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ON n-PERMUTABLE AND DISTRIBUTIVE AT 0 VARIETIES

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Abstract. Mal'cev conditions characterizing varieties which are 3-permutable at 0, distributive and n-permutable at 0, and having weakly parallel classes, are presented. Every of these conditions is completed by an example of variety having this condition.

Some of congruence conditions were "localized at 0" by the author in **[2]**. However, the most useful "at 0" conditions are distributivity at 0, see **[1]**, **[4]** and permutability at 0, see e.g. **[5]**. Also congruence modularity was localized at 0 in **[3]**, however no Mal'cev condition characterizing varieties with this property was found. Moreover, recently P. Lipparini **[6]** characterized varieties which are simultaneously congruence distributive and n-permutable. We also try this attempt for the localized version to obtain a simple and useful Mal'cev condition. It is worth to say that, contrary to the case of distributivity or permutability, the so called n-permutability cannot be localized (the characterization of n-permutability at 0 presented in **[2]** is unfortunately wrong; an essential error in it was found out by Frank Lindauer). Hence, we do it here at least for 3-permutability at 0 (our Theorem 1) and, suprisingly, such a characterization can be derived by the standard way when it is taken together with distributivity at 0 (Theorem 2). Finally, we will describe varieties having weakly parallel congruence classes which, under certain conditions, satisfy also a version of distributivity.

From now on, every algebra will be considered with a constant, which is either a nullary operation of the similarity type or a nullary term. In every algebra or variety, this constant will be denoted by the symbol 0 although for some special algebras (e.g. for implicative algebras) it is usually denoted by another symbol (e.g. by 1).

Recall that an algebra $\mathcal{A}$ is **n-permutable at** 0 if for every $\Theta, \Phi \in \mathrm{Con}\ \mathcal{A}$,

$$[0]_{\Theta\circ\Phi\circ\Theta\cdots} = [0]_{\Phi\circ\Theta\circ\Phi\cdots}$$

where the symbol $\circ$ denotes relational product and there is exactly n factors on every side of the foregoing equality. A variety $\mathcal{V}$ is **n-permutable at** 0 if $\mathcal{V}$ has

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a constant 0 and every $\mathcal{A} \in \mathcal{V}$ has this property with respect to 0. Especially, $\mathcal{A}$ (and also $\mathcal{V}$) is **3-permutable at** 0 if

$$[0]_{\Theta\circ\Phi\circ\Theta} = [0]_{\Phi\circ\Theta\circ\Phi} \ .$$

It was shown in [**3**] that if $\mathcal{A}$ is 3-permutable at 0 then $\mathcal{A}$ is also modular at 0. Varieties of 3-permutable at 0 algebras can be characterized by a (strong) Mal'cev condition:

Theorem 1. *A variety $\mathcal{V}$ with* 0 *is* 3*-permutable at* 0 *if and only if there exist a ternary term f and a binary term d satisfying*

$$f(x,y,y) = x, \quad f(x,x,0) = d(x,0), \quad d(x,x) = 0\,.$$

Proof. Let $\mathcal{V}$ be 3-permutable at 0 and $F_{\mathcal{V}}(x,y,z)$ be a free algebra of $\mathcal{V}$ with three free generators x,y,z. Set $\Theta = \Theta(x,y) \vee \Theta(z,0)$ and $\Phi = \Theta(y,z)$. Then $x \in [0]_{\Theta\circ\Phi\circ\Theta}$ since $x\Theta y\Phi z\Theta 0$. Thus, by 3-permutability at 0, there are $f,g \in F_{\mathcal{V}}(x,y,z)$ such that

$$x\Phi f\Theta g\Phi 0\,.$$

Since $F_{\mathcal{V}}(x,y,z)$ is a free algebra with three generators, f and g are ternary terms, say $f = f(x,y,z)$, $g = g(x,y,z)$.

From $\langle x,f\rangle \in \Phi$ we deduce $f(x,y,y) = x$, from $\langle f,g\rangle \in \Theta$ we have $f(x,x,0) = g(x,x,0)$ and the last relation $\langle g,0\rangle \in \Phi$ yields $g(x,y,y) = 0$. We can set $d(x,y) = g(x,x,y)$. Then $d(x,0) = g(x,x,0) = f(x,x,0)$ and $d(x,x) = g(x,x,x) = 0$.

Conversely, let $\mathcal{V}$ have the terms f,d satisfying the identities of Theorem 1 and let $\mathcal{A} = (A,F) \in \mathcal{V}, a \in A,\ \Theta,\Phi \in \mathrm{Con}\ \mathcal{A}$. Suppose $a \in [0]_{\Theta\circ\Phi\circ\Theta}$. Then there are $b,c \in \mathcal{A}$ such that $a\Theta b\Phi c\Theta 0$. We infer immediately:

$$a = f(a,b,b)\Phi f(a,b,c)\Theta f(b,b,0) = d(b,0)\Theta d(b,c)\Phi d(b,b) = 0$$

i.e. $[0]_{\Theta\circ\Phi\circ\Theta} \subseteq [0]_{\Phi\circ\Theta\circ\Phi}$. The converse inclusion can be shown analogously, i.e. $\mathcal{A}$, and hence also $\mathcal{V}$, is 3-permutable at 0. □

Example 1. Consider a variety $\mathcal{V}$ of type $(2,2,0)$ where 0 is the nullary operation and the binary operations are denoted by $+,\cdot$. Let $\mathcal{V}$ satisfies the identities $x \cdot x = 0$, $x + 0 = x$, $x + (x \cdot 0) = x \cdot 0$. Set

$$\begin{aligned} f(x,y,z) &= x + (y \cdot z) \\ d(x,y) &= x \cdot y\,. \end{aligned}$$

Then

$$\begin{aligned} f(x,y,y) &= x + (y \cdot y) = x + 0 = x \\ d(x,x) &= x \cdot x = 0 \\ f(x,x,0) &= x + (x \cdot 0) = x \cdot 0 = d(x,0) \end{aligned}$$

thus $\mathcal{V}$ is 3-permutable at 0. □

Let us recall that an algebra $\mathcal{A}$ is **distributive at** 0 if

$$[0]_{\Theta\cap(\Phi\vee\Psi)} = [0]_{(\Theta\cap\Phi)\vee(\Theta\cap\Psi)} \tag{d}$$

for every $\Theta, \Phi, \Psi \in \operatorname{Con}\mathcal{A}$. A variety $\mathcal{V}$ with 0 is **distributive at** 0 if every $\mathcal{A} \in \mathcal{V}$ has this property.

It is worth saying that the identity (d) is not equivalent to its dual, see [**4**]. E.g. the variety of $\vee$-semilattices with 0 is distributive at 0 but there is a five element semilattice which does not satisfy the dual of (d). It was shown by J. Duda [**5**] that if $\mathcal{V}$ is *permutable at* 0, (i.e. 2-permutable at 0) then $\mathcal{V}$ satisfies (d) if and only if $\mathcal{V}$ satisfies the dual of (d).

Theorem 2. *For a variety $\mathcal{V}$ with* 0*, the following conditions are equivalent:*

(1) *$\mathcal{V}$ is n-permutable at* 0 *and distributive at* 0*;*
(2) *For each $\mathcal{A} \in \mathcal{V}$ and every $\alpha, \beta, \gamma \in$ Con $\mathcal{A}$,*

$$[0]_{\gamma\cap(\alpha\circ\beta\circ\alpha\circ\cdots)} = [0]_{(\gamma\cap\beta)\circ(\gamma\cap\alpha)\circ(\gamma\cap\beta)\circ\cdots}$$

(with n factors in both sides);
(3) *There exist ternary terms $q_0, \dots, q_{n-2}$ and a binary term d such that*

$$\begin{aligned} q_0(x,y,x) &= x, \quad d(x,x) = 0 = d(0,x),\\ q_i(0,x,0) &= 0 \quad \textit{for} \;\; i = 0,\dots,n-2,\\ q_{i-1}(x,x,z) &= q_i(x,z,z) \quad \textit{for}\; i = 1,\dots,n-2\\ q_{n-2}(x,x,0) &= d(x,0)\,. \end{aligned}$$

Proof. (1)$\Longleftrightarrow$(2) is evident. Prove (2) $\Rightarrow$ (3): Let $\mathcal{A} = F_{\mathcal{V}}(x_1,\dots,x_n)$ be a free algebra of $\mathcal{V}$ with n free generators $x_1,\dots,x_n$. We set

$$\begin{aligned} \gamma &= \Theta(x_1,0)\\ \alpha &= \Theta(x_1,x_2) \vee \Theta(x_3,x_4) \vee \cdots \vee \rho\\ \beta &= \Theta(x_2,x_3) \vee \Theta(x_4,x_5) \vee \cdots \vee \sigma\,, \end{aligned}$$

where $\rho = \Theta(x_n,0)$, $\sigma = \omega_A$ for n odd and $\rho = \omega_A$, $\sigma = \Theta(x_n,0)$ for n even (ω_A denotes the identity relation on A). Clearly $x_1\gamma 0$ and $x_1\alpha x_2\beta x_3\cdots 0$, whence

$$x_1 \in [0]_{\gamma\cap(\alpha\circ\beta\circ\alpha\circ\cdots)}.$$

By (2), there exist $a_0, a_1, \dots, a_n \in A$ such that $a_0 = x$, $a_n = 0$ and

$$a_0(\gamma\cap\beta)a_1(\gamma\cap\alpha)a_2(\gamma\cap\beta)\cdots a_n = 0.$$

Applying the standard procedure, there are n-ary terms $p_1, \dots, p_n$ satisfying

$$\begin{aligned}
x_1 = p_0(x_1, \dots, x_n), \quad 0 = p_n(x_1, \dots, x_n) \\
p_i(0, x_2, \dots, x_{n-1}, 0) = 0 \quad \text{for } i = 0, \dots, n \\
p_{i-1}(x_1, x_1, x_3, x_3, \dots, x_{n-1}, x_{n-1}) &= p_i(x_1, x_1, x_3, x_3, \dots, x_{n-1}, x_{n-1}) \\
&\qquad \text{for } n \text{ even and } i \text{ even,} \\
p_{i-1}(x_1, x_3, x_3, \dots, x_{n-1}, x_{n-1}, 0) &= p_i(x_1, x_3, x_3, \dots, x_{n-1}, x_{n-1}, 0) \\
&\qquad \text{for } n \text{ even and } i \text{ odd,} \\
p_{i-1}(x_1, x_1, x_3, x_3, \dots, x_{n-2}, x_{n-2}, 0) &= p_i(x_1, x_1, x_3, x_3, \dots, x_{n-2}, x_{n-2}, 0) \\
&\qquad \text{for } n \text{ odd and } i \text{ even,} \\
p_{i-1}(x_1, x_3, x_3, \dots, x_n, x_n) &= p_i(x_1, x_3, x_3, \dots, x_n, x_n) \\
&\qquad \text{for } n \text{ odd and } i \text{ odd.}
\end{aligned}$$

We can set $d(x, y) = p_{n-1}(x, \dots, x, y)$ and for $i = 1, \dots, n-2$

$$q_i(x, y, z) = p_1(\underbrace{x, \dots, x}_{i \text{ times}}, y, z, \dots, z, 0) \quad \text{if} \quad i \not\equiv n \bmod 2$$

and

$$q_i(x, y, z) = p_i(\underbrace{x, \dots, x}_{i \text{ times}}, y, z, \dots, z) \quad \text{if} \quad i \equiv n \bmod 2\,.$$

It is a routine way to check (3).

(3) $\Rightarrow$ (2): Let $\mathcal{A} = (A, F) \in \mathcal{V}$ and $\alpha, \beta, \gamma \in \operatorname{Con} \mathcal{A}$. Suppose $x \in [0]_{\gamma \cap (\alpha \circ \beta \circ \alpha \cdots)}$. Then there exist $a_0, a_1, \dots, a_{n-1} \in A$ such that

$$x = a_0 \alpha a_1 \beta a_2 \alpha \cdots 0\,.$$

Set $v_i = q_i(a_{i-1}, a_i, a_{i+1})$ for $i = 1, \dots, n-2$ and $v_{n-1} = d(a_{n-2}, a_{n-1})$. Applying (3), we have

$$\begin{aligned}
x = a_0 &= q_1(a_0, a_2, a_2) \beta q_1(a_0, a_1, a_2) = v_1 \alpha q_1(a_1, a_1, a_3) \\
&= q_2(a_1, a_3, a_3) \alpha q_2(a_1, a_2, a_3) = v_2 \beta q_2(a_2, a_2, a_4) = \cdots = 0\ ,
\end{aligned}$$

i.e. $x \beta v_1 \alpha v_2 \beta \cdots 0$. Further, $q_i(x, v_j, 0) \gamma q_i(0, v_j, 0) = 0$ for all i, j, thus also $q_i(x, v_j, 0) \gamma q_i(x, v_{j+1}, 0)$.

This yields

$$q_i(x, x, 0)(\gamma \cap \beta) q_i(x, v_1, 0)(\gamma \cap \alpha) q_i(x, v_2, 0)(\gamma \cap \beta) \cdots q_i(x, 0, 0)\,,$$

thus

$$q_i(x, x, 0) \mu q_i(x, 0, 0)$$

for $\mu = (\beta \cap \gamma) \circ (\alpha \cap \gamma) \circ (\beta \cap \gamma) \cdots$ (n factors).

Hence, we conclude

$$x = q_0(x, x, 0) = q_1(x, 0, 0) \mu q_1(x, x, 0) = q_2(x, 0, 0) \mu \cdots 0\,,$$

proving (2). $\square$

Example 2. Consider a variety of groupoids with 0 satisfying the identities

$$x \cdot x = 0, \quad 0 \cdot x = x, \quad x \cdot 0 = 0$$

(e.g. every reduct of an algebra of logic where instead of 0 we have the true-value 1 and $x \cdot y$ is $x \Rightarrow y$). We can take $n = 2$ (thus $i = 0$ is the only possibility) and $q_0(x, y, z) = x$, $d(x, y) = y \cdot x$. One can easily verify that $\mathcal{V}$ is distributive at 0 and permutable at 0 (= 2-permutable at 0), i.e. arithmetical at 0 in the terminology of [**5**].

Example 3. Let $\mathcal{V}$ be a variety of type $(2, 2, 0)$ where the binary operations are denoted by $+$ and $\cdot$, and let $\mathcal{V}$ satisfies the identities

$$\begin{aligned} x \cdot x = 0 &= 0 \cdot x \\ x + 0 &= x \\ x + (x \cdot 0) &= x \cdot 0 \\ 0 + (x \cdot 0) &= 0 \end{aligned}$$

We can set $n = 3$, $d(x, y) = x \cdot y$, $q_0(x, y, z) = x$ and $q_1(x, y, z) = x + (y \cdot z)$.

Then $d(x, x) = 0 = d(0, x)$ and

$$\begin{aligned} q_0(0, x, 0) = 0, \quad q_1(0, x, 0) &= 0 + (x \cdot 0) = 0 \\ q_0(x, x, z) = x = x + 0 &= x + (z \cdot z) = q_1(x, x, z) \\ q_1(x, x, 0) = x + (x \cdot 0) &= x \cdot 0 = d(x, 0) \end{aligned}$$

Hence, $\mathcal{V}$ is 3-permutable at 0 and distributive at 0.

In the remaining part of the paper, we introduce a new concept which is related with distributivity at 0: A variety $\mathcal{V}$ with 0 has **weakly parallel classes** if for every $\mathcal{A}, \mathcal{B} \in \mathcal{V}$ and each $\Theta \in \mathrm{Con}\ \mathcal{A} \times \mathcal{B}$ and for each $x, y \in A$, $z, v \in B$ it holds:

$$\begin{aligned} \langle [0, z], [0, v] \rangle \in \Theta &\Longrightarrow \langle [y, z], [y, v] \rangle \in \Theta \\ \langle [x, 0], [y, 0] \rangle \in \Theta &\Longrightarrow \langle [x, v], [y, v] \rangle \in \Theta\,. \end{aligned}$$

Theorem 3. *A variety $\mathcal{V}$ with 0 has weakly parallel classes if and only if there exist $(2 + k)$-ary terms $q_0, \dots, q_n$ and unary terms $s_1, \dots, s_k$ and binary terms $r_1, \dots, r_k$ $(k \geq 0)$ such that*

$$\begin{aligned} x &= q_i(0, 0, s_1(x), \dots, s_k(x)) \quad \textit{for} \quad i = 0, 1, \dots, n \\ x &= q_0(x, y, r_1(x, y), \dots, r_k(x, y)) \\ q_i(y, x, r_1(x, y), \dots, r_k(x, y)) &= q_{i+1}(x, y, r_1(x, y), \dots, r_k(x, y)) \\ &\qquad \textit{for } i = 1, \dots, n-1 \\ y &= q_n(y, x, r_1(x, y), \dots, r_k(x, y)). \end{aligned}$$

Proof. Let $\mathcal{V}$ have weakly parallel classes and let $\mathcal{A} = F_{\mathcal{V}}(x)$, $\mathcal{B} = F_{\mathcal{V}}(x,y)$, the free algebras of $\mathcal{V}$ with one or two free generators, respectively. Let $\theta = \theta([0,x],[0,y]) \in \operatorname{Con}\mathcal{A}\times\mathcal{B}$. Then $\langle [x,x],[x,y]\rangle \in \theta$, i.e. there exist elements $z_0, z_1, \dots, z_n \in \mathcal{A}\times\mathcal{B}$ and binary polynomials over $\mathcal{A}\times\mathcal{B}$ $\varphi_0,\dots,\varphi_n$ such that $z_0 = [x,x]$, $z_n = [x,y]$ and $z_i = \varphi_i([0,x],[0,y])$, $z_{i+1} = \varphi_i([0,y],[0,x])$ for $i = 0,\dots,n-1$. Hence, there exist $(2+k)$-ary terms $q_0,\dots,q_n$ and elements $e_1,\dots,$ $e_k \in \mathcal{A}\times\mathcal{B}$ such that

$$\varphi_i(a,b) = q_i(a,b,e_1,\dots,e_k)\,.$$

Since $e_i \in F_{\mathcal{V}}(x)\times F_{\mathcal{V}}(x,y)$, there are unary terms $s_1,\dots,s_k$ and binary terms $r_1,\dots,r_k$ with $e_j = [s_j(x), r_j(x,y)]$ for $j = 1,\dots,k$. Hence

$$\begin{aligned}
[x,x] &= q_0([0,x],[0,y],[s_1(x),r_1(x,y)],\dots,[s_k(x),r_k(x,y)])\\
q_i([0,y],&[0,x],[s_1(x),r_1(x,y)],\dots,[s_k(x),r_k(x,y)])\\
&= q_{i+1}([0,x],[0,y],[s_1(x),r_1(x,y)],\dots,[s_k(x),r_k(x,y)])\\
[x,y] &= q_n([0,y],[0,x],[s_1(x),r_1(x,y)],\dots,[s_k,r_k(x,y)]).
\end{aligned}$$

If we write it componentwise, we obtain the identities of Theorem 3.

Conversely, let $\mathcal{A},\mathcal{B} \in \mathcal{V}$ and $\theta \in \operatorname{Con}\mathcal{A}\times\mathcal{B}$. Suppose $x,y \in A$ and $z,v \in B$ and

$$\langle [0,z],[0,v]\rangle \in \theta\,.$$

Applying the identities of Theorem 3, we easily derive $\langle [y,z],[y,v]\rangle \in \theta$. Analogously it can be shown the second condition of the definition, i.e $\mathcal{V}$ has weakly parallel classes. ☐

Remark. It is well-known that if a variety $\mathcal{V}$ is congruence-permutable then, instead of the $n+1$ elements $z_0, z_1, \dots, z_n$ of the foregoing proof, one can take only one z_0 because $n = 0$ (the reason is that instead of the congruence Θ, only the least reflexive and compatible relation containing the pair $\langle [0,x],[0,y]\rangle$ is considered).

By a quite routine modification of the proof of Theorem 3, we obtain the proof of the following

Theorem 4. *Let $\mathcal{V}$ be a congruence-permutable variety with* 0. *Then $\mathcal{V}$ has weakly parallel classes if and only if there exist a $(1+k)$-ary term q and unary terms $s_1,\dots,s_k$ and binary terms $r_1,\dots,r_k$ such that*

$$\begin{aligned}
x &= q(0, s_1(x),\dots,s_k(x))\\
x &= q(x, r_1(x,y),\dots,r_k(x,y))\\
y &= q(y, r_1(x,y),\dots,r_k(x,y))\,.
\end{aligned}$$

Example 4. Consider a variety $\mathcal{V}$ of type $(2,2,0)$ where the nullary operation is 0 and the binary operations are denoted by $\vee, \wedge$ and $\mathcal{V}$ satisfies the identities:

$$\begin{aligned} 0 \vee x &= x \\ x \vee (x \wedge y) &= x \\ y \vee (x \wedge y) &= y\,. \end{aligned}$$

We can set $s_1(x) = x$, $r_1(x,y) = x \wedge y$, $k = 1$, $n = 0$ and $q_0(x_1, x_2, z) = x_1 \vee z$. Then

$$\begin{aligned} q_0(0,0,s_1(x)) &= 0 \vee x = x \\ q_0(x,y,r_1(x,y)) &= x \vee (x \wedge y) = x \\ q_0(y,x,r_1(x,y)) &= y \vee (x \wedge y) = y \end{aligned}$$

thus $\mathcal{V}$ has weakly parallel classes.

Remark. If some congruence condition is characterized by a Mal'cev condition then all terms of this Mal'cev condition are idempotent. It has appeared firstly in **[1]**, that there are Mal'cev conditions characterizing conditions at 0 (alias conditions on 0-classes) which contain non-idempotent terms. Other such Mal'cev conditions are presented in this paper. All terms p of these conditions satisfies only $p(0,\ldots,0) = 0$.

References

1. Chajda I., *Congruence distributivity in varieties with constants*, Archiv. Math. (Brno) **22** (1986), 121–124.

2. ______, *A localization of some congruence conditions in varieties with nullary operations*, Annales Univ. Sci. Budapest, sectio Math. **30** (1987), 17–23.

3. Chajda I. and Halaš R., *Congruence modularity at* 0, Discussione Math., Algebra and Stoch. Math. **17** (1997), 57–65.

4. Czédli G., *Notes on congruence implications*, Archiv. Math. (Brno) **27** (1991), 149–153.

5. Duda J., *Arithmeticity at* 0, Czech. Math. J. **37** (1987), 197–206.

6. Lipparini P., *n-Permutable varieties satisfy non trivial congruence identities*, Algebra Universalis **33** (1995), 159–168.

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