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# GENERAL ALGEBRAIC GEOMETRY AND FORMAL CONCEPT ANALYSIS

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## 0. Introduction

This paper describes the interaction between classical Algebraic Geometry, General Algebra, and Formal Concept Analysis. The mathematical foundations of Formal Concept Analysis can be found in (**[GW99]**). Its goal is to elaborate the general core of the basic results of Algebraic Geometry using this interaction. We start from a general polynomial context of the form $\mathbb{K}_{n,A} := (A^n, F_n(X,A) \times F_n(X,A), \perp)$. Here $A$ is a general algebra, $F_n(X,A)$ is the free algebra in $n$ variables in the variety[1] $\operatorname{Var} A$ generated by $A$, and we have $\vec{a} \perp (p,q)\colon \iff p(\vec{a}) = q(\vec{a})$ for $\vec{a} \in A^n$ and $p, q \in F_n(X,A)$. Extents of this formal context will be called $A$-**algebraic sets**. We find that the intents of $\mathbb{K}$ are certain congruence relations on $F_n(X,A)$, which we will call **radical congruences** (cf. Section 1). We conclude that the lattice of $A$-algebraic sets in $A^n$ and the lattice of radical congruences on $F_n(X,A)$ are dually isomorphic. When we choose a general algebra such that $F_n(X,A)$ is the ring of polynomials $K[x_1,\dots,x_n]$ over an algebraically closed field, we obtain the classical correspondence between algebraic varieties in $K^n$ and reduced ideals in $K[x_1,\dots,x_n]$. In Algebraic Geometry we have a functorial correspondence between algebraic varieties and **coordinate algebras** $K[V] := K[x_1,\dots,x_n]/V^{\perp}$. (Here $V^{\perp}$ is the ideal of polynomials that vanish on $V$). For $A$-algebraic sets $V$, we define a **coordinate algebra** $\Gamma(V)$ by $\Gamma(V) := F_n(X,A)/\Phi$, where $\Phi := V^{\perp}$ is the congruence relation corresponding to $V$. Since $A$-algebraic sets can be understood as homomorphisms from $F_n(X)/\Phi$ to $A$ and since coordinate algebras can be understood as finitely generated subalgebras of a power of $A$, we get a dual equivalence between the category of $A$-algebraic sets with polynomial morphisms – yet to be defined – and the category of of finitely generated subalgebras of a power of $A$ with homomorphisms. This result is due to H. Bauer (**[Ba83]**). In the classical case we get the dual equivalence mentioned afore. We will use the general results to deduce the classical results of Algebraic

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[1] In the sense of General Algebra.

Geometry. Along with basic results from Formal Concept Analysis we determine which classical results follow from the general treatment and which results require Commutative Algebra. The importance of $A$-algebraic sets is due to the role within mathematical theories and their role in applications. They can be used in the modelling of databases, since answers to queries can be interpreted as $A$-algebraic sets ([**Pl96a**]). $A$-algebraic sets are treated, for instance, in ([**PL96a**]), ([**Pl96b**]), and ([**Pl96c**]). B. Plotkin ([**Pl96a**]) does not resrict himself to the case $\mathrm{Var}(A)$, he considers arbitrary varieties $\mathcal{V}$ (in the sense of General Algebra). He asks when two algebras $G_1, G_2 \in \Theta$ are $X$-equivalent, i.e. when for all finite free algebras $F_n(X)$ in $\mathcal{V}$ the contexts $(G_1^n, F_n(X) \times F_n(X), \perp)$ and $(G_2^n, F_n(X) \times F_n(X), \perp)$ yield the same concept lattice. If they are $X$-equivalent for all finite sets $X$, they are called geometrically equivalent. For instance, all free non-commutative groups are mutually equivalent.

## 1. The Basic Situation

Let $K$ be a field. In classical Algebraic Geometry so-called **algebraic varieties** $V \subseteq K^n$ are investigated. They can be described in the form $V = V(f_1, \dots, f_s) := \{a \in K^n \mid f_i(a) = 0,\ i = 1, \dots s\}$ where the $f_i \in K[x_1, \dots, x_n]$ are polynomials in $n$ variables. The corresponding ideal $I(V) := \{f \in K[x_1, \dots, x_n] \mid \forall a \in V : f(a) = 0\}$ plays a decisive role in the analysis of algebraic varieties. When $K$ is algebraically closed then the ideals defined by the vanishing on algebraic varieties are exactly the reduced ideals in $K[x_1, \dots, x_n]$, i.e., those ideals $I$ for which $f^m \in I$ implies $f \in I$. This situation has been formulated and elaborated in the language of Formal Concept Analysis in the first part of this dissertation. There the context $\mathbb{K} := (K, K[x_1, \dots, x_n], \perp)$ with $a \perp f :\iff f(a) = 0$ has been studied. We observe that its formal concepts are exactly the pairs of the form $(V, V^\perp)$ where $V \subseteq K^n$ is an algebraic variety and where $V^\perp = I(V)$ is the corresponding ideal. In particular, we conclude that the lattice of all algebraic varieties contained in $K^n$ and the lattice of all reduced ideals in $K[x_1, \dots, x_n]$ are dually isomorphic, when $K$ is algebraically closed (cf. Section 3).

A natural generalization of the classical problem is to consider "algebraic sets" which are determined by terms from arbitrary free algebras. Since polynomials can be understood as terms in the free algebra $F_n(X, A)$ over an algebra $A$ which contains the constant operations, we will cover the classical case as well. To formulate the problem we consider pairs $(p, q)$ of terms from a free algebra in $n$ variables over a given general algebra $A$. In this case, algebraic sets $V \subseteq A^n$ can be written in the form $V = \{a \in A^n \mid \forall i \in I : p_i(a) = q_i(a)\}$. The question arises how the sets of the form $V^\perp := \{(p, q) \mid \forall a \in V : p(a) = q(a)\}$ can be described. A complete answer is given in ([**Ba83**]). mathematical theories and their role in applications. They can be used in the modelling of databases, since answers to queries can be interpreted as $A$-algebraic sets ([**Pl1**]). $A$-algebraic sets are treated, for instance, in ([**PL1**]), ([**Pl2**]), ([**Pl3**]) and ([**Be97**]). B. Plotkin ([**Pl1**]) does not resrict himself to the case $\mathrm{Var}(A)$, he considers arbitrary varieties $\Theta$ (in the sense of general algebra). He asks when two algebras $G_1, G_2 \in \Theta$ are $X$-equivalent, i.e. when do they generate the same congruence relation $T^{\perp\perp} \subseteq W(X) \times W(X)$ for all sets $T \subseteq W(X) \times W(X)$ and all finite free finite sets $X$, they are called geometrically equivalent. For instance, all free non-commutative groups are mutually equivalent.

The formulation of the generalized situation in the language of Formal Concept Analysis forms the basis of this paper. The classical case has been treated in [**Be99**]. Many ideas carry over. We start now with the basic formalizations.

Let $A$ be a fixed general algebra and let $\mathrm{Var}(A)$ be the variety generated by $A$. Let $X := \{x_1, \dots, x_n\}$ be a finite set of variables and let $F(X) := F_n(X, A)$ be the free algebra over $X$ in $\mathrm{Var}(A)$. We consider the formal context $\mathbb{K} := \mathbb{K}_{n,A} := (A^n, F(X) \times F(X), \perp)$ with the relation $\vec{a} \perp (p,q) :\iff p(\vec{a}) = q(\vec{a})$. This is a substitute for the relation $a \perp f :\iff f(a) = 0$ for classical polynomials. Analogous to the classical case, the extents of $\mathbb{K}$ are called $A$-algebraic sets. Thus an $A$-**algebraic set** is of the form $D^\perp \subseteq A^n$, $D^\perp := \{\vec{a} \in A^n \mid \forall (p,q) \in D\colon p(\vec{a}) = q(\vec{a})\}$ for some $D \subseteq F(X) \times F(X)$.

We wish to determine the intents of $\mathbb{K}$. Therefore, we have to check all sets of the form $C^\perp$ where $C \subseteq A^n$. For $C \subseteq A^n$, the set $C^\perp = \{(p,q) \in F(X) \times F(X) \mid \forall \vec{c} \in C\colon p(\vec{c}) = q(\vec{c})\} \subseteq F(X) \times F(X)$ is a congruence relation: clearly, $C^\perp$ is an equivalence relation. If $f$ is a $t$-ary operation on $F(X)$ and if $(p_1, q_1), \dots (p_t, q_t) \in C^\perp$ we compute $f(p_1, \dots, p_t)(\vec{c}) = f(p_1(\vec{c}), \dots, p_t(\vec{c})) = f(q_1(\vec{c}), \dots, q_t(\vec{c})) = f(q_1, \dots, q_t)(\vec{c})$ for all $\vec{c} \in C$ and so $(f(p_1, \dots, p_t), f(q_1, \dots, q_t)) \in C^\perp$. (We use the same notation for an operation on $A$ and for the corresponding operation on $F(X, A)$).

In order to determine which congruence relations occur as intents of $\mathbb{K}$, we follow an unpublished preprint by Heiko Bauer ([**Ba83**]), "About Hilbert's and Rückert's Nullstellensatz".

**Definition 1.1.** Let $\Phi$ be a congruence relation on $F(X)$. The smallest congruence relation $\sqrt{\Phi}$ on $F(X)$ containing $\Phi$ such that $F(X)/\sqrt{\Phi}$ can be embedded into a power of $A$ is called the **radical** of $\Phi$. (We will see in the next section that $\sqrt{\Phi}$ always exists).

**Theorem 1.2.** $\Phi^{\perp\perp} = \sqrt{\Phi}$ *holds for all congruence relations* $\Phi$ *on* $F(X)$.

The proof is given in the next section.

**Example 1.3.** Consider the group $\mathbb{Z}_4 := \{0,1,2,3\}$ with the operations $+$, $-$, $0$. Then we have $F := F(\{x\}, A) = \{0, x, 2x, 3x\}$ and the context $(\mathbb{Z}_4, F \times F, \perp)$

has the following form:

| $\perp$ | $(0,0)$ | $(0,x)$ | $(0,2x)$ | $(0,3x)$ | $(x,0)$ | $(x,x)$ | $(x,2x)$ | $(x,3x)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 1 | $\times$ | | | | | $\times$ | | |
| 2 | $\times$ | | $\times$ | | | $\times$ | | $\times$ |
| 3 | $\times$ | | | | | $\times$ | | |

| $\perp$ | $(2x,0)$ | $(2x,x)$ | $(2x,2x)$ | $(2x,3x)$ | $(3x,0)$ | $(3x,x)$ | $(3x,2x)$ | $(3x,3x)$ |
|---|---|---|---|---|---|---|---|---|
| 0 | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ | $\times$ |
| 1 | | | $\times$ | | | | | $\times$ |
| 2 | $\times$ | | $\times$ | | | $\times$ | | $\times$ |
| 3 | | | $\times$ | | | | | $\times$ |

Let $\nabla := F \times F$ be the all-congruence and let $\triangle := \{(x,x) \,|\, x \in F\}$ be the diagonal congruence. It is easily seen that $\{0\}^{\perp\perp} = \{0\}$, $\{1\}^{\perp\perp} = \mathbb{Z}_4$, $\{2\}^{\perp\perp} = \{0,2\}$ and $\{3\}^{\perp\perp} = \mathbb{Z}_4$. Hence, the formal concepts of $(\mathbb{Z}_4, F \times F, \perp)$ are

$$(\{0\}, \nabla), (\{0,2\}, \triangle \cup \{(0,2x),(2x,0),(x,3x),(3x,x)\}), \text{ and } (\mathbb{Z}_4, \triangle).$$

Note that the extents are exactly the subgroups of $\mathbb{Z}_4$ and that all intents are congruence relations. In this case, also every congruence relation is an intent. In general, this is not true.

## 2. Proof of the Theorem

In this section we will not only prove Theorem 1.2, we will show a stronger result. A functorial correspondence between the category of $A$-algebraic sets – yet to be defined – and the category of finitely generated subalgebras of a power of $A$ is established. This is a generalization of the classical situation where we have a functorial correpondence between algebraic varieties and reduced finite $K$-algebras. Theorem 1.2 will follow from this general result. The proof of 1.2 and of the categorial correspondence given in this paragraph is an elaboration of the afore mentioned paper by H. Bauer ([**Ba83**]). Although some introductory knowledge from General Algebra, as it is presented in ([**We78**]), is presupposed, we try to give detailed proofs which address also to non-specialists. Notions from the theory of categories such as functors, morphisms, and natural transformations are defined in ([**Br73**]). A result analogous to the result we give, within a more general setting, can be found in ([**Pl96a**]).

**Definition 2.1.** Let $V \subseteq A^n$ and $W \subseteq A^m$ be $A$-algebraic sets. A map $g\colon V \longrightarrow W$ is a **morphism** of $A$-algebraic sets, if there are $n$-ary terms $p_1, \dots, p_m \in F_n(X, A)$ such that $g(\vec{a}) = (p_1(\vec{a}), \dots, p_m(\vec{a}))$ holds for all $\vec{a} \in V$. Although the $p_i \in F_n(X, A)$ are terms and not polynomials, we call morphisms of $A$-algebraic sets **polynomial morphisms**, because it is the terminology used in Algebraic Geometry.

Let $\mathcal{K}_A(V)$ be the category of all $A$-algebraic sets with polynomial morphisms and let $S_f P(A)$ denote the category of all finitely generated subalgebras of a power of $A$ with homomorphisms. We establish the first step towards a functorial correspondence between the two categories.

**Definition 2.2.** Let $V$ be an $A$-algebraic set. The **coordinate algebra** $\Gamma(V)$ of $V$ is defined by $\Gamma(V) := F(X)/V^\perp$.

We wish to show that every coordinate algebra can be interpreted as a finitely generated subalgebra of a power of $A$. This will yield the possibility to use $\Gamma(-)$ to define a functor from $\mathcal{K}_A(V)$ to $S_f P(A)$. Therefore, let $\mathrm{HOM}(V, A)$ be the set of all polynomial morphisms from $V$ to $A$, endowed with pointwise operations. $\mathrm{HOM}(V, A)$ is an algebra of the same type as $A$.

**Proposition 2.3.** *Let $V$ be an $A$-algebraic set. Then $\Gamma(V) \cong \mathrm{HOM}(V, A)$ holds.*

*Proof.* Let $[p] \in \Gamma(V)$ be the congruence class of a given $p \in F(X)$. Let $\alpha$ be the map that sends $[p]$ to the restriction map $\underline{p}\colon V \longrightarrow A$ which is defined by $\underline{p}(\vec{a}) := p(\vec{a})$ for all $\vec{a} \in V$. This map is well-defined and injective because we have $\underline{p} = \underline{q} \iff \forall \vec{a} \in V : p(\vec{a}) = q(\vec{a}) \iff (p, q) \in V^\perp \iff [p] = [q]$. Furthermore, $\alpha$ is surjective by the definition of morphisms from $V$ to $A$. Finally, it is a homomorphism since $\mathrm{HOM}(V, A)$ is endowed with pointwise operations: If $f_i$ is a $t$-ary operation on $\Gamma(V) := F(X)/V^\perp$ then $f_i$ is defined on $\mathrm{HOM}(V, A)$ as follows:[2] if $g_1, \dots, g_t \in \mathrm{HOM}(V, A)$ and if $\vec{b} \in A^n$, then $f_i(g_1, \dots, g_t)(\vec{b}) := f_i(g_1(\vec{b}), \dots, g_t(\vec{b}))$. We show that for $[p_1], \dots, [p_t] \in \Gamma(V)$ we have $\alpha(f_i([p_1], \dots, [p_t])) = f_i((\alpha([p_1]), \dots, \alpha([p_t])))$. For $\vec{b} \in V$ we have $f_i(\alpha([p_1]), \dots, \alpha([p_t]))(\vec{b}) = f_i(\underline{p_1}, \dots, \underline{p_t})(\vec{b}) = f_i(p_1(\vec{b}), \dots, p_t(\vec{b}))$. On the other hand $\alpha(f_i([p_1], \dots, [p_t]))(\vec{b}) = \alpha([f_i(p_1, \dots, p_t)])(\vec{b}) = \underline{f_i(p_1, \dots, p_t)}(\vec{b}) = f_i(p_1, \dots, p_t)(\vec{b}) = f_i(p_1(\vec{b}), \dots, p_t(\vec{b}))$ for all $\vec{b} \in V$, and the proof is finished. □

**Corollary 2.4.** *For each $A$-algebraic set $V$, the coordinate algebra $\Gamma(V)$ can be embedded into a power of $A$.*

The next lemma is needed in order to define a functor from $S_f P(A)$ to $\mathcal{K}_A(V)$. For $B \in S_f P(A)$ let $\mathrm{Hom}(B, A)$ be the set of all homomorphisms from $B$ to $A$.

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[2] If $f$ is a fundamental operation on $A$, we write "$f$" for the corresponding operations on $A^n$ and $F_n(X)/\Phi$ as well.

**Lemma 2.5.** *Let $\Phi$ be a congruence relation on $F(X)$ and let $X{:=}\{x_1, \dots, x_n\}$. Then the map $\alpha\colon \operatorname{Hom}(F(X)/\Phi, A) \longrightarrow \Phi^\perp \subseteq A^n$ defined by $\alpha(g) := x_g$, where $x_g := (g([x_1]), \dots, g([x_n]))$, is a bijection.*

*Proof.* Let $g \in \operatorname{Hom}(F(X)/\Phi, A)$. Then $\alpha(g) = x_g \in \Phi^\perp$ because we have $p(x_g) = p(g([x_1]), \dots, g([x_n])) = g(p([x_1], \dots, [x_n])) = g(q([x_1], \dots, [x_n])) = q(g([x_1]), \dots, g([x_n])) = q(x_g)$ for all $(p, q) \in \Phi$.

We define an inverse mapping $\beta\colon \Phi^\perp \longrightarrow \operatorname{Hom}(F(X)/\Phi, A)$ by $\beta(\vec{a}) := g_{\vec{a}}$ where $g_{\vec{a}} : F(X)/\Phi \longrightarrow A$ is given by $g_{\vec{a}}([p]) := p(\vec{a})$. $\beta$ is well-defined because $g_{\vec{a}}$ will be well-defined for all $\vec{a} \in \Phi^\perp$: if $p\Phi q$ we have $p(\vec{a}) = q(\vec{a})$ and $g_{\vec{a}}([p]) = g_{\vec{a}}([q])$.

$\alpha \circ \beta$ is the identity on $\Phi^\perp$: let $\vec{a} := (a_1, \dots, a_n)$. We have $\alpha(\beta(\vec{a})) = \alpha(g_{\vec{a}}) = x_{g_{\vec{a}}} = (g_{\vec{a}}([x_1]), \dots, g_{\vec{a}}([x_n])) = (x_1(\vec{a}), \dots, x_n(\vec{a})) = (a_1, \dots, a_n) = \vec{a}$. $\beta \circ \alpha$ is the identity on $\operatorname{Hom}(F(X)/\Phi, A)$ since $\beta(\alpha(h)) = \beta(h([x_1]), \dots, h([x_n])) = g_{(h([x_1]), \dots, h([x_n]))}$ with $g_{(h([x_1]), \dots, h([x_n]))}([p]) = p(h([x_1]), \dots, h([x_n])) = h(p([x_1], \dots, [x_n])$. $\square$

**Remark 2.6.** a) The lemma shows that a point $\vec{a} := (a_1, \dots, a_n) \in \Phi^\perp \subseteq A^n$ can be thought of as a homomorphism from $F(X)/\Phi$ to $A$. It is the unique homomorphism which sends $[x_i]$ to $a_i$. Vice versa, a homomorphism $\delta$ from $F(X)/\Phi$ to $A$ can be identified with the point $\vec{a} := (\delta([x_1]), \dots, \delta([x_n])) \in \Phi^\perp$. This fact is exploited by B. Plotkin ([**Pl96a**]) who starts his consideration right from the beginning with the context $(\operatorname{Hom}(F(X)/\Phi, A), F(X) \times F(X), \perp)$ where $\sigma \perp (\omega_1, \omega_2)\colon \iff (\omega_1, \omega_2) \in \operatorname{Kern}\ \sigma$.

b) It can be shown that if $\operatorname{Hom}(F(X)/\Phi, A)$ is a general algebra of the same type as $A$, for instance when $A$ is an abelian group, the mapping above is an isomorphism. In particular, $\Phi^\perp \subseteq A^n$ is a subalgebra.

We wish to use $\operatorname{Hom}(\ , A)$ as a functor from $S_f P(A)$ to $\mathcal{K}_A(V)$. Therefore let $B$ be an object from $S_f P(A)$. We observe that $B \cong F_n(X)/\Phi$ for some $n \in \mathbb{N}$ and some congruence relation $\Phi$ on $F(X)$. Hence we can consider $\operatorname{Hom}(B, A)$ as an $A$-algebraic set by means of the isomorphism $\operatorname{Hom}(B, A) \cong \operatorname{Hom}(F(X)/\Phi, A)$ and the bijection between $\operatorname{Hom}(F(X)/\Phi, A)$ and $\Phi^\perp$.

**Theorem 2.7.** $\operatorname{Hom}(\ , A)$ *is a contra-variant functor from $S_f P(A)$ to $\mathcal{K}_A(V)$ where, for a homomorphism $g\colon B \longrightarrow C$, the morphism* $\operatorname{Hom}(g, A)\colon \operatorname{Hom}(C, A) \longrightarrow \operatorname{Hom}(B, A)$ *is given by $h \mapsto h \circ g$.*

*Proof.* Let $g\colon B \longrightarrow C$ be a morphism. Since $B, C \in S_f P(A)$, we have $B \cong F_n(X)/\Phi$ for some $n \in \mathbb{N}$ and some congruence relation $\Phi$ on $F_n(X)$ and $C \cong F_m(X)/\Psi$ for some $m \in \mathbb{N}$ and some congruence relation $\Psi$ on $F_m(X)$. We have bijections from $\operatorname{Hom}(B, A)$ to $\Phi^\perp$ and from $\operatorname{Hom}(C, A)$ to $\Psi^\perp$ (cf. 2.5). We wish to interpret $\operatorname{Hom}(g, A)$ as a polynomial morphism. Consider the induced homomorphism $g\colon F_n(X)/\Phi \longrightarrow F_m(X)/\Psi$. We choose $n$ $m$-ary polynomials $p_i$ such that $g(x_i) = [p_i]$ for $i = 1, \dots n$. We claim that the following diagram commutes.

$$\begin{array}{ccc} \Psi^{\perp} & \xrightarrow{(p_1,\dots p_n)} & \Phi^{\perp} \\ {\scriptstyle\beta_C}\big\downarrow & & \big\downarrow{\scriptstyle\beta_B} \\ \mathrm{Hom}(C,A) & \xrightarrow[\mathrm{Hom}(g,A)]{} & \mathrm{Hom}(B,A) \end{array}$$

(Here $\beta_B$ and $\beta_C$ are the maps belonging to $B \cong F_n(X)/\Phi$ and $C \cong F_m(X)/\Psi$ defined in 2.5).

We must show that $\mathrm{Hom}(g,A) \circ \beta_C = \beta_B \circ (p_1,\dots,p_n)$. Choose $\vec{a} \in \Psi^{\perp}$. We have

$$(\mathrm{Hom}(g,A) \circ \beta_C)(\vec{a}) = g_{\vec{a}} \circ g$$

and

$$(\beta_B \circ (p_1,\dots,p_n))(\vec{a}) = g_{(p_1(\vec{a}),\dots,p_n(\vec{a}))}.$$

In order to show that $g_{\vec{a}} \circ g = g_{(p_1(\vec{a}),\dots,p_n(\vec{a}))}$ it is sufficient to show the equality for a fixed $x_j \in X = \{x_1,\dots,x_n\}$ because we are dealing with homomorphisms. Now $(g_{\vec{a}} \circ g)([x_j]) = g_{\vec{a}}(g([x_j])) = g_{\vec{a}}([p_j]) = p_j(\vec{a})$ and in the same way $g_{(p_1(\vec{a}),\dots,p_n(\vec{a}))}([x_j]) = x_j(p_1(\vec{a}),\dots,p_n(\vec{a})) = p_j(\vec{a})$. Hence $\mathrm{Hom}(g,A) = (p_1,\dots,p_n)$ is indeed a morphism between the algebraic sets $\Psi^{\perp}$ and $\Phi^{\perp}$.

Finally, it is clear from the definition of $\mathrm{Hom}(\ ,A)$ for homomorphisms that we have a contra-variant functor: If $f\colon B \longrightarrow C$ and $g\colon C \longrightarrow D$ and $g \circ f\colon B \longrightarrow D$ are morphisms then $\mathrm{Hom}(g \circ f, A)\colon \mathrm{Hom}(D,A) \longrightarrow \mathrm{Hom}(B,A)$ is given by $h \mapsto h \circ (g \circ f)$ which is equal to $\mathrm{Hom}(f,A) \circ \mathrm{Hom}(g,A)\colon \mathrm{Hom}(D,A) \longrightarrow \mathrm{Hom}(B,A)$ since $\mathrm{Hom}(g,A)(h) = h \circ g$ and $\mathrm{Hom}(f,A)(h \circ g) = (h \circ g) \circ f$. □

**Theorem 2.8.** *$\Gamma(-)$ is a contra-variant functor from $\mathcal{K}_A(V)$ to $S_f P(A)$ where for $g\colon V \longrightarrow W$ the map $\Gamma(g)\colon \Gamma(W) \longrightarrow \Gamma(V)$ is given by $[p](W^{\perp}) \mapsto [p \circ g](V^{\perp})$.*

*Proof.* We must show that $\Gamma(g)\colon \Gamma(W) \longrightarrow \Gamma(V)$ is a homomorphism. First we have $[p \circ g](V^{\perp}) \in \Gamma(V)$: if $\vec{a} \in V$ then $g\colon V \longrightarrow W$ is given by $g(\vec{a}) = (g_1(\vec{a}),\dots,g_m(\vec{a}))$, which is an element of $W$. Since $(p \circ g)(\vec{a}) = p(g(\vec{a})) = p(g_1(\vec{a}),\dots,g_m(\vec{a})) = p(g_1,\dots,g_m)(\vec{a})$ with $g_i \in F_n(X)$, we see that $[p \circ g](V^{\perp})$ is an element of $\Gamma(V)$. $\Gamma(g)$ is well-defined because $[p \circ g](W^{\perp}) = [q \circ g](W^{\perp})$ implies $[p(g(\vec{a}))](V^{\perp}) = [q(g(\vec{a}))](V^{\perp})$, since $g(\vec{a}) \in W$. $\Gamma(g)$ is a homomorphism: $\Gamma(g)(f_i([p_1],\dots,[p_t]))(\vec{b}) = f_i(p_1(g_1(\vec{b}),\dots,g_n(\vec{b})),\dots p_t(g_1(\vec{b}),\dots,g_n(\vec{b}))) = f_i(\Gamma(g)([p_1]),\dots,\Gamma(g)([p_t]))(\vec{b})$. Since $\Gamma(g)$ is given by composition, we see as before that $\Gamma(-)$ is a contra-variant functor. □

Finally, we define natural transformations in order to show that our categories $\mathcal{K}_V(A)$ and $S_f P(A)$ are dually isomorphic.

**Lemma 2.9.** *Let $V$ be an $A$-algebraic set. Then the map $\eta_V\colon V \longrightarrow \mathrm{Hom}(\Gamma(V),A)$ with $\eta_V(\vec{a}) := g_{\vec{a}}$, where $g_{\vec{a}}\colon \Gamma(V) \longrightarrow A$ is given by $g_{\vec{a}}([p]) := p(\vec{a})$, is a polynomial isomorphism.*

*Proof.* We now already from 2.5 that $\eta_V$ is bijective. The homomorphism $g_{\vec{a}}\colon \Gamma(V) \longrightarrow A$ corresponds to the point $(g_{\vec{a}}([x_1]), \dots, g_{\vec{a}}([x_n]))$ which is equal to $(x_1(\vec{a}), \dots, x_n(\vec{a})) = \vec{a} \in V^{\perp\perp} = V$. So $\eta_V$ clearly is a polynomial isomorphism. $\square$

**Lemma 2.10.** *Let $B \cong F_n(X)/\Phi$ be a finitely generated subalgebra of a power of $A$. (Note that in this case we have $\Phi = \sqrt{\Phi}$ by the definition of $\sqrt{\Phi}$ and $S_f P(A)$). Then there is an isomorphism $\varepsilon_B\colon F_n(X)/\Phi \longrightarrow \Gamma(\mathrm{Hom}(F(X)/\Phi, A))$.*

*Proof.* First note that $\Gamma(\mathrm{Hom}(F(X)/\Phi, A)) \cong \mathrm{HOM}(\mathrm{Hom}(F(X)/\Phi, A), A)$ by means of 2.3 and 2.5. Therefore we can define $\varepsilon_B\colon F_n(X)/\Phi \longrightarrow \Gamma(\mathrm{Hom}(F(X)/\Phi, A))$ as follows: For $[p] \in F_n(X)/\Phi$ let $\varepsilon_B([p]) := \beta_p$ with $\beta_p\colon \mathrm{Hom}(F_n(X)/\Phi, A) \longrightarrow A$ where $\beta_p(h) := h([p])$ which is a polynomial morphism by means of 2.5. Since the operations on HOM are definede pointwise, $\varepsilon_B$ is a homomorphism: Let $f$ be a $t$-ary operation. Then we have $\varepsilon_B(f([p_1], \dots, [p_t])) = \varepsilon_B([f(p_1, \dots, p_t)]) = \beta_{f(p_1,\dots,p_t)}$ with $\beta_{f(p_1,\dots,p_t)}(h) = h([f(p_1, \dots, p_t)]) = h(f([p_1], \dots, [p_t])) = f(h[p_1], \dots, h[p_t])$, since $h$ is a homomorphism. On the other hand, we have $f(\varepsilon_B([p_1]), \dots, \varepsilon_B([p_t])) = f(\beta_{p_1}, \dots, \beta_{p_t})$ with $f(\beta_{p_1}, \dots, \beta_{p_t})(h) = f(\beta_{p_1}(h), \dots, \beta_{p_t}(h))$ since the operations are defined pointwise. The latter expression is equal to $f(h[p_1], \dots, h[p_t])$ and $\varepsilon_B$ is a homomorphism. $\varepsilon_B([p])$ is surjective: Let $g\colon \mathrm{Hom}(F_n(X)/\Phi, A) \longrightarrow A$ be a polynomial morphism. Then there is a $n$-ary polynomial $p$ such that $g(h) = p(h([x_1]), \dots, h([x_n]))$ for all $h \in \mathrm{Hom}(B, A)$. We compute $\varepsilon_B(p([x_1], \dots, [x_n]))(h) = h(p([x_1], \dots, [x_n])) = p(h([x_1]), \dots h([x_n])) = g(h)$, hence $\varepsilon_B(p([x_1], \dots, [x_n])) = g$. $\varepsilon_B([p])$ is injective: $B \cong F_n(X)/\Phi$ is a subalgebra of a power $A^M$ of $A$. Therefore, if $b \neq c$ are distinct elements of $B$, then there is an element $m \in M$ with $b(m) \neq c(m)$. Hence there is a homomorphism $h\colon F_n(X)/\Phi \longrightarrow A$ with $h(b) \neq h(c)$. (Here we need that $F_n(X)/\Phi$ is in $S_f P(A)$). Then $\beta_b \neq \beta_c$ and $\varepsilon_B$ is injective. $\square$

**Theorem 2.11** (Hilbert's Nullstellensatz for General Algebras)**.** *Let $A$ be a general algebra. The category of all $A$-algebraic sets with polynomial morphisms is dually equivalent to the category of all finitely generated subalgebras of a power of $A$ with homomorphisms. The dual equivalence is given by the coordinate algebra functor.*

*Proof.* We claim that the family of mappings $\eta := (\eta_V)_V$ determines a natural transformation $\eta\colon \mathrm{Id}_{\mathcal{K}_A(V)} \longrightarrow \mathrm{Hom}(\Gamma(-), A)$ and that the family $\varepsilon := (\varepsilon_B)_B : \mathrm{Id}_{S_f P(A)} \longrightarrow \Gamma(\mathrm{Hom}(-, A))$ is a natural transformation as well. From the preceeding lemmas we know that $V \cong \mathrm{Hom}(\Gamma(V), A)$ and $B \cong \Gamma(\mathrm{Hom}(B, A))$ hold for all $A$-algebraic sets $V$ and for all finitely generated subalgebras $B$ of a power of $A$. We have to show that we have commutative diagrams.

a) Let $V \subseteq A^n$ and $W \subseteq A^m$ be $A$-algebraic sets and let $g\colon V \longrightarrow W$ be a polynomial morphism. We must show that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\eta_V} & \mathrm{Hom}(\Gamma(V), A) \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{Hom}(\Gamma(g),A)} \\ W & \xrightarrow{\eta_W} & \mathrm{Hom}(\Gamma(W), A) \end{array}$$

Let $\vec{a} \in V$ and let $g$ be represented by $f_1, \ldots, f_m$. Then $\eta_W(g(\vec{a})) = \eta_W(f_1(\vec{a}), \ldots, f_m(\vec{a})) = g_{(f_1(\vec{a}),\ldots,f_m(\vec{a}))}$ with $g_{(f_1(\vec{a}),\ldots,f_m(\vec{a}))}(q) = q(f_1(\vec{a}), \ldots, f_m(\vec{a}))$ for $q \in \Gamma(V)$.

On the other hand $(\mathrm{Hom}(\Gamma(g), A))(\eta_V(\vec{a})) = \mathrm{Hom}(\Gamma(g), A)(g_{\vec{a}})$. Here we have $g_{\vec{a}}\colon \Gamma(V) \longrightarrow A$, given by $g_{\vec{a}}(p) := p(\vec{a})$. $\Gamma(g)$ from $\Gamma(W)$ to $\Gamma(V)$ is given by $q \mapsto [q \circ g](V^{\perp})$. Now, by definition, $(\mathrm{Hom}(\Gamma(g), A))(g_{\vec{a}}) = g_{\vec{a}} \circ \Gamma(g)$ with $g_{\vec{a}} \circ \Gamma(g)(q) = g_{\vec{a}}(q \circ g) = (q \circ g)(\vec{a}) = q(g(\vec{a})) = q(f_1(\vec{a}), \ldots, f_m(\vec{a}))$.

b) Let $B \cong F_n(X)/\Phi$ and $C \cong F_m(X)/\Psi$ be objects from $S_f P(A)$ and let $h\colon F_n(X)/\Phi \longrightarrow F_m(X)/\Psi$ be a homomorphism. We must show that the diagram

$$\begin{array}{ccc} F_n(X)/\Phi & \xrightarrow{\varepsilon_B} & \Gamma(\mathrm{Hom}(F_n(X)/\Phi, A)) \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle \Gamma(\mathrm{Hom}(h,A))} \\ F_m(X)/\Psi & \xrightarrow{\varepsilon_C} & \Gamma(\mathrm{Hom}(F_m(X)/\Psi, A)) \end{array}$$

commutes.

Let $p \in F_n(X)/\Phi$. We have $\varepsilon_C(h(p)) = \beta_{h(p)}$ with $\beta_{h(p)}\colon F_m(X)/\Psi \longrightarrow A$ given by $\beta_{h(p)}(q) = q(h(p))$.

Similarly, $\Gamma(\mathrm{Hom}(h, A))(\varepsilon_B(p)) = \Gamma(\mathrm{Hom}(h, A))(\beta_p)$ with $\beta_p\colon F_n(X)/\Phi \longrightarrow A$, given by $\beta_p(t) := t(p)$. Hence $\Gamma(\mathrm{Hom}(h, A))(\varepsilon_B(p)) = \varepsilon_B(p) \circ \mathrm{Hom}(h, A)$ with $(\varepsilon_B(p) \circ \mathrm{Hom}(h, A))(q) = \varepsilon_B(p)(\mathrm{Hom}(h, A)(q)) = \varepsilon_B(p)(q \circ h) = (q \circ h)(p)$, and the proof is completed. $\square$

*Proof of Theorem* 1.2. We know that every intent of $\mathbb{K} := (A^n, F \times F, \perp)$ is a congruence relation. Therefore, let $\Phi$ be a fixed congruence relation. Since $\Phi^{\perp}$ is an $A$-algebraic set, we conclude from 2.3 that $F(X)/\Phi^{\perp\perp} \cong \mathrm{HOM}(\Phi^{\perp}, A)$, which means that $F(X)/\Phi^{\perp\perp}$ can be embedded into a power of $A$. Since $\Phi \subseteq \Phi^{\perp\perp}$ and since $\sqrt{\Phi}$ is the smallest congruence relation containing $\Phi$ such that $F(X)/\sqrt{\Phi}$ can be embedded into a power of $A$ we conclude $\sqrt{\Phi} \subseteq \Phi^{\perp\perp}$.

In order to show that $\Phi^{\perp\perp} \subseteq \sqrt{\Phi}$, it is sufficient to show that $\sqrt{\Phi} = \sqrt{\Phi}^{\perp\perp}$. (Then $\Phi \subseteq \sqrt{\Phi}$ implies $\Phi^{\perp\perp} \subseteq \sqrt{\Phi}^{\perp\perp}$, hence $\Phi^{\perp\perp} \subseteq \sqrt{\Phi}$). $F(X)/\sqrt{\Phi}$ is a finitely generated subalgebra of a power of $A$. Our functorial correspondence tells us that $F(X)/\sqrt{\Phi} \cong \Gamma(\mathrm{Hom}(F(X)/\sqrt{\Phi}, A))$ and the latter expression is isomorphic to $F(X)/\sqrt{\Phi}^{\perp\perp}$, since $\mathrm{Hom}(F(X)/\sqrt{\Phi}, A) \cong \sqrt{\Phi}^{\perp}$ as sets. Now $F(X)/\sqrt{\Phi} \cong F(X)/\sqrt{\Phi}^{\perp\perp}$ shows that $\sqrt{\Phi} = \sqrt{\Phi}^{\perp\perp}$ and the proof is finished. $\square$

We observe that for the radical $\sqrt{\Phi}$ of a congruence relation $\Phi$ we have the following formula:

$$\sqrt{\Phi} = \bigcap \Big\{ \operatorname{kern} h \;\Big|\; h\colon F(X) \longrightarrow A,\ \Phi \subseteq \operatorname{kern} h \Big\}$$

Indeed, the set of all homomorphisms $h_j\colon F(X) \longrightarrow A$ with $\Phi \subseteq \operatorname{kern} h_j$, $j \in J$, induces a homomorphism $h\colon F(X) \longrightarrow A^J$ with kern $h = \bigcap_{j\in J} \operatorname{kern}(\pi_j \circ h)$. We have $F(X)/\text{kern } h \cong h(F(X)) \subseteq A^J$, which shows that $\bigcap\{\operatorname{kern} h \mid h\colon F(X) \longrightarrow A,\ \Phi \subseteq \operatorname{kern} h\}$ is a radical congruence containing $\Phi$. On the other hand, if $h\colon F(X)/\sqrt{\Phi} \longrightarrow A^J$ is an isomorphism onto its image, then there is a homomorphism $h\colon F(X) \longrightarrow A^J$ with $\operatorname{kern} h = \sqrt{\Phi}$. We conclude $\sqrt{\Phi} = \bigcap_{j\in J} \operatorname{kern}(\pi_j \circ h)$ and the other inclusion follows.

**Corollary 2.13.** *The lattice of all $A$-algebraic subsets of an $A$-algebraic set $V$ is dually isomorphic to the lattice of all radical congruences of the coordinate algebra $\Gamma(V)$ of $V$.*

*Proof.* From 2.12 we conclude that the lattice of $A$-algebraic subsets of an algebraic set $V$ is dually isomorphic to the lattice of all radical congruences on $F(X)$ containing $V^{\perp}$. Since $\Phi$ is a radical congruence in $F(X)$ if and only if $\Phi/V^{\perp}$ is a radical congruence on $F(X)/V^{\perp}$, the desired dual equivalence follows. $\square$

We wish to formulate several notions which are analogous to notions from algebraic geometry in the new setting.

**Definition 2.14.** Let $V$ be an $A$-algebraic set. The **geometric dimension** of $V$ is the length of a longest chain of distinct $\bigvee$-irreducible $A$-algebraic subsets of $V$. The **algebraic dimension** of $V$ is defined as the length of a longest chain of $\bigwedge$-irreducible elements in the lattice of all radical congruences of $\Gamma(V)$.[3]

**Corollary 2.15.** *Algebraic dimension and geometric dimension are equal.*

In the remaining part of this section, we characterize the notion of isomorphism betweeen algebraic sets via isomorphisms between suitable contexts which we define now. Let $V \subseteq A^n$ be an $A$-algebraic set. We may consider the subcontext $(V, F(X) \times F(X), \perp)$ of $\mathbb{K}$, where we write $\perp$ for reasons of simplicity for the relation $\perp \cap (V \times (F(X) \times F(X)))$. We can clarify this context if we identify those pairs $(p_1, q_1), (p_2, q_2)$ for which we have $p_1(\vec{a}) = q_1(\vec{a})$ if and only if $p_2(\vec{a}) = q_2(\vec{a})$ for all $\vec{a} \in V$. Thus the clarified version of our subcontext is canonically isomorphic to the context $(V, \Gamma(V) \times \Gamma(V), \perp)$ with $\vec{a} \perp ([p],[q])\colon \iff p(\vec{a}) = q(\vec{a})$.[4] We will denote this context by $\mathbb{K}_V$.

[3] The length of a chain of the form $V_0 \subseteq V_1 \subseteq \ldots \subseteq V_m$ with distinct $V_i$ is defined to be $m$.

[4] Strictly speaking, the discussion concerns the context where only the set of attributes is clarified.

**Corollary 2.16.** *Let $V \subseteq A^n$ and $W \subseteq A^m$ be $A$-algebraic sets. Then the following statements are equivalent:*

a) $V \cong W$.

b) $\Gamma(V) \cong \Gamma(W)$.

*In this case, $\mathbb{K}_V \cong \mathbb{K}_W$ also holds. If $F(X)$ contains all constant operations $c_a$ for $a \in A$, then $\mathbb{K}_V \cong \mathbb{K}_W$ implies $V \cong W$.*

*Proof.* The equivalence of a) and b) follows immediately from the Nullstellensatz for general algebras. Therefore, let $g\colon V \longrightarrow W$ be an isomorphism. Then so is $\Gamma(g^{-1})\colon \Gamma(V) \longrightarrow \Gamma(W)$. In particular, $g$ and $\Gamma(g^{-1})$ are bijective and $(g, \Gamma(g^{-1}))\colon \Gamma(V) \longrightarrow \Gamma(W)$ is an isomorphism of contexts: $\vec{a} \perp (p,q) \iff p(\vec{a}) = q(\vec{a}) \iff (p \circ g^{-1})(g(a)) = (q \circ g^{-1})(g(a)) \iff (\Gamma(g^{-1})(p))(g(a)) = (\Gamma(g^{-1})(q))(g(a))$.

Now let the extra condition on $F(X)$ be satisfied and let $(\alpha, \beta)\colon \mathbb{K}_V \longrightarrow \mathbb{K}_W$ be an isomorphism. Consider a pair $([p],[p]) \in \Gamma(V) \times \Gamma(V)$. Then $\beta([p],[p]) \perp \vec{b}$ holds for all $\vec{b} \in W$ and $\beta([p],[p])$ belongs to the diagonal congruence. Hence, $\beta([p],[p]) = ([q],[q])$ for some $q \in F(X,m)$. We obtain a bijective map $\beta\colon \Gamma(V) \longrightarrow \Gamma(W)$ if we send $[p]$ to $[q]$. We claim that $\beta$ is an isomorphism. Since $F(X)$ contains all constant operations we observe that $p(\vec{a}) = c \in A$ is always equivalent to $\beta(p)(\alpha(\vec{a})) = c$. Therefore, we conclude $\beta(f_i([p_1], \ldots, [p_t]))(\alpha(\vec{a})) = f_i([p_1], \ldots, [p_t])(\vec{a}) = f_i(p_1(\vec{a}), \ldots, p_t(\vec{a})) = f_i((\beta([p_1]))(\alpha(\vec{a})), \ldots, (\beta([p_t]))(\alpha(\vec{a}))) = f_i(\beta([p_1]), \ldots, \beta([p_t]))(\alpha(\vec{a}))$ for all $\vec{a} \in A$, and $\beta$ is an isomorphism. $\square$

**Remark.** The question arises whether there are weaker conditions on $F(X)$ that guarantee that $\mathbb{K}_V \cong \mathbb{K}_W$ implies $V \cong W$.

## 3. Application to Algebraic Geometry

The two following sections work out which basic results from Algebraic Geometry already follow from our general observations and which of the results need Commutative Algebra. Let $K$ be a field and let $K[x_1, \ldots, x_n]$ be the ring of polynomials (in the classical sense) in $n$ variables over $K$. In Algebraic Geometry one is interested in the solution sets of systems of equations of the form $f_i(x_1, \ldots, x_n) = 0$, $i \in I$. Such sets are called **algebraic varieties** (over $K$). In order to be able to apply the results from the second section let $A$ be the general algebra with carrier set $K$ endowed with addition and multiplication and with the constant operations. Then we have $F_n(X, A) = K[x_1, \ldots, x_n]$ and we obtain the context $\mathbb{K}_A := (K^n, K[x_1, \ldots, x_n] \times K[x_1, \ldots, x_n], \perp)$ with $\vec{a} \perp (f,g) \iff f(\vec{a}) = g(\vec{a})$. For $C \subseteq K^n$, we consider $C^\perp = \{(f,g) \mid \forall \vec{c} \in C : f(\vec{c}) = g(\vec{c})\}$. Since $C^\perp$ is a congruence relation, we conclude $(0,0) \in C^\perp$ and that, if $(f_1, g_1), (f_2, g_2) \in C^\perp$ and if $r \in K$, we have $(f_1 + f_2, g_1 + g_2) \in C^\perp$ and $(rf_1, rg_1) \in C^\perp$. Consider the congruence class $[0]C^\perp$ belonging to the zero polynomial, $[0]C^\perp = \{f \mid \forall \vec{a} \in C : f(\vec{a}) = 0\}$. $[0]C^\perp =: I$ is an ideal in $K[x_1, \ldots, x_n]$, because of the above reasoning, and we have $(f,g) \in C^{\perp} \iff f-g \in I$. According to our previous observations, we have $C^{\perp\perp} = \{\vec{a} \mid f(\vec{a}) = 0 \text{ for all } f \text{ with } f(\vec{c}) = 0 \text{ for all } \vec{c} \in C\}$. Therefore we can consider the simpler context $\mathbb{K} := (K^n, K[x_1, \dots, x_n], \perp)$ with $\vec{a} \perp f :\iff f(\vec{a}) = 0$.

**Corollary 3.1.** *Let $V \subseteq K^n$ and let $A$ be as above. Then $V$ is an $A$-algebraic set in the sense of Section 2 if and only if it is an algebraic variety over $K$ in the sense of Algebraic Geometry. The concept lattices $\underline{\mathfrak{B}}(\mathbb{K}_A)$ and $\underline{\mathfrak{B}}(\mathbb{K})$ are isomorphic via the isomorphism which sends $(V, V^{\perp})$ to $(V, I(V))$ where $I(V)$ is the ideal $[0]V^{\perp}$.*

**Example.** Consider the ideal $I := \langle x^2 \rangle \subseteq K[x]$. We have $I^{\perp} = \{0\}$ and $I^{\perp\perp} = \langle x \rangle$. This phenomenon is due to the fact that $f(a) = 0$ is equivalent to $f^m(a) = 0$ for all integers $m$. If $K$ is algebraically closed, this is the only reason why $I^{\perp\perp}$ can become larger than $I$.

**Definition 3.2.** Let $I \subseteq K[x_1, \dots, x_n]$ be an ideal. The **radical** $\sqrt{I}$ of $I$ is defined as $\sqrt{I} := \{f \mid f^m \in I \text{ for some } m \in \mathbb{N}\}$.

**Theorem 3.3** (Hilbert's Nullstellensatz)**.** *Let $K$ be algebraically closed and let $I \subseteq K[x_1, \dots, x_n]$ be an ideal. Then we have $I^{\perp\perp} = \sqrt{I}$ in $\mathbb{K} := (K^n, K[x_1, \dots, x_n], \perp)$.*

*Proof.* A proof can be found in (**[Ku80**, I.3]). Note that this result does not follow from 1.2. $\square$

**Corollary 3.4.** *Let $K$ be algebraically closed. Let $\Phi \subseteq K[x_1, \dots, x_n] \times K[x_1, \dots, x_n]$ be a congruence relation. Then $[0]\sqrt{\Phi} = \sqrt{[0]\Phi}$ holds.*

We see that not every congruence relation of $\mathbb{K}_A$ is closed. When $I$ is an ideal that is not equal to its radical, then the congruence relation $\Phi_I$ defined by $f \Phi_I g :\iff f - g \in I$ is not closed. Instead, its closure will be the congruence relation $\Phi$ determined by the radical of $I$, $f \Phi g :\iff f - g \in \sqrt{I}$.

**Definition 3.5.** Let $V \subseteq K^n$ and $W \subseteq K^n$ be algebraic varieties. A **polynomial morphism** from $V$ to $W$ is a mapping $\varphi$ for which there are polynomials $f_1, \dots, f_m \in K[x_1, \dots, x_n]$ such that $\varphi(a) = (f_1(a), \dots, f_m(a))$ holds for all $a \in K^n$.

**Corollary 3.6.** *Let $K$ be algebraically closed. The category of all algebraic varieties over $K$ with polynomial morphisms*[5] *is dually equivalent to the category of all coordinate algebras $K[V] := K[x_1, \dots, x_n]/V^{\perp}$ over $K$ with $K$-algebra homomorphisms, i.e. ring homomorphisms which are the identity on $K$.*

[5] Here we talk about polynomials in the usual sense.

**Corollary 3.7.** *Let $K$ be algebraically closed and let $V \subseteq K^n$ be an algebraic variety. Consider the context $\mathbb{K}_V := (V, K[V], \perp)$ with $\vec{a} \perp [f]\colon \iff f(\vec{a}) = 0$. The extents of $\mathbb{K}_V$ are exactly the subvarieties of $V$ and the intents are exactly the* **reduced** *ideals of $K[V]$, i.e. those ideals which are equal to its radical. Hence the lattice of all algebraic varieties contained in $V$ and the lattice of all reduced ideals of $K[V]$ are dually isomorphic.*

*Proof.* It remains to show that for an ideal $[I] \subseteq K[V]$ its radical $\sqrt{[I]} := \{[g] \mid [g]^m \in [I] \text{ for some } m \in \mathbb{N}\}$ in $K[V]$ is equal to $[\sqrt{I}]$. This is a standard result which follows immediately from the homomorphism theorem for rings. □

**Corollary 3.8.** *Let $V \subseteq K^n$ and $W \subseteq K^m$ be algebraic varieties. Then the following statements are equivalent:*

a) *$V \cong W$.*
b) *$K[V] \cong K[W]$ as $K$-algebras.*
c) *$\mathbb{K}_V \cong \mathbb{K}_W$ via an isomorphism $(\alpha, \beta)$ which satisfies the additional condition that $f(\vec{a}) = \beta(f)(\alpha(\vec{a}))$ holds for all $\vec{a} \in V$ and $f \in K[V]$.*

*Proof.* This follows from 2.16. The extra conditions stem from the fact that $A$, defined in the beginning of this section, contains the constant operations. □

**Definition 3.9.** An algebraic variety $V \subseteq K^n$ is said to be **irreducible** if it is not the union of two proper subvarieties.

**Theorem 3.10.** *Every finite union of algebraic varieties is an algebraic variety. Every algebraic variety allows for a unique decomposition $V = V_1 \cup \dots V_m$ into irreducible varieties.*

*Proof.* A proof can be found in ([**Ku80**, I.2]). □

**Example.** Consider the algebraic variety $V := \langle xy \rangle^\perp = \{(a, b) \mid a = 0 \vee b = 0\}$. It is the union of the $x$-axis and the $y$-axis. These are its irreducible components, which can be deduced from the next corollary: The vanishing ideal of the $x$-axis is $langley\rangle$ and the vanishing ideal of the $y$-axis is $\langle x \rangle$. Both ideals are prime ideals.

**Corollary 3.11.** *$V$ is irreducible in the sense of* 3.9 *if and only if $V^\perp$ is not a finite intersection of two or more reduced ideals. Hence $V$ is irreducible if and only if $V^\perp$ is prime.*

*Proof.* The first equivalence is obvious. Let $V^\perp$ be prime. Suppose $V^\perp = I_1 \cap \dots \cap I_m$ with reduced ideals $I_j$. Choose $f_j \in I_j \setminus V^\perp$. Then $f_1 f_2 \dots f_m \in I_1 \cap \dots \cap I_m = V^\perp$ but $f_1 \notin V^\perp$ and $f_2 \dots f_m \notin V^\perp$. This is a contradiction to $V^\perp$ being prime. We conclude that $V$ is irreducible. For the other implication let $V$ be irreducible. Let $f_1 f_2 \in V^\perp$. Then $V = (V \cap f_1^\perp) \cup (V \cap f_2^\perp)$. Since $V$ is irreducible we conclude without loss of generality that $V = V \cap f_1^\perp$. Hence $f_1 \in V^\perp$ and $V^\perp$ is prime. □

**Definition 3.11.** Let $R$ be a commutative ring and let $P \subseteq R$ be a prime ideal. The **height** $h(P)$ of $P$ is the length of a longest chain of prime ideals contained in $P$. The **dimension** of $R$ is the supremum over all $h(P)$ for $P \subseteq R$.

**Corollary 3.12.** *Let $V$ be an algebraic variety. Then* $\dim V = \dim K[V] = \dim K[x_1, \dots, x_n] - h(V^\perp)$.

All the results listed in this section are standard facts from Algebraic Geometry. Some of the proofs can be given in the non-standard setting used here, as we have seen so far. Hilbert's Nullstellensatz and the existence of a decomposition into irreducible varieties are intrinsic properties of the classical situation. The remaining properties can be settled within the more general framework. This idea is carried on in the last section, where we give a general description of the construction of the spectrum related to an algebraic variety.

## 4. A Modification of the Set of Objects

In classical Algebraic Geometry the context $\mathbb{K} := (K^n, K[x_1, \dots, x_n], \perp)$ is often replaced by the context $\mathbb{K}_{\mathrm{Spec}} := (\operatorname{Spec} K[x_1, \dots, x_n], K[x_1, \dots, x_n], \ni)$ where the **spectrum** $\operatorname{Spec} K[x_1, \dots, x_n]$ of $K[x_1, \dots, x_n]$ is the set of all prime ideals contained in $K[x_1, \dots, x_n]$. This is justified by the fact that $I^{\perp\perp} = I^{\ni\ni}$ holds for all ideals $I \subseteq K[x_1, \dots, x_n]$. Usually, one understands the spectrum as follows: each point $a \in K^n$ corresponds to exactly one maximal ideal $\mathfrak{m}_a$ in $Spec\, K[x_1, \dots, x_n]$. In this way, we can consider the maximal ideals as "points" of $\operatorname{Spec} K[x_1, \dots, x_n]$. For each irreducible variety $V$, its ideal $\mathfrak{p}_V := V^\perp$ is a prime ideal. For such "points" we have $(\mathfrak{p}_V)^{\ni\ni} = \{\mathfrak{p} \mid \mathfrak{p} \supseteq \mathfrak{p}_V\}$. In particular, we have $a \in V$ if and only if $(\mathfrak{p}_V)^{\ni\ni} \ni \mathfrak{m}_a$. If we identify $V$ and $(\mathfrak{p}_V)^{\ni\ni}$ we can consider the extents in the spectrum as algebraic varieties. Now, every irreducible "variety" $V$ is the extent $g^{II}$ of some **object concept** $(g^{II}, g^I)$, indeed it is the object concept of $\mathfrak{p}_V$. $\mathfrak{p}_V$ is called a **generic point** of the variety $V$. Hence, the spectrum consists of closed points (maximal ideals), which are in one-to-one correspondence with the points of the affine space $K^n$, and additionally of generic points, whose topological closure can be regarded as an irreducible algebraic variety.

The idea to introduce "general objects" can also be carried out for formal contexts in general (cf. [**Pr99**]). Using the interpretation of General Algebraic Geometry in terms of formal contexts we get an analogue to the spectrum of classical algebraic varieties for $A$-algebraic sets. In order to do this, we model the situation from classical Algebraic Geometry described above.

**Definition 4.1.** Let $\mathbb{K} := (G, M, I)$ be a formal context. An extent of $\mathbb{K}$ is called $\cup$-**irreducible** if it is not the union of two proper subextents. Let $\mathcal{U}_{\mathrm{irr}}$ be the set of all $\cup$-irreducible extents. Let $\hat{G} := \{B \subseteq M \mid B = A^I \text{ for some } A \in \mathcal{U}_{\mathrm{irr}}\}$ and $\hat{K} := (\hat{G}, M, \ni)$.

**Lemma 4.2.** *Let $\mathbb{K}$ be a formal context. If each extent of $\mathbb{K}$ is the union of $\cup$-irreducible extents, then the concept lattices $\underline{\mathfrak{B}}(\mathbb{K})$ and $\underline{\mathfrak{B}}(\hat{\mathbb{K}})$ are isomorphic with identical intents. Conversely, if $\underline{\mathfrak{B}}(\mathbb{K})$ and $\underline{\mathfrak{B}}(\hat{\mathbb{K}})$ are isomorphic then every extent of $\mathbb{K}$ is the union of $\cup$-irreducible extents.*

*Proof.* A proof can be found in **[Be99]**. □

The reader who is familiar with Formal Concept Analysis will observe that the result above is trivial for finite contexts because in this case the reduced versions of $\hat{\mathbb{K}}$ and $\mathbb{K}$ are isomorphic.

**Corollary 4.3.** *Let $A$ be a general algebra and $n \in \mathbb{N}$ such that every $A$-algebraic set of $\mathbb{K}_A := (A^n, F_n(X,A) \times F_n(X,A), \perp)$ is the union of $\cup$-irreducible $A$-algebraic sets. Then $\underline{\mathfrak{B}}(\mathbb{K}_A)$ and $\underline{\mathfrak{B}}(\hat{\mathbb{K}}_A)$ are isomorphic. Hence we have $\sqrt{\Phi} = \Phi^{\ni\ni}$, which means that $\sqrt{\Phi}$ is equal to the intersection of all congruence relations $V^{\perp}$ where $V$ is a $\cup$-irreducible $A$-algebraic set and where $\Phi \subseteq V^{\perp}$.*

We return to the classical case.

**Corollary 4.4.** *Let $K$ be algebraically closed. Then we have $\underline{\mathfrak{B}}(\mathbb{K}_{\mathrm{Spec}}) \cong \underline{\mathfrak{B}}(\mathbb{K})$. Hence, for an ideal $I \subseteq K[x_1, \dots, x_n]$ its radical $\sqrt{I}$ is equal to the intersection of all prime ideals containing $I$.*

*Proof.* Because of 3.11 and because every prime ideal is an intent when $K$ is algebraically closed, $\{\mathfrak{p} \mid \mathfrak{p} \subseteq K[x_1, \dots, x_n]\,,\ \mathfrak{p} \text{ is a prime ideal}\}$ is the set of general objects of $(K^n, K[x_1, \dots, x_n], \perp)$. Therefore we can apply 4.3. In particular, we have $I^{\perp\perp} = I^{\ni\ni}$ where $I^{\ni} = \{\mathfrak{p} \mid \mathfrak{p} \supseteq I\}$ and $I^{\ni\ni} = \bigcap \{\mathfrak{p} \mid \mathfrak{p} \supseteq I\}$ by the definition of $\ni$. □

Finally, it should be mentioned that the introduction of general objects makes good sense in particular in the classical case, mainly for two reasons. Since finite unions of algebraic varieties are again algebraic varieties, the notions of $\cup$-irreducibility and (finite) irreducibility with respect to suprema coincide. Moreover, it can be seen easily that every single point in the affine space forms an algebraic variety. Therefore, the original objects can still be identified within the spectrum. In particular, irreducible closed sets in the spectrum can be thought as usual algebraic varieties with an additional generic point.

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