) because for any functions u and v in the domain of l_α and L_α, respectively, and for every $c \in \mathrm{R}$

$$l_\alpha[cu] = |c|^{\alpha-1}c\,l_\alpha[u]$$

and

$$L_\alpha[cv] = |c|^{\alpha-1}c\,L_\alpha[v]$$

that is, if x and y are respective solutions of the corresponding equations $l_\alpha[x] = 0$ and $L_\alpha[y] = 0$, then for any real constant c the functions cx and cy are the solutions of the same equations, too.

2. Picone Type Identity and Leightonian Comparison Theorems

Define $\varphi(u) := |u|^{\alpha-1}u$, $\alpha > 0$, and consider second order nonlinear ordinary differential operators of the form

$$l_\alpha[x] \equiv (p\varphi(x'))' + q\varphi(x)$$

and

$$L_\alpha[y] \equiv (P\varphi(y'))' + Q\varphi(y)$$

where p, q, P and Q are continuous functions on a given interval I and $p(t) > 0$ and $P(t) > 0$ for all $t \in I$. The domains $D_l(I)$ and $D_L(I)$ of the operators l_α and L_α, respectively, are defined to be the sets of all continuous real-valued functions x (resp. y) defined on I such that x and $p\varphi(x')$ (resp. y and $P\varphi(y')$) are continuously differentiable on I.

The following lemma is of basic importance for our later considerations.

Lemma 1 (Picone type identity). *If $x \in D_l(I_0)$ and $y \in D_L(I_0)$ for some non-degenerate subinterval $I_0 \subset I$ and $y(t) \neq 0$ for $t \in I_0$, then*

$$\begin{aligned}(8)\qquad &\frac{d}{dt}\left\{\frac{x}{\varphi(y)}[\varphi(y)p\varphi(x') - \varphi(x)P\varphi(y')]\right\}\\ &= (p-P)|x'|^{\alpha+1} + (Q-q)|x|^{\alpha+1} + P[|x'|^{\alpha+1} + \alpha|xy'/y|^{\alpha+1}\\ &\quad - (\alpha+1)x'\varphi(xy'/y)] \quad + \frac{x}{\varphi(y)}\{\varphi(y)l_\alpha[x] - \varphi(x)L_\alpha[y]\}.\end{aligned}$$

The identity (8) may be verified by a straightforward differentiation and the verification is left to the reader.

The following simple lemma will be also used in proving our main resuls.

Lemma 2. *If $X, Y \in R$ and $\alpha > 0$, then*

$$X\varphi(X) + \alpha Y\varphi(Y) - (\alpha+1)X\varphi(Y) \geq 0 \tag{9}$$

where equality holds if and only if $X = Y$.

Proof. If $XY \leq 0$, then (9) is obvious. If $XY \geq 0$, then the inequality (9) is essentialy the well known inequality from [**4**] applied to $|X|$ and $|Y|$.

For our first result based on the identity (8) let

$$U = \{\eta \in C^1[a,b] : \eta(a) = \eta(b) = 0\}$$

and define the functional $J_\alpha : U \to R$ by

$$J_\alpha[\eta] = \int_a^b \left[P(t)|\eta'(t)|^{\alpha+1} - Q(t)|\eta(t)|^{\alpha+1}\right] dt. \qquad \square$$

Theorem 1 (Wirtinger type inequality). *If there exists a solution y of $L_\alpha[y] = 0$ such that $y(t) \neq 0$ on (a,b), then for all $\eta \in U$*

$$J_\alpha[\eta] \geq 0 \tag{10}$$

where equality holds if and only if η is a constant multiple of y.

Proof. From Picones's identity (8) applied to the case $p(t) \equiv P(t)$, $q(t) \equiv Q(t)$ and $x(t) = \eta(t)$ we obtain

$$\begin{aligned}
&\frac{d}{dt}\left[\eta P\varphi(\eta') - \eta\varphi(\eta) P\frac{\varphi(y')}{\varphi(y)}\right] \\
&\quad = P\left[|\eta'|^{\alpha+1} + \alpha|\eta y'/y|^{\alpha+1} - (\alpha+1)\eta'\varphi(\eta y'/y)\right] + \eta L_\alpha[\eta] - \frac{\eta\varphi(\eta)}{\varphi(y)} L_\alpha[y].
\end{aligned}$$

Now, using the fact that y is a solution of $L_\alpha[y] = 0$ and cancelling $\eta(P\varphi(\eta'))'$ we get

$$\begin{aligned}
P|\eta'|^{\alpha+1} - Q|\eta|^{\alpha+1} &= \frac{d}{dt}\left[\eta\varphi(\eta)\frac{P\varphi(y')}{\varphi(y)}\right] \\
&\quad + P\left[|\eta'|^{\alpha+1} + \alpha|\eta y'/y|^{\alpha+1} - (\alpha+1)\eta'\varphi(\eta y'/y)\right].
\end{aligned} \tag{11}$$

If both $y(a) \neq 0$ and $y(b) \neq 0$, then integrating (11) from a to b and using Lemma 2 we obtain

$$\int_a^b \left[P(t)|\eta'(t)|^{\alpha+1} - Q(t)|\eta(t)|^{\alpha+1}\right] dt \geq 0$$

which is the desired inequality (10).

If $y(a) = 0$, then due to the fact that zeros of nontrivial solutions of second-order half-linear equations are simple (see, for example, [**14**, Lemma 2.3]) $y'(a)$ must be a nonzero finite value. Since, obviously, $\lim_{t\to a+} P(t)\eta(t)\varphi(y'(t)) = 0$ and also

$$\lim_{t\to a+} \varphi(\eta(t)/y(t)) = \varphi(\lim_{t\to a+} \eta'(t)/y'(t)) < \infty$$

by l'Hospital rule, we have

$$\lim_{t\to a+} P(t)\eta(t)\frac{\varphi(y'(t))\varphi(\eta(t))}{\varphi(y(t))} = 0.$$

Similarly

$$\lim_{t\to b-} P(t)\eta(t)\frac{\varphi(y'(t))\varphi(\eta(t))}{\varphi(y(t))} = 0,$$

if $y(b) = 0$.

Thus, integrating (11) over the interval $[a+\epsilon, b-\epsilon]$, letting $\epsilon \to 0+$ and using Lemma 2, we again obtain (10).

Obviously, equality in (10) holds if and only if

$$|\eta'|^{\alpha+1} + \alpha|\eta y'/y|^{\alpha+1} - (\alpha+1)\eta'\varphi(\eta y'/y) \equiv 0$$

which according to Lemma 2 is possible only if $\eta' \equiv \eta y'/y$, or equivalently, if η is a constant multiple of y. □

From Theorem 1 we immediately have the following Corollary which is a straightforward extension of variational Lemma 1.3 from [**17**] valid for linear second order equations to the case of half-linear equations.

Corollary 1. *If there exists an $\eta \in U$ such that*

$$J_\alpha[\eta] \le 0 \tag{12}$$

then every solution y of $L_\alpha[y] = 0$ has a zero in (a, b) except possibly when $y = c\eta$ for some nonzero constant c.

Now, along with the equation

$$L_\alpha[y] = 0 \tag{A}$$

consider also the equation

$$l_\alpha[x] = 0 \tag{B}$$

and define

$$V_\alpha[\eta] = \int_a^b \left[(p(t) - P(t))|\eta'(t)|^{\alpha+1} + (Q(t) - q(t))|\eta(t)|^{\alpha+1}\right] dt, \quad \eta \in U.$$

The following comparison theorem is the main result in this section.

Theorem 2 (Leighton-type comparison theorem). *If there exists an $x \in U$ such that $l_\alpha[x] = 0$ and*

$$V_\alpha[x] \ge 0 \tag{13}$$

then every solution y of (A) has a zero in (a, b) except possibly it is a constant multiple of x.

Proof. Assume for the sake of contradiction that Eq. (A) has a solution which is nonzero on (a, b). Then from the Picone's identity (8) it follows that

$$\begin{aligned} \frac{d}{dt}\left\{\frac{x}{\varphi(y)}[\varphi(y)p\varphi(x') - \varphi(x)P\varphi(y')]\right\} &= (p - P)|x'|^{\alpha+1} + (Q - q)|x|^{\alpha+1} \\ &\quad + P[|x'|^{\alpha+1} + \alpha|xy'/y|^{\alpha+1} - (\alpha+1)x'\varphi(xy'/y)] \end{aligned} \tag{14}$$

where we have used that x and y are solutions of (B) and (A), respectively.

As in the proof of Theorem 1 we can show that the function $(x/\varphi(y))[\varphi(y)p\varphi(x') - \varphi(x)P\varphi(y')]$ tends to zero as $t \to a+$ or $t \to b-$, regardless $y(a) = 0$ or $y(a) \neq 0$ ($y(b) = 0$ or $y(b) \neq 0$). Thus, integrating (14) from $a + \varepsilon$ to $b - \varepsilon$, letting $\varepsilon \to 0+$ and using Lemma 2 we obtain

$$V_\alpha \leq 0$$

which contradicts (13) except possibly $V_\alpha[x] = 0$ that corresponds to the case where $p \equiv P$, $q \equiv Q$ and $x' \equiv xy'/y$, i.e. x is a constant multiple of y. $\square$

Corollary 2 (Sturm-Picone comparison theorem). *If*

$$p(t) \geq P(t) > 0 \tag{15}$$

and

$$Q(t) \geq q(t) \tag{16}$$

on a given interval I and there exists an $x \in U$ such that $l_\alpha[x] = 0$, then any solution of Eq. (A) either has a zero in (a, b) or it is a constant multiple of x.

Remark 1. If, in addition to (15) and (16) in Corollary 2, we suppose that on any non-degenerate subinterval I_0 of I neither (15) nor (16) becomes an identity, then the later possibility is excluded and any solution y of (A) must vanish in (a, b).

Example 1. Along with the half-linear equation

$$(|x'|^{\alpha-1}x')' + \alpha|x|^{\alpha-1}x = 0, \qquad \alpha > 0, \tag{17}$$

which is a natural extension of the linear harmonic oscillator equation consider the generalized Airy's equation

$$(|y'|^{\alpha-1}y')' + \beta t^\gamma |y|^{\alpha-1}y = 0, \qquad t \geq 0, \tag{18}$$

where α, β and γ are positive constants.

The first equation has the generalized sine function $S(t)$ as an oscillatory solution satisfying the initial conditions $S(0) = 0$ and $S'(0) = 1$. The function $S(t)$ has the properties

$$|S(t)|^{\alpha+1} + |S'(t)|^{\alpha+1} = 1 \quad \text{and} \quad S(t + \pi_\alpha) = -S(t)$$

for all $t \in R$ where

$$\pi_\alpha = \frac{\frac{2\pi}{\alpha+1}}{\sin \frac{\pi}{\alpha+1}}$$

(see [**1**]). As easily seen, for any $k \in N$ the function $x_k(t) = S(kt)$ is a solution of the equation

$$(|x'|^{\alpha-1}x')' + \alpha k^{\alpha+1}|x|^{\alpha-1}x = 0$$

which has an infinite sequence of zero points $\frac{n\pi_\alpha}{k}$, $n = 1, 2, \ldots$.

Since for any $k \in N$ there exists a T_k such that $\beta t^\gamma > \alpha k^{\alpha+1}$ for $t \geq T_k$, an application of Corollary 2 yields that any nontrivial real solution y of the equation (18) has a zero in the interval $\left(\frac{n\pi_\alpha}{k}, \frac{(n+1)\pi_\alpha}{k}\right)$ if $n \in N$ is so large that $\frac{n\pi_\alpha}{k} > T_k$. Consequently, all nontrivial solution of the Airy's equation (18) are oscillatory and the distance $t_{n+1} - t_n$ between consecutive zeros of any solution tends to 0 as $n \to \infty$.

Example 2. Consider a pair of half-linear equations

$$\left(|x'|^{\alpha-1}x'\right)' + \alpha\mu^{\alpha+1}|x|^{\alpha-1}x = 0 \tag{19}$$

and

$$\left(P(t)|y'|^{\alpha-1}y'\right)' + Q(t)|y|^{\alpha-1}y = 0 \tag{20}$$

where $\alpha > 0$, $\mu = n\pi_\alpha/(b-a)$ for some given natural number n and P and Q are continuous functions defined on the interval $[a, b]$ with $P(t) > 0$ on $[a, b]$.

The equation (19) has the function $x(t) = S(\mu(t-a))$ where $S(t)$ is the generalized sine function defined in Example 1, as a solution having exactly $(n+1)$ zeros

$$t_0 = a, \quad t_k = \frac{b-a}{n-k+1} + a, \quad k = 1, \ldots, n,$$

in the interval $[a, b]$. Denote $P^* = \max\{P(t) : t \in [a, b]\}$. Then the equation

$$\left(P^*|x'|^{\alpha-1}x'\right)' + \alpha P^*\mu^{\alpha+1}|x|^{\alpha-1}x = 0 \tag{21}$$

has the same function $x(t) = S(\mu(t-a))$ as a solution and since obviously

$$P(t) \leq P^*$$

from Corollary 2 it follows that any solution y of Eq. (20) has at least n zeros in the interval (a, b) if

$$Q(t) > \alpha P^*\mu^{\alpha+1}. \tag{22}$$

The next comparison result is an alternative to Theorem 2.

Theorem 3. *Let $P/p \in C^1(a,b)$. Then if there exists an $x \in U$ such that $l_\alpha[x] = 0$ and*

$$\int_a^b \left[(Q - \frac{P}{p}q)|x|^{\alpha+1} + p\left(\frac{P}{p}\right)' x\varphi(x') \right] dt > 0 \tag{23}$$

every solution y of Eq. (A) has a zero in (a,b).

Proof. Putting $p \equiv P$ on the left-hand side of (8) and rewriting (B) as

$$\left(\frac{p}{P}P|x'|^{\alpha-1}x'\right)' + q|x|^{\alpha-1}x = 0 \tag{24}$$

or, equivalently,

$$(P\varphi(x'))' - p\left(\frac{P}{p}\right)'\varphi(x') + \frac{P}{p}q\varphi(x) = 0, \tag{25}$$

the identity (8) becomes

$$\begin{aligned} \frac{d}{dt}\left\{\frac{x}{\varphi(y)}[\varphi(y)P\varphi(x') - \varphi(x)P\varphi(y')]\right\} &= \left(Q - \frac{P}{p}q\right)|x|^{\alpha+1} \\ + p\left(\frac{P}{p}\right)' x\varphi(x') &+ P\left[|x'|^{\alpha+1} + \alpha|xy'/y|^{\alpha+1} - (\alpha+1)x'\varphi(xy'/y)\right]. \end{aligned} \tag{26}$$

Again, as in the proof of Theorem 1, we can show that the function on the left-hand side inside {} tends to zero as $t \to a+$ or $t \to b-$, so that after integrating (26) from $a+\epsilon$ to $b-\epsilon$, letting $\epsilon \to 0+$ and using Lemma 2 we are led to the contradiction with (23). □

Similarly, assuming that $q \neq 0$, $Q \neq 0$ and $Q/q \in C'(a,b)$ we can rewrite $(p\varphi(x'))'$ as

$$\frac{q}{Q}\left(\frac{pQ}{q}\varphi(x')\right)' + \left(\frac{q}{Q}\right)'\frac{pQ}{q}\varphi(x')$$

so that (B) becomes

$$\left(\frac{pQ}{q}\varphi(x')\right)' + \frac{Q}{q}\left(\frac{q}{Q}\right)'\frac{Q}{q}p\varphi(x') + Q\varphi(x) = 0$$

or, equivalently,

$$\left(\frac{pQ}{q}\varphi(x')\right)' - \left(\frac{Q}{p}\right)' p\varphi(x') + Q\varphi(x) = 0 \tag{27}$$

so that the coefficients in front of $\varphi(x)$ and $\varphi(y)$ in (27) and (A), respectively, are the same.

From (8) with $p \equiv pQ/q$ we then have

$$\frac{d}{dt}\left\{\frac{x}{\varphi(y)}[\varphi(y)\frac{pQ}{q}\varphi(x') - \varphi(x)P\varphi(y')]\right\} = \left(\frac{pQ}{q} - P\right)|x'|^{\alpha+1} \tag{28}$$
$$+ p\left(\frac{Q}{q}\right)' x\varphi(x') + P\Big[|x'|^{\alpha+1} + \alpha|xy'/y|^{\alpha+1} - (\alpha+1)x'\varphi(xy'/y)\Big]$$

and integrating it from $a+\epsilon$ to $b-\epsilon$, letting $\epsilon \to 0+$ and using Lemma 2 we have proven the following dual comparison result to Theorem 3.

Theorem 4. *Let $q \neq 0$, $Q \neq 0$ and $Q/q \in C^1[a,b]$. If there exists an $x \in U$ such that $l_\alpha[x] = 0$ and*

$$\int_a^b \left[\left(\frac{pQ}{q} - P\right)|x'|^{\alpha+1} + p\left(\frac{Q}{q}\right)' x\varphi(x')\right] dt > 0, \tag{29}$$

then every solution y of (A) has a zero in (a,b).

3. Forced Oscillations

In this section we consider along with the half-linear equation (B) the forced second order differential equation

$$(P(t)\varphi(y'))' + f(t,y) = h(t) \tag{30}$$

where $P, h\colon [t_0,\infty) \to R$ and $f\colon [t_0,\infty)\times R \to R$ are continuous and $P(t) > 0$ for $t \geq t_0$.

By a solution of (30) we mean a function $y\colon [t_0,\infty) \to R$ which is continuously differentiable together with $P(t)\varphi(y')$ and satisfies (30) on $[t_0,\infty)$. A non-trivial solution of (30) is called **oscillatory** if there exists a sequence of zeros clustering at $t = \infty$; otherwise a solution is called **nonoscillatory**.

In the sequel, by **consecutive sign change points** of a function $h \in C([t_0,\infty),R)$ we will understand points $t_1, t_2 \in [t_0,\infty)$ such that $t_1 < t_2$ and there exists an $\varepsilon > 0$ such that $h(t) \geq 0$ (resp. $h(t) \leq 0$) on $[t_1,t_2]$ and $h(t) < 0$ (resp. $h(t) > 0$) on $(t_1-\varepsilon, t_1) \cup (t_2, t_2+\varepsilon)$.

In the case $\alpha = 1$, it has been shown by several authors including [**9**] that all nontrivial solutions of the equation

$$(P(t)y')' + f(t,y) = h(t), \qquad t \geq t_0, \tag{31}$$

where $P\colon [t_0,\infty) \to (0,\infty)$, $h\colon [t_0,\infty) \to R$ and $f\colon [t_0,\infty)\times R \to R$ are continuous and such that

$$\int_{t_0}^{\infty} [P(t)]^{-1}\, dt = \infty,$$

$yf(t,y) > 0$ for $y \neq 0$ and f is non-decreasing in the second variable, are oscillatory if and only if the corresponding unforced equation is oscillatory and there exists a continuous oscillatory function $\rho\colon [t_0,\infty) \to R$ such that

$$(P(t)\rho'(t))' = h(t) \tag{32}$$

and

$$\lim_{t\to\infty} \rho(t) = 0. \tag{33}$$

The idea of employing the function ρ satisfying (32) and (33) cannot be used for Eq. (30) because the second-order differential operator $L_\alpha[u] = (P(t)\varphi(u'))'$ does not have, in general, the additivity property so that we cannot consider a function $z(t)$ defined by $z(t) = y(t) - \rho(t)$ as a solution of the perturbed equation without forcing term. The technique used in proving our comparison theorem that enables us to deduce the oscillatory character of the forced equation (30) from oscillation of solutions of the unforced half-linear second order equation (1) is based on a Picone-type identity developed in Section 1. Roughly speaking, the result says that the oscillatory nature of the second order half-linear equation is maintained when an oscillatory forcing term is added to the equation provided that the distance between consecutive sign change points of the forcing function is greater that the distance between consecutive zeros of any solution of the equation without forcing term. As a consequence, the generalized harmonic oscillator

$$(\varphi(y'))' + \alpha\varphi(y) = 0 \tag{34}$$

remains to be oscillatory equation if the external force h with a distance between consecutive sign change points greater than

$$\frac{\frac{2\pi}{\alpha+1}}{\sin\frac{\pi}{\alpha+1}}$$

is added to the right-hand side.

Theorem 5. *Assume that*

$$p(t) \geq P(t) \quad \textit{for} \quad t \geq t_0, \tag{34}$$

$$u[f(t,u) - q(t)\varphi(u)] \geq 0 \quad \textit{for} \quad u \neq 0 \quad \textit{and} \quad t \geq t_0 \tag{35}$$

and either (34) *or* (35) *does not become an identity on any open interval where* $h(t) \equiv 0$. *Moreover, suppose that Eq.* (1) *is oscillatory and the distance between consecutive zeros of any solution of* (1) *is less than the distance between consecutive* *sign change points of the forcing function $h(t)$. Then every nontrivial solution of Eq.* (30) *is oscillatory.*

Proof. Assume for the sake of contradiction that Eq. (30) has a nonoscillatory solution $y(t)$ defined on some interval $[T_y, \infty), T_y \geq t_0$. Without loss of generality we may assume that $y(t)$ is positive for $t \geq T_y$. Now, let $t_2 > t_1 \geq T_y$ be consecutive sign change points of the forcing function $h(t)$ such that $h(t) \leq 0$ on $[t_1, t_2]$. According to the fundamental existence-uniqueness theorem for second order half-linear equations (see [**1**]), there exists a solution $x(t)$ of Eq. (1) satisfying $x(t_1) = 0$ and $x'(t_1) > 0$. Since according to the assumptions of Theorem 5, all solutions of Eq. (1) are oscillatory and the distance between consecutive zeros of any of these solutions is less than the distance between t_1 and t_2, there exists a t_3 such that $t_1 < t_3 \leq t_2$ and $x(t_3) = 0$. Integrating over $[t_1, t_3]$, using the Picone identity

$$\begin{aligned}(36)\qquad \frac{d}{dt}\left\{\frac{x}{y^\alpha}[y^\alpha p\varphi(x') - \varphi(x)P\varphi(y')]\right\} &= (p-P)|x'|^{\alpha+1} \\ &+ \frac{|x|^{\alpha+1}}{y^\alpha}[f(t,y(t)) - q(t)y^\alpha] + P[|x'|^{\alpha+1} + \alpha|xy'/y|^{\alpha+1} \\ &- (\alpha+1)x'\varphi(xy'/y)] - \frac{|x|^{\alpha+1}}{y^\alpha}h(t)\end{aligned}$$

and taking Lemma 2 with $X = x'(t)$ and $Y = x(t)y'(t)/y(t)$ into account, we obtain a contradiction. $\square$

Remark. In the above Theorem 5 we needed to know the distance between consecutive zeros of any solution of Eq. (1) explicitly. However, for the half-linear equations which possess only solutions that are quickly oscillatory in the sense that the distance $|t_{n+1} - t_n|$ between consecutive zero points t_n and t_{n+1} of any solution tends to zero as $n \to \infty$, we only need to assume that the forcing term $h(t)$ is a moderately oscillating function, i.e. the distance between any two consecutive sign change points of h is greater than some positive constant c. Then, as an immediate consequence of Theorem 5, we have the following

Corollary 3. *If* (34) *and* (35) *hold, the forcing $h(t)$ is moderately oscillating function and every solution of Eq.* (1) *is quickly oscillatory, then the equation* (30) *is oscillatory, too.*

Example 3. Consider the equation

$$(37)\qquad (|y'|^{\alpha-1}y')' + \alpha|y|^{\alpha-1}y = h(t)$$

where $\alpha > 0$. The unforced equation

$$(38)\qquad (|x'|^{\alpha-1}x')' + \alpha|x|^{\alpha-1}x = 0$$

is the generalized harmonic oscillator considered in Example 1 with the generalized sine function $S(t)$ as a solution satisfying the initial conditions $x(0) = 0$ and $x'(0) = 1$.

It is easy to see that as a solution x of (38) in the proof of Theorem 5 satisfying $x(t_1) = 0$ and $x'(t_1) > 0$ we can take

$$x(t) = kS(t - t_1)$$

where k is arbitrary positive constant. So that it is sufficient to assume that the distance between any two consecutive sign change points of h is greater than π_α and we have the conclusion of Theorem 5, i.e. all nontrivial solutions of Eq. (37) are oscillatory.

As examples of such functions h we give

$$\sin \beta t \quad (0 < \beta < \pi/\pi_\alpha), \quad \sin(\ln t), \quad S(\gamma t) \quad (0 < \gamma < 1), \quad S(\ln t).$$

In our second result in this section we will prove that all solutions of the forced equation (30) are oscillatory regardless of oscillation or nonoscillation of the corresponding unforced equation if the forcing term h oscillates and its amplitude is sufficiently large in the sense specified below. The result will be formulated and proved for slightly more general forced equations than (1), namely, for equations of the form

$$(P(t)\psi(y'(t)))' + f(t, y(t)) = h(t), \quad t \geq t_0, \tag{39}$$

where ψ is not neccessarily the nonlinearity of Emden-Fowler type considered above. We assume that:

(a) $P\colon [t_0, \infty) \to (0, \infty)$ is continuous;
(b) $\psi\colon R \to R$ is continuous, strictly increasing and such that sgn $\psi(u) =$ sgn u and $\psi(R) = R$;
(c) $f\colon [t_0, \infty) \times R \to R$ is continuous and such that sgn $f(t, v) =$ sgn v for each fixed $t \geq t_0$;
(d) $h\colon [t_0, \infty) \to R$ is oscillatory.

Theorem 6. *Suppose that*

$$\liminf_{t\to\infty} \left\{ k + \int_T^t \psi^{-1}\Big[\frac{1}{P(s)}\Big(l + \int_T^s h(\sigma)d\sigma\Big)\Big]\, ds \right\} < 0 \tag{40}$$

and

$$\limsup_{t\to\infty} \left\{ -k + \int_T^t \psi^{-1}\Big[\frac{1}{P(s)}\Big(l + \int_T^s h(\sigma)d\sigma\Big)\Big]\, ds \right\} > 0 \tag{41}$$

for all sufficiently large $T \geq t_0$ and all constants k and l with $k > 0$, where ψ^{-1} denotes the inverse function of ψ. Then all nontrivial solutions of Eq. (39) *are oscillatory.*

Proof. Let $y(t)$ be a positive solution of (39) defined on $[t_1, \infty), t_1 \geq t_0$. From (39) and (c) we have

$$(P(t)\psi(y'(t)))' = h(t) - f(t, y(t)) \leq h(t), \quad t \geq t_1.$$

Integrating the above inequality from t_1 to t, we get

$$P(t)\psi(y'(t)) \leq c_1 + \int_{t_1}^{t} h(s)\,ds, \tag{42}$$

where $c_1 = P(t_1)\psi(y'(t_1))$. Dividing (42) by $P(t)$, taking the inverse function ψ^{-1} of ψ on both sides of (42) and integrating the resulting inequality from t_1 to t again, we obtain

$$y(t) \leq c_2 + \int_{t_1}^{t} \psi^{-1}\Big[\frac{1}{P(s)}\Big(c_1 + \int_{t_1}^{s} h(\sigma)d\sigma\Big)\Big]\,ds,$$

where $c_2 = y(t_1)$. Taking the lower limit as $t \to \infty$ and using (40), we get the contradiction with the assumption that $y(t)$ is positive eventually.

The proof in the case that $y(t)$ is negative on $[t_1, \infty)$ is similar and is omitted.□

Example 4. Consider the equation

$$((y')^{1/3})' + q(t)y^{1/3} = t\sin t, \quad t \geq t_0, \tag{43}$$

where $q\colon [t_0, \infty) \to (0, \infty)$ is a continuous function. The equation (43) is a special case of (39) with

$$P(t) \equiv 1, \quad \psi(u) = u^{1/3}, \quad f(t, v) = q(t)v^{1/3}, \quad h(t) = t\sin t.$$

The conditions (40) and (41) written for (43) read

$$\liminf_{t\to\infty} \Big\{k + \int_T^t \Big(l + \int_T^s \sigma\sin\sigma\,d\sigma\Big)^3 ds\Big\} < 0$$

and

$$\limsup_{t\to\infty} \Big\{-k + \int_T^t \Big(l + \int_T^s \sigma\sin\sigma\,d\sigma\Big)^3 ds\Big\} > 0$$

for all constants $k > 0$ and l, respectively. These conditions are clearly satisfied since

$$\liminf_{t\to\infty} \int_T^t \Big(l + \int_T^s \sigma\sin\sigma\,d\sigma\Big)^3 ds = -\infty$$

and

$$\limsup_{t\to\infty} \int_T^t \Big(l + \int_T^s \sigma \sin \sigma \, d\sigma\Big)^3 ds = +\infty.$$

Therefore, by Theorem 6, all solutions of (43) are oscillatory as long as the coefficient q is continuous and positive on $[t_0, \infty)$.

It is known [**5**], [**10**] that all solutions of the unforced equation

$$((x')^{1/3})' + q(t)x^{1/3} = 0$$

are oscillatory if

$$\liminf_{t\to\infty} t^{1/3} \int_t^\infty q(s)\, ds > 3\Big(\frac{1}{4}\Big)^{4/3}$$

and nonoscillatory if

$$\limsup_{t\to\infty} t^{1/3} \int_t^\infty q(s)\, ds < 3\Big(\frac{1}{4}\Big)^{4/3}.$$

The qualitative study of half-linear differential equations of the type

$$\text{(B)} \qquad (p(t)|x'|^{\alpha-1}x')' + q(t)|x|^{\alpha-1}x = 0$$

was initiated by Elbert [**1**] and Mirzov [**15**] and followed by several authors including Kusano et al. [**5**], [**6**] and Li and Yeh [**14**]. Their study shows a surprising similarity between the qualitative properties of solutions of (B) and those of the linear equation

$$(p(t)x')' + q(t)x = 0.$$

It is natural to expect that the qualitative behaviour of the Emden-Fowler equation

$$(p(t)x')' + q(t)|x|^{\beta-1}x = 0 \qquad (\beta > 0,\ \beta \neq 1)$$

is similar to that of its generalization

$$(p(t)|x'|^{\alpha-1}x')' + q(t)|x|^{\beta-1}x = 0, \qquad (\alpha > 0,\ \beta > 0,\ \alpha \neq \beta).$$

For attempts confirming the truth of this expectation the reader is referred to the papers [**2**], [**7**], [**11**] and [**12**].

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