Acta Math. Univ. Comenianae
Vol. LXVIII, 1(1999), pp. 49–70

A TOPOLOGICAL REPRESENTATION OF POLARITY LATTICES

G. HARTUNG, M. KAMARA AND C. SĂCĂREA

1. Introduction

In [**Ha92**] a representation of bounded lattices within so called topological contexts has been developed, representation which gives rise to a duality between the category of bounded lattices with onto homomorphisms and the category of standard topological contexts with the so-called standard embeddings. This representation includes Stone's representation theorem of Boolean algebras by totally disconnected compact spaces [**St37**], Priestley's representation of distributive 0-1-lattices by totally order-disconnected spaces [**Pr70**] as well as Urquhart's representation of arbitrary lattices by so-called L-spaces [**Ur78**].

This duality was extended in [**Ha93**] to arbitrary 0-1-lattice homomorphisms while the appropriate morphisms in the category of standard topological contexts were defined using the idea of multivalued functions [**HW81**].

In the present paper we consider bounded lattices with an additional unary operation called polarity, generalizing orthocomplementation, and we give a representation by standard topological contexts where the polarity operation is captured in a convenient matter.

We characterize the congruence lattice of a polarity lattice within its standard topological polarity context, obtaining in a diferent way the well known result that the congruence lattice of a polarity lattice is distributive. As another application, we prove Varlet's conjecture, that a distributive polarity lattice has a Boolean congruence lattice if and only if the given polarity lattice is finite.

Further we prove that in the polarity case, the representation of polarity lattices by polarity contexts captures the representation given by Urquhart in [**Ur79**].

We extend then the duality to arbitrary 0-1 polarity lattice homomorphisms recapturing the polarity operation within the standard topological context as a multivalued polarity pair.

Received July 23, 1997; revised March 15, 1999.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 06B15; Secondary 54H10.

2. Preliminaries

A **polarity lattice** is a pair (L,p) where L is a bounded lattice and p is a unary operation on L satisfying

(1) $p^2 = \mathrm{id}$;
(2) $p(x \vee y) = p(x) \wedge p(y)$;
(3) $p(x \wedge y) = p(x) \vee p(y)$, for all $x, y \in L$.

We call a map $\phi\colon (L,p) \to (M,q)$ between polarity lattices a **polarity homomorphism** if ϕ is a 0-1-lattice homomorphism which in addition fulfills

$$\phi(p(x)) = q(\phi(x)).$$

We briefly sketch the duality between bounded lattices and standard topological contexts developed in [**Ha92**]. This approach is based on the theory of Formal Concept Analysis. We recall some definitions and basic facts, for other definitions and results we refer to [**GW96**].

By (X,τ) we denote a topological space, where X is the underlying set and $\mathcal{T}$ is the family of all closed sets of that space. We start with a triple $\mathbb{K}^{\mathcal{T}} := ((G,\rho),(M,\sigma),I)$ consisting of two topological spaces $(G,\rho),(M,\sigma)$ and a binary relation $I \subseteq G \times M$.

For $A \subseteq G$ and $B \subseteq M$ we define two derivations by

$$A' := \{m \in M \mid gIm \text{ for all } g \in A\}$$
$$B' := \{g \in G \mid gIm \text{ for all } m \in B\}.$$

These form a Galois-connection which gives rise to a complete lattice

$$\underline{\mathfrak{B}}(\mathbb{K}^{\mathcal{T}}) := \{(A,B) \mid A \subseteq G,\ B \subseteq M,\ A' = B,\ B' = A\}$$

which is known as the **concept lattice** of the **context** $\mathbb{K}^{\mathcal{T}}$. The elements of $\underline{\mathfrak{B}}(\mathbb{K}^{\mathcal{T}})$ are called (formal) **concepts**. If (A,B) is a concept of $\mathbb{K}^{\mathcal{T}}$, the sets A and B are called the **extent** and the **intent** of the concept (A,B). For two concepts the relation subconcept-superconcept is given by

$$(A,B) \le (C,D) \Leftrightarrow A \subseteq B \ (\Leftrightarrow B \supseteq B).$$

A **closed concept** is a concept consisting in each component of a closed set with respect to the corresponding topology. The set of all closed concepts is denoted by

$$\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) := \{(A,B) \in \underline{\mathfrak{B}}(\mathbb{K}^{\mathcal{T}}) \mid A \in \rho \text{ and } B \in \sigma\}.$$

The triple $\mathbb{K}^{\mathcal{T}} := ((G,\rho),(M,\sigma),I)$ is called a **topological context** if the following two conditions are satisfied:

(i) $A \in \rho \Rightarrow A'' \in \rho; B \in \sigma \Rightarrow B'' \in \sigma$.
(ii) $\mathcal{S}_\rho := \{A \subseteq G \mid (A,A') \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})\}$ is a subbasis of ρ and $\mathcal{S}_\sigma := \{B \subseteq M \mid (B,B') \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})\}$ is a subbasis of σ.

Under these assumptions, $\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with the induced order is a 0-1-lattice in which infima and suprema can be described as follows

$$(A_1, B_1) \wedge (A_2, B_2) = (A_1 \cap A_2, (B_1 \cup B_2)'');$$
$$(A_1, B_1) \vee (A_2, B_2) = ((A_1 \cup A_2)'', B_1 \cap B_2).$$

For each $g \in G$, the concept $\gamma g := (g'', g')$ is called the **object concept** of G and for each $m \in M$ the concept $\mu m := (m', m'')$ is called the **attribute concept** of m. We call a context **clarified** if $g, h \in G$ with $g' = h'$ implies $g = h$ and $m, n \in M$ with $m' = n'$ implies $m = n$. A clarified context is called **reduced** if each object concept is completely join-irreducible and each attribute concept is completely meet-irreducible. For each context $\mathbb{K} := (G, M, I)$, $g \in G$ and $m \in M$ we define:

$$g \swarrow m \Leftrightarrow g \not{I} m \text{ and } (g' \subset h' \Rightarrow m \in h');$$
$$g \nearrow m \Leftrightarrow g \not{I} m \text{ and } (m' \subset n' \Rightarrow g \in n');$$
$$g \swarrow\!\!\!\!\nearrow m \Leftrightarrow g \swarrow m \text{ and } g \nearrow m.$$

We call two contexts $\mathbb{K}_1$ and $\mathbb{K}_2$ **isomorphic** if there are bijective maps $\alpha\colon G_1 \to G_2$ and $\beta\colon M_1 \to M_2$ such that for all $g \in G_1$ and $m \in M_1$ the following condition is fulfilled:

$$g I_1 m \Leftrightarrow \alpha(g) I_2 \beta(m).$$

If $(L, \leq)$ is a lattice we shall denote by L^δ the dual of this lattice, i.e., the lattice $(L, \geq)$. If $\mathbb{K} := (G, M, I)$ is a context, by $\mathbb{K}^d$ we mean the context obtained by permuting G and M and having as incidence the inverse of I, namely $\mathbb{K}^d := (M, G, I^{-1})$.

For each $H \subseteq G$ and $N \subseteq M$, the context $(H, N, I \cap (H \times N))$ is called a **subcontext** of $\mathbb{K}$. This subcontext is **compatible** if $(A, B) \in \underline{\mathfrak{B}}(\mathbb{K})$ implies $(A \cap H, B \cap N) \in \underline{\mathfrak{B}}(H, N, I \cap (H \times N))$.

Proposition 1. *A subcontext* $(H, N, I \cap (H \times N))$ *of* $\mathbb{K}$ *is compatible if and only if*

$$\Pi_{H,N} \colon \underline{\mathfrak{B}}(\mathbb{K}) \to \underline{\mathfrak{B}}(H, N, I \cap (H \times N)) \text{ with } (A, B) \mapsto (A \cap H, B \cap N)$$

is a surjective complete lattice homomorphism.

A subcontext $(H, N, I \cap (H \times N))$ of a purified context $\mathbb{K}$ is called **arrow-closed** if for $h \in H$ the relation $h \swarrow m$ implies $m \in N$ and for $n \in N$ the relation $g \nearrow n$ implies $g \in H$.

A topological context is called a **standard topological context** if in addition the following hold:

(R) $\mathbb{K}^{\mathcal{T}}$ is reduced;

(S) $gIm \Rightarrow \exists (A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ with $g \in A$ and $m \in B$;

(Q) $(I^c, (\rho \times \sigma)_{|I^c})$ is a quasicompact space where $I^c := (G \times M) \backslash I$ and $\rho \times \sigma$ denotes the product topology on $G \times M$.

Let now L be a 0-1-lattice. A nonempty lattice filter F of L is called a **I-maximal filter** [**Ur78**] if there exists a nonempty lattice ideal I of L such that $F \cap I = \emptyset$ and every proper superfilter $\tilde{F} \supset F$ already contains an element of I. We denote the set of all I-maximal proper filters of L by $\mathfrak{F}_0(L)$. Dually the set $\mathfrak{I}_0(L)$ of all F-maximal ideals is introduced. The dual space of L, called the standard topological context of L is defined by

$$\mathbb{K}^{\mathcal{T}}(L) := ((\mathfrak{F}_0(L), \rho_0), (\mathfrak{I}_0(L), \sigma_0), \Delta)$$

where $F \Delta I :\Leftrightarrow F \cap I \neq \emptyset$ and the topologies ρ_0 and σ_0 are given by the subbasis

$$\mathcal{S}_{\rho_0} := \{F_a \mid a \in L\}; \quad F_a := \{F \in \mathfrak{F}_0(L) \mid a \in F\},$$
$$\mathcal{S}_{\sigma_0} := \{I_a \mid a \in L\}; \quad I_a := \{I \in \mathfrak{I}_0(L) \mid a \in I\}.$$

$\mathbb{K}^{\mathcal{T}}(L)$ is the reduced context of all filters and ideals of L and it is a standard topological context. The 0-1-lattice L is isomorphic to $\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$ via the isomorphism $\iota_A : L \to \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$; $\iota_A(a) = (F_a, I_a)$.
Conversely, every standard topological context $\mathbb{K}^{\mathcal{T}}$ is isomorphic to $\mathbb{K}^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ via the pair of homeomorphisms

$$\psi_{\mathbb{K}^{\mathcal{T}}} : G \to \mathfrak{F}_0(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \quad g \mapsto \{(A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \mid g \in A\},$$
$$\phi_{\mathbb{K}^{\mathcal{T}}} : M \to \mathfrak{I}_0(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})), \quad m \mapsto \{(A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \mid m \in B\}.$$

Let $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ be standard topological contexts. A pair of maps (α, β) with $\alpha : G_1 \to G_2$ and $\beta : M_1 \to M_2$ is called a **context embedding of $\mathbb{K}_1^{\mathcal{T}}$ into $\mathbb{K}_2^{\mathcal{T}}$** if the contexts $\mathbb{K}_1^{\mathcal{T}}$ and $((\alpha(G_1), \rho_{2|\alpha(G_1)}), (\beta(M_1), \sigma_{2|\beta(M_1)}), I_2 \cap (\alpha(G_1) \times \beta(M_1)))$ are isomorphic as topological contexts with respect to (α, β).

If $\mathbb{K}^{\mathcal{T}}$ is a topological context, a subcontext $((H, \rho_{|H}), N, \sigma_{|N}), I \cap H \times N))$ is called **weakly compatible** if

$$(A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \Rightarrow (A \cap H, B \cap N) \in \underline{\mathfrak{B}}(H, N, I \cap (H \times N)).$$

A context embedding (α, β) between two standard topological contexts $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ is called a **standard embedding of $\mathbb{K}_1^{\mathcal{T}}$ into $\mathbb{K}_2^{\mathcal{T}}$** if the following conditions are satisfied:

(a) $((\alpha(G_1), \rho_{2|\alpha(G_1)}), (\beta(M_1), \sigma_{2|\beta(M_1)}), I_2 \cap (\alpha(G_1) \times \beta(M_1)))$ is a weakly compatible subcontext of $\mathbb{K}_2^{\mathcal{T}}$;

(b) For $(A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ there exists $(C,D) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$ such that

$$(\alpha(A), \beta(B)) = ((C \cap \alpha(G_1)), (D \cap \beta(M_1))).$$

We briefly sketch the extended duality developed in [**Ha93**]. Since preimages of I-maximal filters (resp. ideals) are not maximal again, we have to define appropriate morphisms between standard topological contexts to improve a categorical dual equivalence between the category of bounded lattices and the category of standard topological contexts.

A multivalued function $F\colon X \to Y$ from a set X to a set Y is a binary relation $F \subseteq X \times Y$ such that $pr_X(F) = X$, where pr_X denotes the projection onto X. For $A \subseteq X$ and $B \subseteq Y$ we define

$$\begin{aligned} FA &:= pr_Y(F \cap (A \times Y)) = \{y \in Y \mid (a,y) \in F \text{ for some } a \in A\};\\ F^{-1}B &:= pr_X(F \cap (X \times B)) = \{x \in X \mid (x,b) \in F \text{ for some } b \in B\};\\ F^{[-1]}B &:= \{x \in X \mid Fx \subseteq B\}. \end{aligned}$$

Note that $FA = \bigcup_{a\in A} Fa$ and $F^{-1}B = \bigcup_{b\in B} F^{-1}b$. If $F\colon X \to Y$ and $G\colon Y \to Z$ are multivalued functions their relational product

$$G \circ F := \big\{(x,z) \in X \times Z \mid (x,y) \in F \text{ and } (y,z) \in G \text{ for some } y \in Y\big\}$$

is a multivalued function from X to Z.

We shall call a **multivalued standard morphism** from $\mathbb{K}_1^{\mathcal{T}}$ to $\mathbb{K}_2^{\mathcal{T}}$ a pair $(R,S)\colon \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$, where $\mathbb{K}_1^{\mathcal{T}}$ and $\mathbb{K}_2^{\mathcal{T}}$ are standard topological contexts, R is a multivalued function from G_1 to G_2 and S is a multivalued function from M_1 to M_2 satisfying the following conditions:

(i) $(R^{[-1]}A, S^{[-1]}B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})$ for every $(A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})$;

(ii) $Rg = Rg'' = \overline{Rg}$ for every $g \in G_1$ and $Sm = Sm'' = \overline{Sm}$ for every $m \in M_1$.

Every multivalued standard morphism induces a 0-1-lattice homomorphism and viceversa. In order to make this assignment functorial we have to modify the relational composition of multivalued standard morphisms, since the relational composition of two multivalued standard morphisms is not necessarily a multivalued standard morphism.

Let $(R_1,S_1)\colon \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$ and $(R_2,S_2)\colon \mathbb{K}_2^{\mathcal{T}} \to \mathbb{K}_3^{\mathcal{T}}$ be multivalued standard morphisms between standard topological contexts. We define

$$(R_2,S_2) \,\Box\, (R_1,S_1) := (R_2 \,\Box\, R_1, S_2 \,\Box\, S_1)$$

where

$$(R_2 \,\Box\, R_1)g_1 := ((R_2 \circ R_1)g_1)'' \text{ and } (S_2 \,\Box\, S_1)m_1 := ((S_2 \circ S_1)m_1)''$$

and $\circ$ denotes the relational product, i.e.,

$$\begin{aligned} (R_2 \circ R_1)g_1 &:= \{g_3 \in G_3 \mid g_3 \in R_2 g_2 \text{ for some } g_2 \in R_1 g_1\} \text{ and dually}\\ (S_2 \circ S_1)g_1 &:= \{m_3 \in M_3 \mid m_3 \in S_2 m_2 \text{ for some } m_2 \in S_1 m_1\}. \end{aligned}$$

The class of all standard topological contexts together with the multivalued standard morphisms with $\Box$ as composition builds up a category which is dually equivalent to the category of 0-1-lattices with 0-1-lattice homomorphisms.

3. The Polarity Operation

For every standard topological context, we call a pair (α, β) of mappings $\alpha\colon G \to M, \beta : M \to G$ a **polarity pair** if it satisfies the following conditions:

(i) α, β are homeomorphisms;
(ii) $gIm \Leftrightarrow \beta(m) I \alpha(g)$;
(iii) $\beta \circ \alpha = \mathrm{id}_G, \alpha \circ \beta = \mathrm{id}_M$.

Let (L, p) be a polarity lattice and let $\mathbb{K}^{\mathcal{T}}(L)$ be the standard topological context of L. Then the polarity p is captured in $\mathbb{K}^{\mathcal{T}}(L)$ as a pair of mappings (α_p, β_p) defined by

$$\alpha_p\colon \mathfrak{F}_0(L) \to \mathfrak{I}_0(L), \quad F \mapsto p(F),$$
$$\beta_p : \mathfrak{I}_0(L) \to \mathfrak{F}_0(L), \quad I \mapsto p(I).$$

Proposition 2. *For every polarity lattice (L, p), the pair of mappings (α_p, β_p) as defined above, is a polarity pair.*

Proof. By the duality between 0-1-lattices and standard topological contexts, since p can be viewed as an isomorphism between L and L^δ, (α_p, β_p) is the induced standard embedding between $\mathbb{K}^{\mathcal{T}}(L)$ and $\mathbb{K}^{\mathcal{T}^d}(L)$, which proves (i) and (ii).
(iii) is obvious since $p^2 = \mathrm{id}$. □

A standard topological context $\mathbb{K}^{\mathcal{T}}$ together with a polarity pair (α, β) is called a **standard topological polarity context** and denoted by $(\mathbb{K}^{\mathcal{T}}, (\alpha, \beta))$. Every such context establishes a polarity lattice.

Proposition 3. *Let $(\mathbb{K}^{\mathcal{T}}, (\alpha, \beta))$ be a standard topological polarity context. Then $(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}), p_{\alpha\beta})$ is a polarity lattice where*

$$p_{\alpha\beta}\colon \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \to \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}),$$
$$(A, B) \mapsto (\alpha^{-1}(B), \beta^{-1}(A)).$$

Proof. Since the polarity pair (α, β) acts as a standard embedding between $\mathbb{K}^{\mathcal{T}}$ and $\mathbb{K}^{\mathcal{T}^d}$, $p_{\alpha\beta}$ is an isomorphism between $\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ and $\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})^\delta$ and therefore

$$\begin{aligned} p_{\alpha\beta}((A, B) \vee (C, D)) &= p_{\alpha\beta}(A, B) \vee^\delta p_{\alpha\beta}(C, D) \\ &= p_{\alpha\beta}(A, B) \wedge p_{\alpha\beta}(C, D). \\ p_{\alpha\beta}((A, B) \wedge (C, D)) &= p_{\alpha\beta}(A, B) \wedge^\delta p_{\alpha\beta}(C, D) \\ &= p_{\alpha\beta}(A, B) \vee p_{\alpha\beta}(C, D). \end{aligned}$$

Moreover

$$\begin{aligned} p_{\alpha\beta}(p_{\alpha\beta}(A, B)) &= (\alpha^{-1}(\beta^{-1}(A)), \beta^{-1}(\alpha^{-1}(B))) \\ &= ((\beta \circ \alpha)^{-1}(A), (\alpha \circ \beta)^{-1}(B)) \\ &= (A, B). \end{aligned}$$

□

Polarity lattices correspond to the standard polarity contexts, as the following two propositions show.

Proposition 4. *For every polarity lattice (L, p), the following diagram*

$$\begin{tikzcd} L \arrow[r, "\iota_A"] \arrow[d, "p"'] & \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)) \arrow[d, "p_{\alpha_p\beta_p}"] \\ L \arrow[r, "\iota_A"'] & \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)) \end{tikzcd}$$

commutes.

Proof. For every $a \in L$ the following holds:

$$\begin{aligned}(p_{\alpha_p\beta_p} \circ \iota_A)(a) &= p_{\alpha_p\beta_p}(F_a, I_a)\\ &= (\alpha_p^{-1}(I_a), \beta_p^{-1}(F_a))\\ &= (F_{p(a)}, I_{p(a)})\\ &= (\iota_A \circ p)(a).\end{aligned}$$ □

Proposition 5. *For every standard topological polarity context $(\mathbb{K}^{\mathcal{T}}, (\alpha, \beta))$, the following diagram*

$$\begin{tikzcd} \mathbb{K}^{\mathcal{T}} \arrow[r, "(\psi_{\mathbb{K}^{\mathcal{T}}}, \phi_{\mathbb{K}^{\mathcal{T}}})"] \arrow[d, "(\alpha,\beta)"'] & \mathbb{K}^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})) \arrow[d, "(\alpha_{p_{\alpha\beta}}, \beta_{p_{\alpha\beta}})"] \\ {\mathbb{K}^{\mathcal{T}}}^d \arrow[r, "(\psi_{\mathbb{K}^{\mathcal{T}}}, \phi_{\mathbb{K}^{\mathcal{T}}})"'] & {\mathbb{K}^{\mathcal{T}}}^d(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})) \end{tikzcd}$$

commutes.

Proof. We have to prove the following relations:

$$\alpha_{p_{\alpha\beta}} \circ \psi_{\mathbb{K}^{\mathcal{T}}} = \phi_{\mathbb{K}^{\mathcal{T}}} \circ \alpha, \tag{1}$$

$$\beta_{p_{\alpha\beta}} \circ \phi_{\mathbb{K}^{\mathcal{T}}} = \psi_{\mathbb{K}^{\mathcal{T}}} \circ \beta. \tag{2}$$

Let g be an arbitrary object of the context $\mathbb{K}^{\mathcal{T}}$, then

$$\begin{aligned}(\alpha_{p_{\alpha\beta}} \circ \psi_{\mathbb{K}^{\mathcal{T}}})(g) &= {p_{\alpha\beta}}^{-1}(\{(A, B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \mid g \in A\})\\ &= \{(C, D) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \mid g \in \alpha^{-1}(D)\}\\ &= \{(C, D) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \mid \alpha(g) \in D\}\\ &= (\phi_{\mathbb{K}^{\mathcal{T}}} \circ \alpha)(g).\end{aligned}$$

Analogous arguments yields the second equality. □

4. Polarity Homomorphisms

The correspondence between polarity lattices and standard topological polarity contexts can be extended to a dual equivalence between categories. As morphisms between polarity lattices we restrict ourself to surjective polarity homomorphisms, since the preimage of a I-maximal filter (resp. F-maximal ideal) is no longer I-maximal if the 0-1-lattice homomorphism is not surjective.

So we are starting with a surjective polarity homomorphism

$$f\colon (L,p) \to (M,q).$$

By [**Ha92**, Prop. 3.1.5] the pair (α_f, β_f) where

$$\begin{aligned} \alpha_f\colon \mathfrak{F}_0(M) &\to \mathfrak{F}_0(L), \quad F \mapsto f^{-1}(F), \\ \beta_f\colon \mathfrak{I}_0(M) &\to \mathfrak{I}_0(L), \quad I \mapsto f^{-1}(I) \end{aligned}$$

is a standard embedding of $\mathbb{K}^{\mathcal{T}}(M)$ into $\mathbb{K}^{\mathcal{T}}(L)$.

We call a standard embedding (α, β) between two standard topological polarity contexts $(\mathbb{K}_1^{\mathcal{T}}, (\alpha_1, \beta_1))$ and $(\mathbb{K}_2^{\mathcal{T}}, (\alpha_2, \beta_2))$ (where $\alpha\colon G_1 \to G_2$ and $\beta\colon M_1 \to M_2$) a **standard polarity embedding** if

$$\text{(i)} \qquad (\alpha_2 \circ \alpha)(g) = (\beta \circ \alpha_1)(g), \text{ for all } g \in G_1,$$

$$\text{(ii)} \qquad (\beta_2 \circ \beta)(m) = (\alpha \circ \beta_1)(m), \text{ for all } m \in M_1.$$

Proposition 6. *For every surjective polarity homomorphism*

$$f\colon (L,p) \to (M,q)$$

the pair (α_f, β_f) defined as above is a standard polarity embedding.

Proof. We have to prove the following equations

$$\text{(i)} \qquad (\alpha_p \circ \alpha_f)(F) = (\beta_f \circ \alpha_q)(F), \text{ for all } F \in \mathfrak{F}_0(M),$$

$$\text{(ii)} \qquad (\beta_p \circ \beta_f)(I) = (\alpha_f \circ \beta_q)(I), \text{ for all } I \in \mathfrak{I}_0(M).$$

Let $F \in \mathfrak{F}_0(M)$, then

$$\alpha_p(\alpha_f(F)) = p^{-1}(f^{-1}(F)) = p(f^{-1}(F)).$$

On the other hand $\beta_f(\alpha_q(F)) = f^{-1}(q^{-1}(F)) = f^{-1}(q(F))$.

Now

$$\begin{aligned} a \in p(f^{-1}(F)) &\Rightarrow \exists x \in F : a \in p(f^{-1}(x)) \\ &\Rightarrow f(a) \in f(p(f^{-1}(x))) = q(f(f^{-1}(x))) \\ &= q(x) \in q(F) \Rightarrow a \in f^{-1}(q(F)). \end{aligned}$$

Conversely,

$$\begin{aligned} a \in f^{-1}(q(F)) &\Rightarrow \exists x \in F : f(a) = q(x) \\ &\Rightarrow f(p(a)) = q(f(a)) = q^2(x) = x \\ &\Rightarrow p(a) \in f^{-1}(x) \\ &\Rightarrow p^2(a) \in p(f^{-1}(x)) \subseteq p(f^{-1}(F)). \end{aligned}$$

Dually the second equation can be verified. □

Proposition 7. *For every standard polarity embedding (α, β) between two standard topological contexts $(\mathbb{K}_1^{\mathcal{T}}, (\alpha_1, \beta_1))$ and $(\mathbb{K}_2^{\mathcal{T}}, (\alpha_2, \beta_2))$, the map*

$$\begin{aligned} f_{\alpha\beta} : \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}) &\to \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}) \\ (A, B) &\mapsto (\alpha^{-1}(A), \beta^{-1}(B)) \end{aligned}$$

is a surjective polarity homomorphism.

Proof. By [**Ha92**, Prop. 3.1.6], $f_{\alpha\beta}$ is a surjective homomorphism. Moreover,

$$\begin{aligned} (f_{\alpha\beta} \circ p_{\alpha_2\beta_2})(A, B) &= f_{\alpha\beta}(\alpha_2^{-1}(B), \beta_2^{-1}(A)) \\ &= ((\alpha^{-1} \circ \alpha_2^{-1})(B), (\beta^{-1} \circ \beta_2^{-1})(A)) \\ &= ((\alpha_2 \circ \alpha)^{-1}(B), (\beta_2 \circ \beta)^{-1}(A)) \\ &= ((\beta \circ \alpha_1)^{-1}(B), (\alpha \circ \beta_1)^{-1}(A)) \\ &= ((\alpha_1^{-1} \circ \beta^{-1})(B), (\beta_1^{-1} \circ \alpha^{-1})(A)) \\ &= p_{\alpha_1\beta_1}(\alpha^{-1}(A), \beta^{-1}(B)) = (p_{\alpha_1\beta_1} \circ f_{\alpha\beta})(A, B). \end{aligned}$$ □

In order to establish a dual equivalence between the category of polarity lattices with surjective polarity homomorphisms as morphisms and the category of standard topological polarity contexts with standard polarity embeddings as morphisms, we have to show the commutativity of two more diagrams.

Proposition 8. *Let $f\colon (L, p) \to (M, q)$ be a surjective polarity homomorphism. Then $f \cong f_{\alpha_f\beta_f}$, i.e., the following diagram commutes:*

$$\begin{array}{ccc} (L, p) & \xrightarrow{\quad f \quad} & (M, q) \\ {\scriptstyle \iota_L}\big\downarrow & & \big\downarrow{\scriptstyle \iota_M} \\ (\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), p_{\alpha_p\beta_p}) & \xrightarrow[f_{\alpha_f\beta_f}]{} & (\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), p_{\alpha_q\beta_q}) \end{array}$$

Proof. For $a \in L$ we have

$$\begin{aligned} (\iota_M \circ f)(a) &= \iota_M(f(a)) = (F_{f(a)}, I_{f(a)}), \\ (f_{\alpha_f\beta_f} \circ \iota_L)(a) &= f_{\alpha_f\beta_f}((F_a, I_a)) = (\alpha_f^{-1}(F_a), \beta_f^{-1}(I_a)). \end{aligned}$$

Since $\alpha_f^{-1}(F_a) = F_{f(a)}$ and $\beta_f^{-1}(I_a) = I_{f(a)}$, the relation $f \cong f_{\alpha_f\beta_f}$ is proved. □

Proposition 9. *Let* $(\alpha,\beta)\colon (\mathbb{K}_1^{\mathcal{T}},(\alpha_1,\beta_1)) \to (\mathbb{K}_2^{\mathcal{T}},(\alpha_2,\beta_2))$ *be a standard polarity embedding between two standard polarity contexts. Then the following diagram commutes:*

$$\begin{tikzcd}
(\mathbb{K}_1^{\mathcal{T}},(\alpha_1,\beta_1)) \arrow[r, "(\alpha,\beta)"] \arrow[d, "(\psi_{\mathbb{K}_1^{\mathcal{T}}},\ \phi_{\mathbb{K}_1^{\mathcal{T}}})"'] & (\mathbb{K}_2^{\mathcal{T}},(\alpha_2,\beta_2)) \arrow[d, "(\psi_{\mathbb{K}_2^{\mathcal{T}}},\ \phi_{\mathbb{K}_2^{\mathcal{T}}})"] \\
(\mathbb{K}_1^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})),(\alpha_{p_{\alpha_1\beta_1}},\beta_{p_{\alpha_1\beta_1}})) \arrow[r, "(\alpha_{f_{\alpha\beta}},\beta_{f_{\alpha\beta}})"'] & (\mathbb{K}_2^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})),(\alpha_{p_{\alpha_2\beta_2}},\beta_{p_{\alpha_2\beta_2}}))
\end{tikzcd}$$

where $(\psi_{\mathbb{K}_1^{\mathcal{T}}},\ \phi_{\mathbb{K}_1^{\mathcal{T}}})$ *and* $(\phi_{\mathbb{K}_2^{\mathcal{T}}},\ \psi_{\mathbb{K}_2^{\mathcal{T}}})$ *are context isomorphisms as defined in Section* 2*.*

Proof. We have to prove the following two relations:

(i) $$\psi_{\mathbb{K}_2^{\mathcal{T}}} \circ \alpha = \alpha_{f_{\alpha\beta}} \circ \psi_{\mathbb{K}_1^{\mathcal{T}}},$$

(ii) $$\phi_{\mathbb{K}_2^{\mathcal{T}}} \circ \beta = \beta_{f_{\alpha\beta}} \circ \phi_{\mathbb{K}_2^{\mathcal{T}}}$$

We only show the first equality. The second is true by analogous arguments. For $g \in G_1$ we get

$$\begin{aligned}
(\psi_{\mathbb{K}_2^{\mathcal{T}}} \circ \alpha)(g) &= \psi_{\mathbb{K}_2^{\mathcal{T}}}(\alpha(g)) \\
&= \{(C,D) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}) \mid \alpha(g) \in C\}, \\
(\alpha_{f_{\alpha\beta}} \circ \psi_{\mathbb{K}_1^{\mathcal{T}}})(g) &= \alpha_{f_{\alpha\beta}}(\{(A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}) \mid g \in A\}) \\
&= f_{\alpha\beta}^{-1}(\{(A,B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}) \mid g \in A\}) \\
&= \{(C,D) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}) \mid g \in \alpha^{-1}(C)\} \\
&= \{(C,D) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}) \mid \alpha(g) \in C\}.
\end{aligned}$$ □

Summarizing the precedent discussion we get the following Theorem.

Representation Theorem. *The category of polarity lattices with surjective polarity homomorphisms is dually equivalent to the category of standard topological polarity contexts with standard polarity embeddings.*

5. Congruences

Now we are ready to describe the congruence lattice of a polarity lattice (L,p) within its standard topological polarity context $(\mathbb{K}^{\mathcal{T}}(L),(\alpha_p,\beta_p))$. As we have seen in the previous section, the surjective polarity homomorphisms with domain L and therefore the congruences of (L,p) correspond to special subcontexts of $\mathbb{K}^{\mathcal{T}}(L)$. In the following, we are going to characterize these subcontexts. Fixing a polarity lattice (L,p), each surjective polarity homomorphism $f\colon (L,p) \to (M,q)$ induces a subcontext of $\mathbb{K}^{\mathcal{T}}(L)$ which is defined by

$$\Pi_f := ((H_f,\rho_{0|H_f}),(N_f,\sigma_{0|N_f}),\Delta_f)$$

where $H_f := \alpha_f(\mathfrak{F}_0(M))$, $N_f := \beta_f(\mathfrak{I}_0(M))$ and $\Delta_f := \Delta \cap (H_f \times N_f)$.

Proposition 10. *Let f be a surjective polarity homomorphism with domain L. Then*

(i) Π_f *is arrow-closed;*
(ii) Δ_f^c *is quasicompact in* Δ^c*;*
(iii) Π_f *is* (α_p, β_p)*-closed, i.e.,* $F \in H_f$ *implies* $\alpha_p(F) \in N_f$ *and* $I \in N_f$ *implies* $\beta_p(I) \in H_f$*.*

Proof. (i), (ii) follow from [**Ha92**, Prop. 4.1], since f is a surjective lattice homomorphism.

To prove (iii) let $F \in H_f$. Then there is $\tilde{F} \in \mathfrak{F}_0(M)$ with $F = \alpha_f(\tilde{F})$. According to [**Ha92**, Prop. 4.1], we get

$$\begin{aligned}\alpha_p(F) &= \alpha_p(\alpha_f(\tilde{F})) = \beta_f(\alpha_q(\tilde{F}))\\ &= \beta_f(p(\tilde{F})) = \beta_f(\tilde{I}) \in N_f.\end{aligned}$$

Dually, $I \in N_f$ implies $\beta_p(I) \in H_f$. □

Conversely, if we consider a subcontext $\Pi := (H, N, \Delta_\Pi)$ of $\mathbb{K}^{\mathcal{T}}(L)$ having the properties (i)–(iii) of Proposition 5.1 then by [**Ha92**], (i) and (ii) guarantee the existence of a surjective lattice homomorphism

$$f_\pi \colon L \to \underline{\mathfrak{B}}^{\mathcal{T}}(\pi) \quad a \mapsto (F_a \cap H, I_a \cap N)$$

such that $\Pi = \Pi_{f_\pi}$. Moreover $(\Pi, (\alpha_{p_{|N}}, \beta_{p_{|N}}))$ is a standard topological polarity context by (iii).

Proposition 11. *Let* Π *be an arrow-closed,* (α_p, β_p)*-closed subcontext of* $(\mathbb{K}^{\mathcal{T}}(L), (\alpha_p, \beta_p))$ *having a quasicompact nonincidence. Then* f_π *is a surjective polarity homomorphism.*

Proof. It remains to show the polarity equation

$$f_\pi(p(a)) = \tilde{p}(f_\pi(a))$$

where $\tilde{p} := p_{\alpha_{p_{|H}}\beta_{p_{|N}}}$.

By definition $f_\pi(p(a)) = (F_{p(a)} \cap H, I_{p(a)} \cap N)$. Now we claim

(i) $F \in F_{p(a)} \cap H \Leftrightarrow F = p(I)$ for some $I \in I_a \cap N$.
(ii) $I \in I_{p(a)} \cap H \Leftrightarrow I = p(F)$ for some $F \in F_a \cap H$.

To prove (i) let $F \in F_{p(a)} \cap H$. Then $p(a) \in F$ and so $a \in p(F) = \alpha_p(F) \in I_a \cap N$, which implies $p(\alpha_p(F)) = p^2(F) = F$.

Conversely, let $I \in I_a \cap N$, it follows that $a \in I$. Applying p we have $p(a) \in p(I) = F \Rightarrow F \in F_{p(a)}$.

$I \in I_a \cap N \Rightarrow \beta_p(I) = p(I) = F \in H \Rightarrow F \in F_{p(a)} \cap H$.

Dually (ii) is proved. Hence

$$\begin{aligned} f_\pi(p(a)) &= (\{p(I)|\ I \in I_a \cap N\}, \{p(F)|\ F \in F_a \cap H\}) \\ &= (\beta_p(I_a \cap N), \alpha_p(F_a \cap H)) \\ &= (\beta_{p_{|N}}(I_a \cap N), \alpha_{p_{|H}}(F_a \cap H)) \\ &= \tilde{p}(f_\pi(a)). \end{aligned}$$ □

If we denote the set of all arrow-closed, (α_p, β_p)-closed subcontexts of $\mathbb{K}^{\mathcal{T}}(L)$ having a quasicompact nonincidence, endowed with the order

$$\Pi_1 \leq \Pi_2 \Leftrightarrow H_1 \subseteq H_2 \text{ and } N_1 \subseteq N_2$$

by $\mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(L))$, we can summarize the previous observations within

Theorem 12. *For every polarity lattice (L, p), the lattice $\mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(L))$ is anti-isomorphic to* $\operatorname{Con} L$.

Proposition 13. *$\mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(L))$ is a distributive lattice for every polarity lattice (L, p).*

Proof. By [**Ha92**, Prop. 4.4] it remains to show that for $\Pi_1, \Pi_2 \in \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(L))$, $\Pi_1 \cap \Pi_2 \in \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(L))$ and $\Pi_1 \cup \Pi_2 \in \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(L))$ are (α_p, β_p)-closed, which is obvious. □

We obtain in this way as an immediately corollary the following well known results.

Corollary 5.1. *The congruence lattice of a polarity lattice is distributive.*

Corollary 5.2. *The variety of polarity lattices is congruence distributive.*

Another application of the methods developed here is the proof of Varlet's conjecture, namely the following Theorem giving a necessary and sufficient condition to a polarity lattice to have a Boolean congruence lattice.

Theorem 14. *The congruence lattice of a distributive polarity lattice (D, p) is a Boolean algebra if and only if D is finite.*

Proof. Let D be a distributive lattice. Then the only arrows in $\mathbb{K}^{\mathcal{T}}(D)$ are double arrows, in every row and column exactly one, indicating the prime filter-prime ideal pairs of D. Therefore the arrow-closed subcontexts of $\mathbb{K}^{\mathcal{T}}(D)$ correspond to all subsets of $\mathfrak{F}_0(D)$. Suppose D is finite, then every subcontext of $\mathbb{K}^{\mathcal{T}}(D)$ has a quasicompact nonincidence.

If $\Pi := (H, N, \Delta) \in \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(D))$, so is $\Pi^c = (\mathfrak{F}_0(D) \backslash H, \mathfrak{I}_0(D) \backslash N, \Delta_{\Pi^c})$, by the definition of (α_p, β_p)-closeness. It is obviously that Π^c is a complement of Π so $\mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(D))$ is a Boolean algebra, hence $\operatorname{Con} D$ is Boolean.

Conversely, suppose $\mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(D))$ is Boolean. We denote by $(\mathfrak{D}(\mathbb{K}^{\mathcal{T}}(D)), (\rho \times \sigma)_{|\mathfrak{D}(\mathbb{K}^{\mathcal{T}}(D))})$, the double arrow space of D, where

$$\mathfrak{D}(\mathbb{K}^{\mathcal{T}}(D)) := \big\{(F,I) \mid F \in \mathfrak{F}_0(D), I \in \mathfrak{I}_0(D), F \nearrow I \text{ and } F \swarrow I\big\}.$$

By [**Ha92**], the double arrow space is compact and D is isomorphic with the lattice of all clopen order-filters of $\mathfrak{I}_0(D)$.

Let now $\Pi := (H, N, \Delta_\Pi) \in \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(D))$. It follows that

$$\chi := \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(D)) \cap \Delta^c$$

is compact in $\mathfrak{D}(\mathbb{K}^{\mathcal{T}}(D))$ and so topologically closed. Since $H = \mathrm{pr}_{\mathfrak{F}_0(D)}\,\chi$ and $N = \mathrm{pr}_{\mathfrak{I}_0(D)}\,\chi$, the sets H, N are closed too. Since $(F, \alpha_p(F), \Delta_{\Pi_F}) \in \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(D))$ and its complement

$$(\mathfrak{F}_0(D)\backslash\{F\}, \mathfrak{I}_0(D)\backslash\{\alpha_p(F)\}, \Delta_c) \in \mathcal{S}^p(\mathbb{K}^{\mathcal{T}}(D)),$$

it follows that the set $\{F\}$ is open in $\mathfrak{F}_0(D)$ and so, the topology on $\mathfrak{F}_0(D)$ is discrete. A similary argument proves that $\mathfrak{I}_0(D)$ is discrete too. Since $\mathfrak{D}(\mathbb{K}^{\mathcal{T}}(D))$ is compact and discrete it is finite and so is also D. □

6. Connection With Known Representations

In [**Ur79**] a topological representation of distributive lattices with a dual homomorphic operation is studied. These lattices are called **Ockham lattices** and the dual space is proved to be a totally order disconnected topological space, where the dual homomorphism is captured in a convenient way.

We will prove that in the polarity case, the representation of polarity lattices by polarity contexts captures the representation made by Urquhart. It turns out (see also [**Ha92**]) that the dual topological space of this representation can be recovered within the double arrow space in the underlying standard topological context of a polarity context.

Definition 1. Let $\mathbb{K}^{\mathcal{T}}$ be a topological context. Denote the product topology on $G \times M$ by $(\rho \times \sigma)$. The topological space

$$\mathfrak{D}(\mathbb{K}^{\mathcal{T}}) := (D(\mathbb{K}^{\mathcal{T}}), (\rho \times \sigma)_{|D(\mathbb{K}^{\mathcal{T}})})$$

where $D(\mathbb{K}^{\mathcal{T}}) := \{(g,m) \in G \times M \mid g \swarrow\!\!\!\nearrow m \text{ in } \mathbb{K}^{\mathcal{T}}\}$ is called the **double arrow space of** $\mathbb{K}^{\mathcal{T}}$.

In the topological duality established by Urquhart the dual space $\mathcal{S}(L) := \{X,\ \tau,\ \leq,\ c\}$ of every Ockham lattice is defined as follows:

(a) X is the set of all 0-1-lattice homomorphisms from L into the two element distributive lattice $\{0,1\}$,
(b) τ is the product topology of $\{0,1\}^L$,
(c) X is ordered by defining $f \leq g$ iff $f(a) \leq g(a)$ for every $a \in L$,
(d) $cf(a) = 1 - f(p(a))$, where p is the dual homomorphic operation on L.

It turns out that X is a compact totally order-disconnected space and c is a continuous order-reversing map from X into X. For all f in X, the sets $f^{-1}(0)$ and $f^{-1}(1)$ are prime ideals, resp. prime filters, and the pair $(f^{-1}(0), f^{-1}(1)) \in \mathfrak{D}(\mathbb{K}^{\mathcal{T}}(L))$.

For a given distributive 0-1-lattice L we can order $D(\mathbb{K}^{\mathcal{T}}(L))$ by

$$(F, I) \leq_L (F^*, I^*) \ :\Leftrightarrow F \subseteq F^* (\Leftrightarrow I \supseteq I^*).$$

Proposition 15. *Let L be a bounded distributive lattice. Then X and $(\mathfrak{D}(\mathbb{K}^{\mathcal{T}}(L)), \leq_L)$ are order homeomorphic.*

Proof. Let $\phi \colon X \to \mathfrak{F}_0(L) \times \mathfrak{I}_0(L)$ be defined by $\phi(f) := (f^{-1}(1), f^{-1}(0))$. It is obviously that ϕ is bijective and, moreover, an order-isomorphism. That ϕ is an homeomorphism follows by [**Ha92**]. □

If the dual homomorphic operation of the Ockham lattice is a polarity, an easy computation shows that $c \circ c = 1_X$. The polarity operation on L is captured in the standard topological space as a pair (α_p, β_p), $\alpha_p \colon \mathfrak{F}_0(L) \to \mathfrak{I}_0(L), \alpha_p(F) = p^{-1}(F)$ and $\beta_p \colon \mathfrak{I}_0(L) \to \mathfrak{F}_0(L), \beta_p(I) = p^{-1}(I)$, having the properties mentioned in Section 3.

Proposition 16. *Let (L, p) be a polarity lattice, X the set of all 0-1-lattice homorphisms from L into $\{0, 1\}$. The following diagram is comutative:*

$$\begin{array}{ccc} X & \xrightarrow{c} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\psi} \\ (\mathfrak{F}_0(L), \mathfrak{I}_0(L)) & \xrightarrow{(\alpha_p, \beta_p)} & (\mathfrak{I}_0(L), \mathfrak{F}_0(L)) \end{array}$$

where $\psi(f) := (f^{-1}(0), f^{-1}(1))$.

Proof. Let $f \in X$ be arbitrary chosen. Then

$$\begin{aligned} (\alpha_p, \beta_p) \circ \phi(f) &= (\alpha_p, \beta_p)(f^{-1}(1), f^{-1}(0)) \\ &= (p^{-1}(f^{-1}(1)), p^{-1}(f^{-1}(0))). \end{aligned}$$

On the other hand, denoting $1 - f \circ p$ by ξ, we have

$$\psi \circ c(f) = \psi(1 - f \circ p) = (\xi^{-1}(0), \xi^{-1}(1))$$

and

$$\begin{aligned} \xi^{-1}(1) &= \{x \in L \mid 1 - f(p(x)) = 1\} \\ &= \{x \in L \mid f(p(x)) = 0\} \\ &= p^{-1}(f^{-1}(0)) \end{aligned}$$

In a similar way we obtain that $\xi^{-1}(0) = p^{-1}(f^{-1}(1))$ and so the diagram commutes. □

7. Multivalued Polarity Pairs

Since the natural morphisms between polarity lattices are arbitrary polarity morphisms, it is quite naturally to ask how can we extend the duality established above, to this case. We built on the results obtained in [**Ha93**].

Let (L, p) be a polarity lattice and $\mathbb{K}^{\mathcal{T}}(L)$ its corresponding standard topological context. We define a pair (R_p, S_p) of multivalued functions where $R_p \subseteq \mathfrak{F}_0(L) \times \mathfrak{I}_0(L)$ and $S_p \subseteq \mathfrak{I}_0(L) \times \mathfrak{F}_0(L)$ by

$$(F, I) \in R_p \Leftrightarrow p^{-1}(F) \subseteq I,$$
$$(I, F) \in S_p \Leftrightarrow p^{-1}(I) \subseteq F.$$

Proposition 17. *For every polarity lattice, the pair of multivalued functions (R_p, S_p) as defined above, satisfies:*

(i) $(R_p^{[-1]}B, S_p^{[-1]}A) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$, *for all* $(A, B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$*;*

(ii) For arbitrary $F \in \mathfrak{F}_0(L)$ and $I \in \mathfrak{I}_0(L)$, R_pF is a closed intent of $\mathbb{K}^{\mathcal{T}}(L)$ and S_pI is a closed extent of $\mathbb{K}^{\mathcal{T}}(L)$;

(iii) $(S_p \square R_p)^{[-1]}(A) = A$, $(R_p \square S_p)^{[-1]}(B) = B$, *for all* $(A, B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$.

Proof. By the extended duality between 0-1-lattices and standard topological contexts, since p can be viewed as a 0-1-lattice homomorphism between L and L^{δ}, (R_p, S_p) is the induced multivalued standard morphism between $\mathbb{K}^{\mathcal{T}}(L)$ and $\mathbb{K}^{\mathcal{T}^d}(L)$, which proves (i) and (ii).

(iii) Let $(A, B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$, we have to prove that $(S_p \square R_p)^{[-1]}(A) = A$ and $(R_p \square S_p)^{[-1]}(B) = B$.

Since every closed concept in $\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$ is of the form (F_a, I_a) for some $a \in L$, it follows that $(A, B) = (F_a, I_a)$. Then

$$\begin{aligned}(S_p \square R_p)^{[-1]}(A) &= (S_p \square R_p)^{[-1]}(F_a)\\ &= R_p^{[-1]} S_p^{[-1]}(F_a)\\ &= R_p^{[-1]}(I_{p(a)}) = F_{p(p(a))} = F_a.\end{aligned}$$

Dually one can prove that $(R_p \square S_p)^{[-1]}(I_a) = I_a$. □

For every standard topological context $\mathbb{K}^{\mathcal{T}} := ((G, \rho), (M, \sigma), I)$ we call a pair (R, S) of multivalued functions $R \colon G \to M$ and $S \colon M \to G$ a **multivalued polarity pair** if it satisfies the conditions (i)–(iii) of the preceding Proposition:

(i) $(R^{[-1]}B, S^{[-1]}A) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$, for every $(A, B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$;

(ii) $Rg = Rg'' = \overline{Rg}$ for all $g \in G$ and $Sm = Sm'' = \overline{Sm}$, for all $m \in M$;

(iii) $(S \square R)^{[-1]}(A) = A, (R \square S)^{[-1]}(B) = B$, for every $(A, B) \in \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$.

A standard topological context $\mathbb{K}^{\mathcal{T}}$ together with a multivalued polarity pair (R, S) is called a **standard topological multivalued polarity context** or simply **topological polarity context** and is denoted by $(\mathbb{K}^{\mathcal{T}}, (R, S))$. Every such context establishes a polarity lattice:

Proposition 18. *Let* $(\mathbb{K}^{\mathcal{T}}, (R, S))$ *be a topological polarity context, defined as above. Then* $(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}), p_{RS})$ *is a polarity lattice, where*

$$\begin{aligned} &p_{RS} : \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}) \to \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}), \\ &(A, B) \mapsto (R^{[-1]}B, S^{[-1]}A). \end{aligned}$$

Proof. By the duality established between 0-1-lattices and standard topological contexts, p_{RS} can be viewed as a 0-1-lattice homomorphism between $\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})$ and $\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})^{\delta}$ and therefore

$$\begin{aligned} p_{RS}((A, B) \vee (C, D)) &= p_{RS}(A, B) \vee^{\delta} p_{RS}(C, D) \\ &= p_{RS}(A, B) \wedge p_{RS}(C, D). \\ p_{RS}((A, B) \wedge (C, D)) &= p_{RS}(A, B) \wedge^{\delta} p_{RS}(C, D) \\ &= p_{RS}(A, B) \vee p_{RS}(C, D). \end{aligned}$$

Moreover, $p_{RS}(p_{RS}(A, B)) = p_{RS}(R^{[-1]}B, S^{[-1]}A) = (R^{[-1]}S^{[-1]}A, S^{[-1]}R^{[-1]}B)$.
By [**Ha93**, Prop. 6],

$$\begin{aligned} &(S \,\square\, R)^{[-1]}A = R^{[-1]}S^{[-1]}A \quad \text{and} \\ &(R \,\square\, S)^{[-1]}B = S^{[-1]}R^{[-1]}B. \end{aligned}$$

We conclude that

$$\begin{aligned} p_{RS}(p_{RS}(A, B)) &= (R^{[-1]}S^{[-1]}A, S^{[-1]}R^{[-1]}B) \\ &= ((S \,\square\, R)^{[-1]}A, (R \,\square\, S)^{[-1]}B) \\ &= (A, B). \end{aligned}$$

□

Polarity lattices correspond to the topological polarity contexts, as the following two Propositions show:

Proposition 19. *Let* (L, p) *be a polarity lattice,* $(\mathbb{K}^{\mathcal{T}}(L), (R_p, S_p))$ *the corresponding topological polarity context. Then* $p \cong p_{R_pS_p}$*, i.e., the following diagram commutes:*

$$\begin{array}{ccc} L & \xrightarrow{\iota_A} & \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)) \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle p_{R_pS_p}} \\ L & \xrightarrow[\iota_A]{} & \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)) \end{array}$$

where $\iota_A \colon L \to \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L))$ *is the isomorphism defined by* $a \mapsto (F_a, I_a)$*, for every* $a \in L$*.*

Proof. For every $a \in L$, we have

$$\begin{aligned}(p_{R_pS_p} \circ \iota_A)(a) &= p_{R_pS_p}(F_a, I_a)\\ &= (R_p^{[-1]} I_a, S_p^{[-1]} F_a)\\ &= (F_{p(a)}, I_{p(a)})\\ &= (\iota_A \circ p)(a)\end{aligned}$$

which proves the commutativity of the above diagram and hence $p \cong p_{R_pS_p}$. $\square$

Proposition 20. *For every topological polarity context, the following diagram commutes:*

$$\begin{tikzcd} \mathbb{K}^{\mathcal{T}} \arrow[r, "(R_\alpha, S_\beta)"] \arrow[d, "(R,S)"'] & \mathbb{K}^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}})) \arrow[d, "(S_{p_{RS}}, R_{p_{RS}})"] \\ {\mathbb{K}^{\mathcal{T}}}^d \arrow[r, "(S_\beta, R_\alpha)"'] & \mathbb{K}^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))^d \end{tikzcd}$$

where $R_\alpha\colon G \to \mathfrak{F}_0(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$ and $S_\beta\colon M \to \mathfrak{I}_0(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}))$, defined by $R_\alpha g := \alpha_{\mathbb{K}^{\mathcal{T}}}(g)$, for $g \in G$ and $S_\beta m := \beta_{\mathbb{K}^{\mathcal{T}}}(m)$, for $m \in M$ is a standard multivalued polarity context isomorphism. The pair $(\alpha_{\mathbb{K}^{\mathcal{T}}}, \beta_{\mathbb{K}^{\mathcal{T}}})$ is the context isomorphism defined in Section 2.

Proof. We have to prove the following two relations:

$$\begin{aligned} R_{p_{RS}} \,\square\, R_\alpha &= R_\alpha \,\square\, R,\\ S_{p_{RS}} \,\square\, S_\beta &= S_\beta \,\square\, S.\end{aligned}$$

We only prove the first relation, the second follows in a similar way.

$R_{p_{RS}} \,\square\, R_\alpha = R_\alpha \,\square\, R$ is equivalent to $(R_\alpha \,\square\, R)g = (R_{p_{RS}} \,\square\, R_\alpha)g$, for every $g \in G$. By the definition of $\square$, $(R_\alpha \,\square\, R)g = (R_\alpha(Rg))''$.

$$\begin{aligned} R_\alpha(Rg) &= \bigcup_{h\in Rg} R_\alpha(g)) = \bigcup_{h\in Rg} (\alpha_{\mathbb{K}^{\mathcal{T}}}(h))''\\ &= \{E \in (\alpha_{\mathbb{K}^{\mathcal{T}}}(h))'' \mid h \in Rg\}\\ &= \{E \in (\alpha_{\mathbb{K}^{\mathcal{T}}}(h))'' \mid \alpha_{\mathbb{K}^{\mathcal{T}}}(h) \supseteq p_{RS}^{-1}(\alpha_{\mathbb{K}^{\mathcal{T}}}(g))\}\\ &= \{E \in F'' \mid F \supseteq p_{RS}^{-1}(\alpha_{\mathbb{K}^{\mathcal{T}}}(g))\}\\ &= \bigcup_{F\in R_{p_{RS}}(\alpha_{\mathbb{K}^{\mathcal{T}}}(g))} F''.\end{aligned}$$

On the other hand, $(R_{p_{RS}} \,\square\, R_\alpha)g = (R_{p_{RS}}(R_\alpha g))''$ and $R_{p_{RS}}(R_\alpha g) = R_{p_{RS}}(\alpha_{\mathbb{K}^{\mathcal{T}}}(g)'')$, for every $g \in G$.

Since $R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g))$ is a closed extent, we have

$$(R_\alpha(Rg)'', R_\alpha(Rg)') = (R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g))'', R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g))').$$

We have to prove that

$$R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g))'' = R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)'')''$$

which is equivalent to

$$R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)) = R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)'')'', g \in G.$$

Since for an arbitrary $g \in G$, $R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)) \subseteq R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)'')''$ we only want to prove that $R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)'') \subseteq R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g))$.

$$\begin{aligned} E \in R_{p_{RS}} &\Leftrightarrow \exists H \in \alpha_{\mathbb{K}^\mathcal{T}}(g)'', E \in R_{p_{RS}}(H) \\ &\Leftrightarrow \exists H \in \alpha_{\mathbb{K}^\mathcal{T}}(g)'', p_{RS}^{-1}(H) \subseteq E \\ &\Leftrightarrow H \in \alpha_{\mathbb{K}^\mathcal{T}}(g)'', E \in \bigcap_{a \in p_{RS}^{-1}(H)} F_a. \end{aligned}$$

Thus we have that

$$\begin{aligned} R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)'') &= \bigcup_{H \in \alpha_{\mathbb{K}^\mathcal{T}}(g)''} \bigcap_{a \in p_{RS}^{1}(H)} F_a \quad \text{and} \\ R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)) &= \bigcap_{a \in p_{RS}^{-1}(\alpha_{\mathbb{K}^\mathcal{T}}(g))} F_a. \end{aligned}$$

If L is a 0-1-lattice, $F \in \mathfrak{F}_0(L)$, then $H \in F''(= F^{\Delta\Delta})$ if and only if $F \subseteq H$. By this, and the considerations above we conclude that $R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g)'') \subseteq R_{p_{RS}}(\alpha_{\mathbb{K}^\mathcal{T}}(g))$ and so the diagram commutes. $\square$

8. Multivalued Polarity Morphisms

In order to give a categorical characterization to the correspondence established above between polarity lattices and standard topological multivalued polarity contexts, we have to study some properties of the involved morphisms.

Let $f\colon (L,p) \to (M,q)$ be a 0-1 polarity lattice homomorphism. By [**Ha93**] this gives rise to a multivalued standard morphism between standard topological contexts

$$R_f\colon \mathfrak{F}_0(M) \to \mathfrak{F}_0(L), \quad (F_2, F_1) \in R_f \Leftrightarrow F_1 \supseteq f^{-1}(F_2)$$

and

$$S_f\colon \mathfrak{I}_0(M) \to \mathfrak{I}_0(L), \quad (I_2, I_1) \in S_f \Leftrightarrow I_1 \supseteq f^{-1}(I_2).$$

Proposition 21. *For every* 0-1 *polarity lattice homomorphism*

$$f\colon (L,p) \to (M,q)$$

the following holds

(i) $(R_p \square R_f)(F) = (S_f \square R_q)(F)$, *for all* $F \in \mathfrak{F}_0(M)$;
(ii) $(S_p \square S_f)(I) = (R_f \square S_q)(I)$, *for all* $I \in \mathfrak{I}_0(M)$.

Proof. Let us consider the following diagrams:

$$\mathbb{K}^{\mathcal{T}}(M) \xrightarrow{(R_q,S_q)} \mathbb{K}^{\mathcal{T}^d}(M) \xrightarrow{(S_f,R_f)} \mathbb{K}^{\mathcal{T}^d}(L)$$

$$\mathbb{K}^{\mathcal{T}}(M) \xrightarrow{(R_f,S_f)} \mathbb{K}^{\mathcal{T}}(L) \xrightarrow{(R_p,S_p)} \mathbb{K}^{\mathcal{T}^d}(L)$$

Since p and q are polarities, by [**Ha93**, Lemma 9], we have

$$\begin{aligned} I \in (R_p \square R_f)(F) &\Leftrightarrow (F,I) \in R_p \square R_f \\ &\Leftrightarrow I \supseteq (f \circ p)^{-1}(F) \\ &\Leftrightarrow I \supseteq (q \circ f)^{-1}(F) \\ &\Leftrightarrow I \in (S_f \square R_q)(F). \end{aligned}$$

A similar argument proves the second part of this Proposition. □

Let (R,S) be a multivalued standard morphism between two standard topological multivalued polarity contexts $(\mathbb{K}_1^{\mathcal{T}},(R_1,S_1))$ and $(\mathbb{K}_2^{\mathcal{T}},(R_2,S_2))$. We call (R,S) a **standard polarity morphism** if the two conditions of the Proposition 21 hold:

(i) $(R_2 \square R)(g) = (S \square R_1)(g)$, for all $g \in G_1$;
(ii) $(S_2 \square S)(m) = (R \square S_1)(m)$, for all $m \in M_1$.

Proposition 22. *For every standard polarity morphism* (R,S) *between two topological polarity contexts* $(\mathbb{K}_1^{\mathcal{T}},(R_1,S_1))$ *and* $(\mathbb{K}_2^{\mathcal{T}},(R_2,S_2))$, *the map*

$$\begin{aligned} f_{RS}\colon \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}}) &\to \underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}}) \\ (A,B) &\mapsto (R^{[-1]}A, S^{[-1]}B) \end{aligned}$$

is a 0-1 *polarity lattice homomorphism.*

Proof. Since every standard polarity morphism (R,S) gives rise to a lattice homomorphism, it follows that f_{RS} is indeed a 0-1-lattice homomorphism.

Moreover, by [**Ha93**, Prop. 6], if $(R_1,S_1)\colon \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_2^{\mathcal{T}}$ and $(R_2,S_2)\colon \mathbb{K}_2^{\mathcal{T}} \to \mathbb{K}_3^{\mathcal{T}}$ are multivalued standard morphisms between standard topological contexts, then

$((R_2, S_2) \square (R_1, S_1)) \colon \mathbb{K}_1^{\mathcal{T}} \to \mathbb{K}_3^{\mathcal{T}}$ is again a multivalued standard morphism and $f_{(R_2,S_2)\,\square\,(R_1,S_1)} = f_{R_1S_1} \circ f_{R_2S_2}$. So we have

$$\begin{aligned} f_{RS} \circ p_{R_2S_2} &= f_{(R_2,S_2)\,\square\,(R,S)} = f_{(R_2\,\square\,R,S_2\,\square\,S)} \\ &= f_{(S\,\square\,R_1,R\,\square\,S_1)} = f_{(S,R)\,\square\,(R_1,S_1)} \\ &= q_{R_1S_1} \circ f_{RS}. \end{aligned}$$

□

In order to establish a dual equivalence between the category of polarity lattices with 0-1 polarity lattice homomorphisms and the category of topological polarity contexts with multivalued standard polarity morphisms, we have to show the commutativity of two more diagrams:

Proposition 23. *Let $f \colon (L, p) \to (M, q)$ be a 0-1 polarity lattice homomorphism. Then $f \cong f_{R_fS_f}$, i.e., the following diagram commutes:*

$$\begin{tikzcd} (L,p) \arrow[r, "f"] \arrow[d, "\iota_L"'] & (M,q) \arrow[d, "\iota_M"] \\ (\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(L)), p_{R_pS_p}) \arrow[r, "f_{R_fS_f}"'] & (\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}^{\mathcal{T}}(M)), p_{R_qS_q}) \end{tikzcd}$$

Proof. By [**Ha93**, Prop. 4]. □

Proposition 24. *Let $(R, S) \colon (\mathbb{K}_1^{\mathcal{T}}, (R_1, S_1)) \to (\mathbb{K}_2^{\mathcal{T}}, (R_2, S_2))$ be a standard polarity morphism between two topological polarity contexts. Then the following diagram commutes:*

$$\begin{tikzcd} \mathbb{K}_1^{\mathcal{T}} \arrow[r, "(R,S)"] \arrow[d, "(R_{\alpha_1}, S_{\beta_1})"'] & \mathbb{K}_2^{\mathcal{T}} \arrow[d, "(S_{\beta_2}, R_{\alpha_2})"] \\ \mathbb{K}^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_1^{\mathcal{T}})) \arrow[r, "(S_{f_{RS}}, R_{f_{RS}})"'] & \mathbb{K}^{\mathcal{T}}(\underline{\mathfrak{B}}^{\mathcal{T}}(\mathbb{K}_2^{\mathcal{T}})) \end{tikzcd}$$

Proof. We only prove the fact that the pair $(R_{\alpha_1}, S_{\beta_1})$ is an isomorphism in the category of standard topological polarity contexts, the rest of the proof can be then obtained easily, by adapting the proof of the Proposition 20 to the requirements of this Proposition.

Since $\alpha_{\mathbb{K}_1^{\mathcal{T}}}$ and $\beta_{\mathbb{K}_1^{\mathcal{T}}}$ are bijections, we claim that

$$(R_{\alpha_1{}^{-1}}, S_{\beta_1{}^{-1}}) \,\square\, (R_{\alpha_1}, S_{\beta_1}) = (R_e, S_e),$$

where $R_e \subseteq G \times G, R_e g := g''$ and $S_e \subseteq M \times M, S_e m := m''$.

By the definition of $\Box$, $(R_{\alpha_1^{-1}}, S_{\beta_1^{-1}}) \Box (R_{\alpha_1}, S_{\beta_1}) = (R_{\alpha_1^{-1}} \Box R_{\alpha_1}, S_{\beta_1^{-1}} \Box S_{\beta_1})$ and we shall only prove that $R_{\alpha_1^{-1}} \Box R_{\alpha_1} = R_e$, the other equality is then similar.

For every $g \in G$, $(R_{\alpha_1^{-1}} \Box R_{\alpha_1})g = (R_{\alpha_1^{-1}}(R_{\alpha_1} g))''$. For the sake of readability, let us denote $\alpha_{\mathbb{K}_1^{\mathcal{T}}}(g)$ by H, where g is an arbitrary element of G.

Now $R_{\alpha_1^{-1}}(R_{\alpha_1} g) = R_{\alpha_1^{-1}}(H'') = \bigcup_{F \in H''} (\alpha^{-1}_{\mathbb{K}_1^{\mathcal{T}}}(F))''$, which implies that

$$(R_{\alpha_1^{-1}}(R_{\alpha_1} g))' = \bigcap_{F \in H''} (\alpha^{-1}_{\mathbb{K}_1^{\mathcal{T}}}(F))'.$$

The following equivalences are true: $m \in g' \Leftrightarrow \alpha_{\mathbb{K}_1^{\mathcal{T}}}(g) \Delta \beta_{\mathbb{K}_1^{\mathcal{T}}}(m) \Leftrightarrow \forall F \in H'', F \Delta \beta_1(m) \Leftrightarrow \forall F \in H'', \alpha^{-1}_{\mathbb{K}_1^{\mathcal{T}}}(F) I m \Leftrightarrow \forall F \in H'', m \in \alpha^{-1}_{\mathbb{K}_1^{\mathcal{T}}}(F)' \Leftrightarrow m \in \bigcap_{F \in H''} \alpha^{-1}_{\mathbb{K}_1^{\mathcal{T}}}(F)'$.

This proves that $R_{\alpha_1^{-1}} \Box R_{\alpha_1} = R_e$. $\Box$

Summarizing the precedent observations, we can state the following representation theorem.

Representation Theorem. *The category of polarity lattices with* 0-1 *polarity lattice homomorphisms is dually equivalent to the category of standard topological multivalued polarity contexts with standard polarity morphisms as categorical morphisms.*

Acknowledgement. We thank the referree for his suggestion to join our two papers to one paper which has increased the understandibility of the presentation of our research. The first part of the paper was initiated by the first two authors while the content of Section 5 to 8 was elaborated by the third author in working for his doctoral thesis.

References

[**GW96**] Ganter B. and Wille R., *Formale Begriffsanalyse*, Mathematische Grundlagen, Springer Verlag, 1996.

[**Ha92**] Hartung G., *A topological representation of lattices*, Algebra Universalis **29** (1992), 273–299.

[**Ha93**] ———, *An extended duality for lattices*, General algebra and applications (Denecke K. und Vogel H.-J., eds.), Heldermann-Verlag, Berlin, 1993, pp. 126-142.

[**Hoch89**] Hoch S., *Begriffsverbände mit Anti-Endomorphismen*, Dissertation, TH Darmstadt.

[**HS73**] Herrlich H. and Strecker G., *Category theory*, Allyn and Bacon Inc. Boston, 1973.

[**HW81**] Hofmann K. and Watkins F., *The spectrum as a functor*, Proceedings of the Bremen Conference on Continuous Lattices and Their Applications, Lecture Notes in Mathematices (Banaschewski B. and Hoffmann R.-E., eds.), vol. 871, Springer Verlag, Berlin–Heidelberg–New York, 1981, pp. 249–263.

[**Pr70**] Priestley H. A., *Representation of distributive lattices by means of ordered Stone spaces*, Bull. London Math. Soc. **2** (1970), 186–190.

[**St37**] Stone H., *Topological representations of distributive lattices and Brouwerian logics*, Časopis pešt.mat. **67** (1937), 1–25.

[**Ur78**] Urquhart A., *A topological representation theory for lattices*, Algebra Universalis **8** (1978), 45–58.

[**Ur79**] ______, *Distributive Lattices with a Dual Homomorphic Operation*, Studia Logica **38(2)** (1979), 201–209..

G. Hartung, Fachbereich Mathematik, Technische Hochschule Darmstadt, Schlossgartenstr. 7, D–64289 Darmstadt

M. Kamara, Fachbereich Mathematik, Technische Hochschule Darmstadt, Schlossgartenstr. 7, D–64289 Darmstadt

C. Săcărea, Fachbereich Mathematik, Technische Hochschule Darmstadt, Schlossgartenstr. 7, D–64289 Darmstadt