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A NOTE ON THE LEBESGUE–DARST DECOMPOSITION THEOREM

N. BOUZAR

Abstract. The purpose of this note is to show that the Lebesgue-Darst decomposition for finitely additive measures on algebras can be derived from its well-known counterpart for countably additive measures on σ-algebras and the Stone representation theorem for algebras.

The Lebesgue decomposition theorem (cf. [**Ro1968**] for example) is a classical result in measure theory. It states that if μ and λ are real-valued, countably additive measures on a σ-algebra $\mathcal{F}$ of subsets of a set Ω, then μ can be decomposed uniquely as $\mu = \mu_0 + \mu_1$, where μ_0 and μ_1 are countably additive measures on $\mathcal{F}$ such that μ_0 is absolutely continuous w.r.t. λ and μ_1 is singular w.r.t. λ. The theorem is essentially a consequence of the Radon-Nikodym theorem. Darst [**Da1962**] extended the decomposition to bounded, finitely additive set functions defined on an algebra. Darst's proof is self-contained but rather intricate. It is based on an approximate Hahn decomposition theorem and a weak convergence argument. Schmidt [**GS1983**] provided a different proof using a Banach lattice argument (cf. his Theorem 2.1.4, p. 64). The purpose of this note is to show that Darst's result can in fact be derived from the countably additive version of the decomposition (recalled above) via the Stone representation theorem for algebras.

In the sequel, $\mathcal{A}$ will always denote an algebra of subsets of a set Ω. By a charge (resp. a measure) we will mean a real-valued, finitely additive (resp. countably additive) set function defined on $\mathcal{A}$. If ν is a bounded charge, then $|\nu|$ is its total variation. Let μ and λ be two bounded charges on $\mathcal{A}$. We recall that μ and λ are said to be mutually singular, or $\mu \perp \lambda$, (resp. μ is said to be absolutely continuous w.r.t. λ, or $\mu \ll \lambda$) if for any $\varepsilon > 0$, there exists $A \in \mathcal{A}$ such that $|\mu|(A) < \varepsilon$ and $|\lambda|(A^c) < \varepsilon$ (resp. for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $|\lambda|(A) < \delta$, then $|\mu|(A) < \varepsilon$).

We first need a lemma.

Lemma 1. *Let μ and λ be bounded measures on $\mathcal{A}$. Then there exist uniquely measures μ_0 and μ_1 on $\mathcal{A}$ such that $\mu = \mu_0 + \mu_1$, $\mu_0 \ll \lambda$ and $\mu_1 \perp \lambda$.*

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Proof. Let $\overline{\mu}$ and $\overline{\lambda}$ be respectively the countably additive extensions of μ and λ to $\sigma(\mathcal{A})$, the σ-field generated by $\mathcal{A}$. The countably additive version of the Lebesgue decomposition theorem (cf. Royden [**Ro1968**]) applied to $\overline{\mu}$ and $\overline{\lambda}$ gives $\overline{\mu} = \overline{\mu}_0 + \overline{\mu}_1$, where $\overline{\mu}_0$ and $\overline{\mu}_1$ are measures on $\sigma(\mathcal{A})$ such that $\overline{\mu}_0 \ll \overline{\lambda}$ and $\overline{\mu}_1 \perp \overline{\lambda}$. Denote by R the operator that restricts a set function from $\sigma(\mathcal{A})$ to $\mathcal{A}$. If ν is a measure on $\sigma(\mathcal{A})$, then $|R\nu| = R|\nu|$. It follows then easily that $R\overline{\mu}_0 \ll R\overline{\lambda}$. It remains to prove the claim that $R\overline{\mu}_1 \perp R\overline{\lambda}$. Let

$$\mathcal{F} = \{A \in \sigma(\mathcal{A}) : \forall\, \varepsilon > 0, \exists\, B \in \mathcal{A}, |\overline{\mu}_1|(A \triangle B) < \varepsilon \text{ and } |\overline{\lambda}|(A \triangle B) < \varepsilon\}.$$

Using the fact that $|\overline{\mu}_0|$ and $|\overline{\mu}_1|$ are finite, nonnegative measures on $\sigma(\mathcal{A})$, it is a simple exercise to show that $\mathcal{F}$ is a σ-field containing $\mathcal{A}$, and hence $\mathcal{F} = \sigma(\mathcal{A})$. Since $\overline{\mu}_1 \perp \overline{\lambda}$ as measures on a σ-field, there exists $A \in \sigma(\mathcal{A})$ such that $|\overline{\mu}_1|(A) = 0$ and $|\overline{\lambda}|(A^c) = 0$. Therefore, for $\varepsilon > 0$, there is $B \in \mathcal{A}$ such that $|R\overline{\mu}_1|(B) = |\overline{\mu}_1|(A \triangle B) < \varepsilon$ and $|R\overline{\lambda}|(B^c) = |\overline{\lambda}|(A \triangle B) < \varepsilon$, which in turn implies the claim. The proof of the uniqueness of the decomposition is omitted. $\square$

Next we use the Stone representation of an algebra to extend Lemma 1 to charges.

Proposition 2. *Let μ and λ be bounded charges on $\mathcal{A}$. Then there exist uniquely charges μ_0 and μ_1 on $\mathcal{A}$ such that $\mu = \mu_0 + \mu_1$, $\mu_0 \ll \lambda$ and $\mu_1 \perp \lambda$.*

Proof. By Stone representation theorem (cf. for example Sikorski [**Si1969**]), there exists a compact, Hausdorff, totally disconnected topological space $\widehat{\Omega}$ and a Boolean isomorphism $\tau\colon \mathcal{A} \to \widehat{\mathcal{A}}$ where $\widehat{\mathcal{A}}$ is the algebra of clopen subsets of $\widehat{\Omega}$. If ν is a charge on $\mathcal{A}$, then the set function $\widehat{\nu}$ defined on $\widehat{\mathcal{A}}$ by

$$\widehat{\nu}(\widehat{\mathcal{A}}) = \nu(A), \quad \widehat{A} = \tau(A),\ A \in \mathcal{A}, \tag{1}$$

is also a charge. Moreover, if ν is bounded, so is $\widehat{\nu}$ and

$$|\widehat{\nu}|(\tau(A)) = |\nu|(A),\ A \in \mathcal{A}. \tag{2}$$

By compactness and clopenness, if $(\widehat{A}_n, n \in \mathbb{N})$ is a sequence of pairwise, disjoint elements of $\widehat{\mathcal{A}}$, then $\widehat{A}_n = \phi$ for all but finitely many $\widehat{A}_n$'s, which implies that $\widehat{\nu}$ is necessarily countably additive on $\widehat{\mathcal{A}}$. Therefore, by applying Lemma 1 to the bounded measures $\widehat{\mu}$ and $\widehat{\lambda}$ on $\widehat{\mathcal{A}}$ (corresponding through (1) to the given μ and λ respectively), we have $\widehat{\mu} = \widehat{\mu}_0 + \widehat{\mu}_1$ with $\widehat{\mu}_0 \ll \widehat{\lambda}$ and $\widehat{\mu}_1 \perp \widehat{\lambda}$. Define for $A \in \mathcal{A}$,

$$\mu_0(A) = \widehat{\mu}_0(\tau(A)) \text{ and } \mu_1(A) = \widehat{\mu}_1(\tau(A)).$$

Clearly, μ_0 and μ_1 are charges on $\mathcal{A}$ and $\mu = \mu_0 + \mu_1$. Since $\widehat{\mu}_0$, $\widehat{\mu}_1$ and $\widehat{\lambda}$ satisfy (2), the assertion that $\mu_0 \ll \lambda$ and $\mu_1 \perp \lambda$ is a straightforward consequence of $\widehat{\mu}_0 \ll \widehat{\lambda}$ and $\widehat{\mu}_1 \perp \widehat{\lambda}$. Again, we omit the proof of the uniqueness of the decomposition. $\square$

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N. Bouzar, Mathematics Department, University of Indianapolis, 1400 E. Hanna Avenue, Indianapolis, Indiana 46227; *e-mail*: nbouzar@uindy.edu