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THE SINGULAR SETS OF A COMPLEX OF MODULES

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ABSTRACT. The concept of the singular set of a complex of modules is introduced and we show some special cases that the singular set is closed in the Zariski topology.

In [**GD**] Grothendieck and Dieudonne defined the singular sets of a module. Let R be a Noetherian ring and M be a finitely generated R-module. Then for any $n \in \mathbb{N}$ the set

$$\mathrm{S}_n^*(M) = \{\mathfrak{p} \in \mathrm{Spec}\,(R) : \mathrm{depth} M_{\mathfrak{p}} + \dim R/\mathfrak{p} \leq n\},$$

is called the n-singular set of M. They showed that when R is a homomorphic image of a biequidimensional regular ring then for any $n \in \mathbb{N}$ the n-singular set is closed in the Zariski topology of $\mathrm{Spec}\, R$. In [**B**] Bijan-Zadeh showed that the above result is true when R is a homomorphic image of a biequidimensional Gorenstein ring. In [**AT**] Ahmadi-Amoli and Tousi showed the same result when R has finite Krull dimension and there exists a Gorenstein R-module N with $\mathrm{Supp}\,(N) = \mathrm{Spec}\, R$ and $\dim R/\mathfrak{p} + \dim R_{\mathfrak{p}} = \dim R$ for all $\mathfrak{p} \in \mathrm{Spec}\, R$.

The extension of homological algebra from modules to complexes of modules was started already in the last chapter of [**CE**] and pursued in [**H**] and [**F**]. The aim of this paper is to introducing the concept of the singular set of a complex of modules and we show some special cases that the singular set is closed in the Zariski topology.

First we bring some definitions about complexes that we use in the rest of this paper. The reader is referred to [**F**] for details of the following brief summary of the homological theory of complexes of modules.

A complex X of R-modules is a sequence of R-linear homomorphisms $\{\partial_\ell : X_\ell \to X_{\ell-1}\}_{\ell \in \mathbb{Z}}$ such that $\partial_\ell \partial_{\ell+1} = 0$ for all ℓ. (We only use subscripts and all differentials have degree -1.) We set

$$\inf X = \inf\bigl\{\ell \in \mathbb{Z} : \mathrm{H}_\ell(X) \neq 0\bigr\},$$
$$\sup X = \sup\bigl\{\ell \in \mathbb{Z} : \mathrm{H}_\ell(X) \neq 0\bigr\}\,.$$

By convention $\sup X = -\infty$ and $\inf X = \infty$ if $X \simeq 0$.

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We identify any module M with a complex of R-modules, which has M in degree zero and is trivial elsewhere.

A homology isomorphism is a morphism $\alpha\colon X \to Y$ such that $\mathrm{H}(\alpha)$ is an isomorphism; homology isomorphisms are marked by the sign $\simeq$, while $\cong$ is used for isomorphisms. The equivalence relation generated by the homology isomorphisms is also denoted by $\simeq$. The **derived category** of the category of modules over R, cf. [**H**], is denoted by $\mathcal{C}$. The full subcategory of $\mathcal{C}$ consisting of complexes with finite homology modules is denoted $\mathcal{C}^{(f)}$, and we write $\mathcal{C}_+$, $\mathcal{C}_-$, $\mathcal{C}_b$, $\mathcal{C}_0$, for the full subcategories defined by $\mathrm{H}_\ell(X) = 0$ for, respectively, $\ell \ll 0$, $\ell \gg 0$, $|\ell| \gg 0$, $\ell \neq 0$.

The left derived functor of the tensor product functor of R-complexes is denoted by $- \otimes_R^{\mathbf{L}} -$, and the right derived functor of the homomorphism functor of complexes of the R-modules is denoted by $\mathbf{R}\mathrm{Hom}_R(-,-)$. Thus, for arbitrary $X, Y \in \mathcal{C}$ there are complexes $X \otimes_R^{\mathbf{L}} Y$ and $\mathbf{R}\mathrm{Hom}_R(X,Y)$ which are defined uniquely up to isomorphism in $\mathcal{C}$, and possess the expected functorial properties.

Familiar invariants of R-modules have been extended to complexes in several non-equivalent ways. We use the notions introduced in [**F**].

The **support** $\mathrm{Supp}\, X$ of the complex X consists of all $\mathfrak{p} \in \mathrm{Spec}\, R$ with the localization $X_\mathfrak{p}$ not homologically trivial. Thus

$$\mathrm{Supp}\, X = \{\mathfrak{p} \in \mathrm{Spec}\, R : \mathrm{H}(X_\mathfrak{p}) \neq 0\}.$$

The (**Krull**) **dimension** of an R-complex is defined in terms of the (Krull) dimensions of its homology modules by the formula:

$$\dim_R X = \sup\{\dim_R \mathrm{H}_\ell(X) - \ell : \ell \in \mathbb{Z}\},$$

with the convention that the dimension of the zero module is equal to $-\infty$. The **depth** of an R-complex X is defined by the formula

$$\mathrm{depth}_R X = -\sup \mathbf{R}\mathrm{Hom}_R(k, X),$$

hence $-\infty \leq \mathrm{depth}_R X \leq \infty$. In case X is an R-module the notions of dimension and depth concide with the standard ones.

Definition 1. Let $X \in \mathcal{C}_b^{(f)}(R)$ and let $n \in \mathbb{N}$. The n-singular set of X is defined by

$$\mathrm{S}_n^*(X) = \{\mathfrak{p} \in \mathrm{Spec}\, R : \dim R/\mathfrak{p} + \mathrm{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \sup X \leq n\}.$$

The next Theorem is a generalization of [**AT**, Theorem 3] for complexes of modules.

Theorem 2. *Let* $\dim R < \infty$ *and let* $Y \in \mathcal{I}^{(f)}(R)$ *such that for any* $\mathfrak{p} \in \operatorname{Supp} Y$

$$\dim R/\mathfrak{p} + \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p} = \dim R.$$

Then for any $X \in \mathcal{C}_b^{(f)}(R)$ *and any* $n \in \mathbb{N}$

$$\mathrm{S}_n^*(X) \cap \operatorname{Supp}_R Y = \bigcup_{\ell=\dim R+\sup X-n}^{\dim(R)+\sup X-\inf Y} \operatorname{Supp}_R \mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}_R(X,Y)).$$

Proof. Set $\mathfrak{p} \in \mathrm{S}_n^*(X) \cap \operatorname{Supp} Y$ then $\dim R/\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} + \sup X \leq n$. Since $\operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p}$ is finite we have that $\inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}) = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p}$ and hence

$$\mathrm{H}_{\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p}}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p})) \neq 0.$$

Therefore $\mathfrak{p} \in \operatorname{Supp}_R(\mathrm{H}_{\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p}}(\mathbf{R}\mathrm{Hom}_R(X,Y))$. Now we have

$$\begin{aligned}
\dim R + \sup X - n &\leq \dim R - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \dim R/\mathfrak{p} \\
&= \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \\
&\leq \operatorname{depth} R_\mathfrak{p} - \inf Y_\mathfrak{p} + \sup X_\mathfrak{p} \\
&\leq \dim R - \inf Y + \sup X
\end{aligned}$$

Suppose that $\mathfrak{p} \in \operatorname{Supp} \mathrm{H}_{-j}(\mathbf{R}\mathrm{Hom}_R(X,Y))$ where

$$\dim R + \sup X - n \leq j \leq \dim R - \inf Y + \sup X.$$

Therefore $\mathrm{H}_{-j}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}})(X_\mathfrak{p}, Y_\mathfrak{p}) \neq 0$ and hence $\mathfrak{p} \in \operatorname{Supp} Y$. On the other hand since $\inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, Y_\mathfrak{p}) = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p}$ we have that $-j \geq \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p}$ and hence $j \leq \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}$. Now we have

$$\begin{aligned}
\dim R/\mathfrak{p} + \operatorname{depth} R_\mathfrak{p} + \sup X - \inf Y_\mathfrak{p} - n &= \dim R/\mathfrak{p} + \operatorname{id}_{R_\mathfrak{p}}(Y_\mathfrak{p}) + \sup(X) - n \\
&= \dim R + \sup X - n \\
&\leq j \\
&\leq \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \\
&= \operatorname{depth} R_\mathfrak{p} - \inf Y_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}.
\end{aligned}$$

Therefore $\dim R/\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \sup X \leq n$ and hence $\mathfrak{p} \in \mathrm{S}_n^*(X)$. □

Remark. All rings admit complexes with bounded finite length homology and finite injective dimension, namely the Matlis dual of kuzul complexes of system of parameters. There exist some complex Y (different from a dualizing complex) satisfying the assumptions on Y in Theorem 2, namely Gorenstein complexes, cf. [**F**].

Corollary 3. *Let* $\dim R < \infty$ *and* $Y \in \mathcal{I}^{(f)}(R)$ *such that* $\operatorname{Supp}_R Y = \operatorname{Spec} R$ *and for all* $\mathfrak{p} \in \operatorname{Spec} R$, $\dim R/\mathfrak{p} + \operatorname{id}_{R_\mathfrak{p}} Y_\mathfrak{p} = \dim R$. *Then for any* $n \in \mathbb{N}$ *and for any* $X \in \mathcal{C}_b^{(f)}(R)$, *the* n*-singular set* $\mathrm{S}_n^*(X)$ *is closed.*

The R-complex $D \in \mathcal{I}^{(f)}(R)$ is said to be a dualizing complex for R when the homothety morphism $\chi_D^R : R \to \mathbf{R}\mathrm{Hom}_R(D, D)$ is an isomorphism. A dualizing complex D is said to be normalized if $\sup D = \dim R$, the Krull dimension of the local ring R. If C is a dualizing complex for R, then the complex $\mathcal{S}^m C$ (the shifted m degrees to the left of C) is a normalized dualizing complex for R for $m = \dim R - \sup C$.

Theorem 4. *Let* $(R, \mathfrak{m})$ *be a local ring and let* D *be a normalized dualizing complex for* R. *Then for any* $X \in \mathcal{C}_b(R)$ *and for any* $n \in \mathbb{N}$, *the singular set*

$$\mathrm{S}_n^*(X) = \bigcup_{\ell = \sup X - n}^{\sup X + \operatorname{cmd} R} \operatorname{Supp}_R(\mathrm{H}_{-\ell}(\mathbf{R}\mathrm{Hom}\,(X, D))).$$

is closed in the Zariski topology of $\operatorname{Spec} R$, *where* $\operatorname{cmd} R = \dim R - \operatorname{depth} R$ *is the Cohen-Macaulay defect of* R.

Proof. Set $\mathfrak{p} \in \mathrm{S}_n^*(X)$. Since $\operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p}$ is finite by the same reason as Theorem 2, we have that $\mathfrak{p} \in \operatorname{Supp}_R \mathrm{H}_{\operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p}}(\mathbf{R}\mathrm{Hom}_R(X, D)$. Set $t = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p}$. Since $\operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p} = -\dim R/\mathfrak{p}$ by [**F**, 15.17(b)], we have that $\sup X - n \leq t$. On the other hand, we have $t \leq \operatorname{depth} R_\mathfrak{p} - \inf D_\mathfrak{p} + \sup X_\mathfrak{p}$ by [**F**, 13.23(I)], and hence $t \leq \dim R - \operatorname{depth} R + \sup X$ by [**F**, 15.18(c)].

Now suppose that $\mathfrak{p} \in \operatorname{Supp} \mathrm{H}_{-j}(\mathbf{R}\mathrm{Hom}_R(X, D)$ where $\sup X - n \leq j \leq \sup X$. Therefore $\mathrm{H}_{-j}(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, D_\mathfrak{p})) \neq 0$. Since $\inf \mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(X_\mathfrak{p}, D_\mathfrak{p}) = \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} - \operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p}$ we have that $j \leq \operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}$. Thus

$$\begin{aligned} \sup X - n &\leq j \\ &\leq \operatorname{id}_{R_\mathfrak{p}} D_\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} \\ &= -\dim R/\mathfrak{p} - \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p}. \end{aligned}$$

Therefore $\dim R/\mathfrak{p} + \operatorname{depth}_{R_\mathfrak{p}} X_\mathfrak{p} + \sup X \leq n$. Since $\mathbf{R}\mathrm{Hom}\,(X, D) \in \mathcal{C}_b^{(f)}(R)$, for any $\ell \in \mathbb{Z}$ the set $\operatorname{Supp}_R \mathrm{H}_\ell(\mathbf{R}\mathrm{Hom}\,(X, D)$ is closed and hence $\mathrm{S}_n^*(X)$ is closed. □

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