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REGULARISING NATURAL DUALITIES

B. A. DAVEY and B. J. KNOX

Abstract. Given an algebra $\mathbf{M}$ we may adjoin an isolated zero to form an algebra $\mathbf{M}_\infty$ satisfying all identities $u \approx v$ true in $\mathbf{M}$ for which u and v contain the same variables. Drawing on the structure theory of Płonka sums, we show that if $\mathbf{M}$ is a finite, dualisable algebra which is strongly irregular, then $\mathbf{M}_\infty$ is also dualisable. Turning the construction of $\mathbf{M}_\infty$ upside-down for the two-element left-zero band, we exhibit a duality for quasi-regular left-normal bands.

1. Introduction

At the present time it is not clear how common algebraic constructions interact with the theory of natural dualities. For example, it is not known in general whether a finite product of dualisable algebras is dualisable. Even the familiar act of passing to a subalgebra may lead to complications – recently, non-dualisable algebras that may be embedded into dualisable algebras have been discovered (see Clark, Davey and Pitkethly [**3**]). In this paper we consider the general algebraic analogue of adding a zero to a semigroup and investigate when this construction preserves dualisability.

We aim to take a finite, dualisable algebra $\mathbf{M}$ and obtain a natural duality for the quasi-variety generated by the algebra $\mathbf{M}_\infty$, formed by adding a zero to $\mathbf{M}$. This has been achieved by Gierz and Romanowska [**7**] in the case that $\mathbf{M}$ is the two-element distributive lattice, thus giving an explicit natural duality for the variety of distributive bisemilattices. Romanowska and Smith, in [**13**] and [**14**], give a more conceptual treatment of the general case and show that a full (not necessarily natural) duality for a strongly irregular variety $\boldsymbol{\mathcal{V}}$ lifts to a full duality for its regularisation. If $\boldsymbol{\mathcal{V}} = \mathbb{ISP}\mathbf{M}$ and its full duality can be realised using a schizophrenic object M (for example, when the duality is natural), then the induced full duality for $\mathbb{ISP}\mathbf{M}_\infty$ can also be realised using a schizophrenic object M_∞. While the algebraic personality of M_∞ is exactly $\mathbf{M}_\infty$, it is not clear how the

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structured topological personality of M_∞ may be distilled from the schizophrenic object M_∞ described by Romanowska and Smith.

In this paper, we give direct algebraic paths in the vain of the Gierz-Romanowska duality from a (not necessarily full nor strong) natural duality for $\mathbf{M}$ to one for $\mathbf{M}_\infty$. However, one result assumes almost nothing about the algebra $\mathbf{M}$ and may lead to a new binary operation in the type of the adjoined-zero algebra. This operation turns out to be superfluous exactly when $\mathbf{M}$ is strongly irregular, so in Section 3 we work under this assumption, allowing use of the beautiful structure theory of Płonka sums and regularisations of strongly irregular varieties.

Finally, we exhibit a bare hands natural duality for a semigroup obtained by "shadowing" an element of the two-element left-zero semigroup. This construction (again a type of Płonka sum) relates to quasi-regularisations of strongly irregular varieties as adjoining a zero relates to regularisations.

We firstly review the setting of Davey and Werner for producing dualities. A leisurely introduction may be found in [**5**] while [**2**] gives a detailed account.

Let $\mathbf{M}$ be a finite algebra and consider a type $G \cup H \cup R$ of **total operation symbols** G, **partial operation symbols** H, and **relation symbols** R. Let $\underset{\sim}{\mathbf{M}} = \langle M; G^M, H^M, R^M; \tau\rangle$ be a topological structure having the same underlying set as $\mathbf{M}$, where

(a) each $g \in G$ is interpreted as a homomorphism $g^M\colon \mathbf{M}^n \to \mathbf{M}$ for some $n \in \mathbb{N} \cup \{0\}$,
(b) each $h \in H$ is interpreted as a homomorphism $h^M\colon\ \mathrm{dom}(h^M) \to \mathbf{M}$ where $\mathrm{dom}(h^M)$ is a subalgebra of $\mathbf{M}^n$ for some $n \in \mathbb{N}$,
(c) each $r \in R$ is interpreted as a subalgebra r^M of $\mathbf{M}^n$ for some $n \in \mathbb{N}$,
(d) τ is the discrete topology.

Whenever (a), (b) and (c) hold, we say that $G^M \cup H^M \cup R^M$ is **algebraic over M**. Under these conditions, there is a naturally defined dual adjunction between the quasi-variety $\boldsymbol{\mathcal{A}} := \mathbb{ISP}\mathbf{M}$ and the topological quasi-variety $\boldsymbol{\mathfrak{X}} := \mathbb{IS}_c\mathbb{P}^+\underset{\sim}{\mathbf{M}}$ consisting of isomorphic copies of topologically closed substructures of non-trivial powers of $\underset{\sim}{\mathbf{M}}$. For each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$ the homset $D(\mathbf{A}) := \boldsymbol{\mathcal{A}}(\mathbf{A}, \mathbf{M})$ (that is, the set of homomorphisms $\mathbf{A} \to \mathbf{M}$) is a closed substructure of $\underset{\sim}{\mathbf{M}}^A$ and for each $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$ the homset $E(\mathbf{X}) := \boldsymbol{\mathfrak{X}}(\mathbf{X}, \underset{\sim}{\mathbf{M}})$ (that is, the set of continuous maps $X \to M$ that preserve each total operation, partial operation and relation symbol in $G \cup H \cup R$) forms a subalgebra of $\mathbf{M}^X$. It follows that the contravariant hom-functors $\boldsymbol{\mathcal{A}}(-, \mathbf{M})\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathfrak{X}}(-, \underset{\sim}{\mathbf{M}})\colon \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{S}}$, where $\boldsymbol{\mathcal{S}}$ is the category of sets, lift to contravariant functors $D\colon \boldsymbol{\mathcal{A}} \to \boldsymbol{\mathfrak{X}}$ and $E\colon \boldsymbol{\mathfrak{X}} \to \boldsymbol{\mathcal{A}}$. For each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, define the **evaluation map** $e_{\mathbf{A}}\colon \mathbf{A} \to ED(\mathbf{A})$ by

$$e_{\mathbf{A}}(a)(x) := x(a)$$

for each $a \in \mathbf{A}$ and each $x \in D(\mathbf{A})$. It may be shown that $e_{\mathbf{A}}$ is an embedding for each $\mathbf{A} \in \boldsymbol{\mathcal{A}}$, as is the similarly defined $\varepsilon_{\mathbf{X}}\colon \mathbf{X} \to DE(\mathbf{X})$ for each $\mathbf{X} \in \boldsymbol{\mathfrak{X}}$.

A simple calculation shows that $e\colon \mathrm{id}_{\mathcal{A}} \to ED$ and $\varepsilon\colon \mathrm{id}_{\mathfrak{X}} \to DE$ are natural transformations.

If, for an algebra $\mathbf{A} \in \mathcal{A}$, the embedding $e_{\mathbf{A}}$ is an isomorphism we say that $\underset{\sim}{\mathbf{M}}$ yields a **duality** on $\mathbf{A}$. In case $e_{\mathbf{A}}$ is an isomorphism for all $\mathbf{A} \in \mathcal{A}$, we say that $\underset{\sim}{\mathbf{M}}$ **dualises M** or that $G^M \cup H^M \cup R^M$ is a **dualising structure** for $\mathbf{M}$. Here, the natural dual adjunction between $\mathcal{A}$ and $\mathfrak{X}$ is actually a dual representation. If there is some choice of $G^M \cup H^M \cup R^M$ such that $\underset{\sim}{\mathbf{M}}$ dualises $\mathbf{M}$, we say that $\mathbf{M}$ is **dualisable**.

One of the aims of natural duality theory is to take the category theoretic and topological conditions for $\underset{\sim}{\mathbf{M}}$ to dualise $\mathbf{M}$ and distill them into (finitary) algebraic conditions. The following three results follow this programme and will be the foundation of the duality results in this paper. The reader is referred to [**2**] for the proofs.

Theorem 1.1 (First Duality Theorem). *$\underset{\sim}{\mathbf{M}}$ yields a duality on $\mathbf{A} \in \mathcal{A} = \mathbb{ISP}\mathbf{M}$ if and only if every morphism $\alpha\colon D(\mathbf{A}) \to \underset{\sim}{\mathbf{M}}$ extends to an A-ary term function $t\colon M^A \to M$ of $\mathbf{M}$.*

Theorem 1.2 (Duality Compactness Theorem). *If $\underset{\sim}{\mathbf{M}}$ is of finite type (that is, $G \cup H \cup R$ is finite) and yields a duality on each finite algebra $\mathbf{A} \in \mathcal{A}$, then $\underset{\sim}{\mathbf{M}}$ dualises $\mathbf{M}$.*

Theorem 1.3 (IC Duality Theorem). *Suppose $G \cup H \cup R$ is finite. Then $\underset{\sim}{\mathbf{M}}$ dualises $\mathbf{M}$ provided the following* **interpolation condition** *is satisfied:*

(IC) *For each $n \in \mathbb{N}$ and each substructure $\mathbf{X}$ of $\underset{\sim}{\mathbf{M}}^n$, every morphism $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}$ extends to an n-ary term function $t\colon M^n \to M$ of $\mathbf{M}$.*

2. Algebras with an Adjoined Isolated Zero

An algebra $\mathbf{A}_\infty$ of type F is said to have a **zero** element, $\infty \in A_\infty$, if there are no nullary operation symbols in F and for every fundamental operation symbol $f \in F$ (f n-ary) we have $f^{\mathbf{A}_\infty}(x_1, \dots, x_n) = \infty$ whenever $x_i = \infty$ for some i. If, in addition, $\mathbf{A} := \langle A_\infty \backslash \{\infty\}; F\rangle$ is a subalgebra of $\mathbf{A}_\infty$, we call ∞ an **isolated** zero.

Clearly, if there is an $f \in F$ with arity greater than 1, then an algebra of type F may have at most one zero.

For the remainder of this section we will work in a fixed type F having no nullary operation symbols. Given an algebra, we may adjoin an isolated zero via the following construction.

Definition 1. Let $\mathbf{M} = \langle M; F\rangle$ be an algebra with $\infty \notin M$. Define $\mathbf{M}_\infty$ to be the algebra with universe $M_\infty = M \dot{\cup} \{\infty\}$ and fundamental operations given by:

$$f^{\mathbf{M}_\infty}(x_1, \dots, x_n) = \begin{cases} f^{\mathbf{M}}(x_1, \dots, x_n), & \text{if } x_1, \dots, x_n \in M, \\ \infty & \text{otherwise,} \end{cases}$$

where $f \in F$ is n-ary and $x_1, \dots, x_n \in M_\infty$.

We will be considering the situation where $\mathbf{M}$ in the above definition is a finite algebra which is dualised by some structure $\underset{\sim}{\mathbf{M}} = \langle M; G^M, H^M, R^M; \tau\rangle$. We would like to dualise $\mathbf{M}_\infty$ by some simple modification of $G^M \cup H^M \cup R^M$.

A result may be obtained using Theorem 3.1 of [**6**] (see also Theorem 7.7.2 in [**2**]) in the case where $\mathbf{M}$ has a one-element subalgebra $\{a\}$ and a zero b with $a \neq b$. We may embed $\mathbf{M}_\infty$ into $\mathbf{M}^2$ via the map sending each $x \in M$ to (x, a) and ∞ to (b, b), hence $\mathbf{M}_\infty \in \mathbb{ISP}\mathbf{M}$. Also, the map $M_\infty \to M$ sending ∞ to b and fixing M is a retraction.

Proposition 2.1. *Suppose that*

$$\underset{\sim}{\mathbf{M}} = \langle M; G^M, H^M, R^M; \tau\rangle$$

dualises $\mathbf{M}$ *where* $\mathbf{M}$ *has a one-element subalgebra* $\{a\}$ *and a zero* b *with* $a \neq b$. *Then*

$$\underset{\sim}{\mathbf{M}}_\infty = \langle M_\infty; \mathrm{End}(\mathbf{M}_\infty), G^M \cup H^M, R^M; \tau\rangle$$

dualises $\mathbf{M}_\infty$.

Returning to the general case, assuming nothing about the algebra $\mathbf{M}$ will lead us to a stronger assumption on $\underset{\sim}{\mathbf{M}}$, namely that it satisfies (IC). As a further detour, we will in this section be producing a duality for the algebra

$$\mathbf{M}^*_\infty := \langle M_\infty; F^{\mathbf{M}_\infty} \cup \{*\}\rangle,$$

where the binary operation $*\colon {M_\infty}^2 \to M_\infty$ given by

$$x * y := \begin{cases} x & \text{if } x, y \in M, \\ \infty & \text{otherwise,} \end{cases}$$

has been added to $\mathbf{M}_\infty$ as a new fundamental operation. The following Lemma indicates when $\mathbf{M}^*_\infty$ is term equivalent to $\mathbf{M}_\infty$.

Lemma 2.2. *The binary operation* $*$ *is a term function of* $\mathbf{M}_\infty$ *if and only if* $\mathbf{M}$ *has a* **left-zero term**, *that is, a binary term* t *involving* v_1 *and* v_2 *such that* $\mathbf{M}$ *satisfies the identity*

$$t(v_1, v_2) \approx v_1.$$

Proof. Suppose $*$ is a term function of $\mathbf{M}_\infty$, that is, $* = t^{\mathbf{M}_\infty}$ for some binary term t. To see that t involves the variable v_1, let $(x, y), (z, y) \in {M_\infty}^2$ with $x, y \in M$ and $z = \infty$. Then

$$t^{\mathbf{M}_\infty}(x, y) = x * y = x \neq \infty = z * y = t^{\mathbf{M}_\infty}(z, y).$$

Similarly, we see that t involves v_2 by letting $(x,y),(x,z) \in {M_\infty}^2$ with $x,y \in M$ and $z = \infty$ and observing

$$t^{\mathbf{M}_\infty}(x,y) = x * y = x \neq \infty = x * z = t^{\mathbf{M}_\infty}(x,z).$$

Now, for all $x,y \in M$,

$$t^{\mathbf{M}}(x,y) = t^{\mathbf{M}_\infty}{\restriction}_{M^2}(x,y) = x * y = x,$$

as $\mathbf{M}$ is a subalgebra of $\mathbf{M}_\infty$, showing that $\mathbf{M}$ satisfies the identity $t(v_1,v_2) \approx v_1$.

Conversely, suppose that $\mathbf{M}$ has a left-zero term t. We observe that t must contain a fundamental operation symbol of arity at least two, hence by an easy induction on the complexity of t,

$$t^{\mathbf{M}_\infty}(x,y) = \begin{cases} t^{\mathbf{M}}(x,y) = x & \text{if } x,y \in M, \\ \infty & \text{otherwise,} \end{cases}$$

for all $x,y \in M_\infty$, therefore $* = t^{\mathbf{M}_\infty}$. □

Note that, regardless of whether or not $*$ is artificially introduced, it is immediately seen to be a homomorphism ${\mathbf{M}_\infty}^2 \to \mathbf{M}_\infty$. Also, for all $x,y,z \in M_\infty$ we have

$$\begin{aligned} x * x &= x \\ (x * y) * z &= x * (y * z) \\ x * y * z &= x * z * y. \end{aligned}$$

That is, $\langle M_\infty; * \rangle$ is a **left normal** idempotent semigroup. From the above identities we may obtain the **entropic** law

$$x * y * w * z = x * w * y * z,$$

so it follows that $*$ is a homomorphism $(\mathbf{M}^*_\infty)^2 \to \mathbf{M}^*_\infty$.

Definition 2. Given a set $G^M \cup H^M \cup R^M$ of operations, partial operations and relations algebraic over $\mathbf{M}$, there is a natural way to lift this structure to $\mathbf{M}_\infty$ via a construction similar to that used in Definition 1.

For each n-ary $g \in G$, with $n > 1$, define $g^{M_\infty} : (\mathbf{M}_\infty)^n \to \mathbf{M}_\infty$ by:

$$g^{M_\infty}(x_1,\dots,x_n) = \begin{cases} g^M(x_1,\dots,x_n), & \text{if } x_1,\dots,x_n \in M, \\ \infty & \text{otherwise,} \end{cases}$$

for all $x_1,\dots,x_n \in M_\infty$. If $g \in G$ is nullary, then we define $g^{M_\infty} := g^M$.

For each n-ary $h \in H$, let $\operatorname{dom}(h^{M_\infty}) = \operatorname{dom}(h^M)\dot\cup\{\underline{\infty}\}$ where $\underline{\infty}$ is the constant n-tuple $(\infty, \dots, \infty)$ and define $h^{M_\infty}\colon \operatorname{dom}(h^{M_\infty}) \to \mathbf{M}_\infty$ by:

$$h^{M_\infty}(x_1, \dots, x_n) = \begin{cases} h^M(x_1, \dots, x_n), & \text{if } x_1, \dots, x_n \in \operatorname{dom}(h^M), \\ \infty & \text{if } (x_1, \dots, x_n) = \underline{\infty}, \end{cases}$$

for all $x_1, \dots, x_n \in \operatorname{dom}(h^{M_\infty})$.

Finally, for each n-ary relation symbol $r \in R$, we let

$$r^{M_\infty} = r^M \dot\cup \{\underline{\infty}\}.$$

If $G^M \cup H^M \cup R^M$ is algebraic over $\mathbf{M}$, it follows that the resulting (partial) operations and relations $G^{M_\infty} \cup H^{M_\infty} \cup R^{M_\infty}$ arising from Definition 2.3 are algebraic over $\mathbf{M}_\infty$. Further, since our constructions are compatible with $*$, we have $G^{M_\infty} \cup H^{M_\infty} \cup R^{M_\infty}$ algebraic over the extension $\mathbf{M}^*_\infty$.

Theorem 2.3. *Let* $\mathbf{M}$ *be a finite algebra and suppose* $G^M \cup H^M \cup R^M$ *satisfies* (IC)*. Then*

$$\underset{\sim}{\mathbf{M}}_\infty := \langle M_\infty; G^{M_\infty} \cup \{*\} \cup \{\infty\}, H^{M_\infty}, R^{M_\infty} \cup \{M\}; \tau\rangle$$

satisfies (IC) *with respect to* $\mathbf{M}^*_\infty$*, hence if* $G^M \cup H^M \cup R^M$ *is finite,* $\underset{\sim}{\mathbf{M}}_\infty$ *dualises* $\mathbf{M}^*_\infty$.

Proof. Let $n \in \mathbb{N}$ and let $\mathbf{X}$ be a (closed) substructure of $(\underset{\sim}{\mathbf{M}}_\infty)^n$ and let $\lambda\colon \mathbf{X} \to \underset{\sim}{\mathbf{M}}_\infty$ be a morphism. Our goal is to extend λ to a term function $(\mathbf{M}^*_\infty)^n \to \mathbf{M}^*_\infty$.

In the first case, if $\lambda(x) = \infty$ for all $x \in X$, we must have $X \cap M^n = \varnothing$ since λ preserves the unary relation M. That is, for every $x \in X$, there is an $i \in \{1, \dots, n\}$ such that $x_i = \infty$. It is then easy to see that for all $x \in X$, $x_1 * \cdots * x_n = \infty = \lambda(x)$, showing the n-ary term function $(v_1 * \cdots * v_n)^{\mathbf{M}^*_\infty}$ extends λ.

If λ is not the constant map onto $\{\infty\}$, the set

$$N := \{x \in X \mid \lambda(x) \neq \infty\}$$

is non-empty. We define

$$x^N := x^1 * \cdots * x^l$$

where $x^1, \dots, x^l$ is some fixed sequence of the elements of N. Since λ preserves $*$, we must have $x^N \in N$, for otherwise

$$\lambda(x^N) = \infty \Rightarrow \lambda(x^1) * \cdots * \lambda(x^l) = \infty \Rightarrow \lambda(x^j) = \infty \text{ for some } x^j \in N,$$

a contradiction. Also, note that $x \in N$ if and only if $x^N * x = x^N$.

Let

$$Z := \{i \in \{1,\dots,n\} \mid x_i \neq \infty \text{ for all } x \in N\}.$$

We claim that $Z \neq \varnothing$. Indeed, $x^N \neq \underline{\infty}$, that is $x^N_i \neq \infty$ for some index i. Supposing that there exists an $x \in N$ with $x_i = \infty$ gives $x^N_i = \infty$, a contradiction. Therefore $i \in Z$. In fact,

$$Z = \{i \in \{1,\dots,n\} \mid x^N_i \neq \infty\}.$$

We have

$$N = \{x \in X \mid x_i \neq \infty \text{ for all } i \in Z\}$$

since $x \in N \iff x^N * x = x^N \iff x_i \neq \infty$ for all $i \in Z$.

Comparing our two descriptions of N, we conclude that for all $x \in X$, one has $\lambda(x) = \infty$ if and only if there exists $i \in Z$ such that $x_i = \infty$. This indicates that an n-ary term function, if it is to extend λ, must involve all of the variables with indices in Z.

Let $\pi_Z \colon {M_\infty}^n \to {M_\infty}^Z$ be restriction to Z. It is clear that $\pi_Z(N) \subseteq M^Z$. We will now argue that $\pi_Z(N)$ is a substructure of $\underset{\sim}{\mathbf{M}}^Z$.

Let $g \in G$ be m-ary and let $y^1,\dots,y^m \in \pi_Z(N)$. To show $g^{M^Z}(y^1,\dots,y^m)$ is in $\pi_Z(N)$, let $x^1,\dots,x^m \in N$ be such that $\pi_Z(x^1) = y^1,\dots,\pi_Z(x^m) = y^m$. Then $g^X(x^1,\dots,x^m) \in X$ and for all $i \in Z$, we have $g^X(x^1,\dots,x^m)_i \in M$. Hence, by our previous claim, $\lambda(g^X(x^1,\dots,x^m)) \neq \infty$, that is $g^X(x^1,\dots,x^m)$ is in N. Since the x^j agree with the y^j on Z, we have

$$g^{M^Z}(y^1,\dots,y^m) = \pi_Z(g^X(x^1,\dots,x^m)) \in \pi_Z(N).$$

Let $h \in H$ be an m-ary partial operation symbol and let $y^1,\dots,y^m \in \pi_Z(N)$ with $y^1,\dots,y^m \in \operatorname{dom}(h^{M^Z})$. That is, $y^1_i,\dots y^m_i \in \operatorname{dom}(h^M)$ for each $i \in Z$. Let $x^1,\dots,x^m \in N$ be such that $\pi_Z(x^1) = y^1,\dots,\pi_Z(x^m) = y^m$. We then have $x^1 * x^N,\dots,x^m * x^N \in X$ and again $\pi_Z(x^1 * x^N) = y^1,\dots,\pi_Z(x^m * x^N) = y^m$. Checking that $x^1 * x^N,\dots,x^m * x^N \in \operatorname{dom}(h^X)$, as before we conclude

$$h^{M^Z}(y^1,\dots,y^m) = \pi_Z(h^X(x^1 * x^N,\dots,x^m * x^N)) \in \pi_Z(N),$$

completing the argument.

Now, suppose $x, y \in N$ agree on Z, that is $\pi_Z(x) = \pi_Z(y)$. Since $x * x^N = y * x^N$ and λ preserves $*$, we have

$$\lambda(x) = \lambda(x) * \lambda(x^N) = \lambda(x * x^N) = \lambda(y * x^N) = \lambda(y) * \lambda(x^N) = \lambda(y).$$

This shows that the map $\lambda' \colon \pi_Z(N) \to M$ given for all $z \in \pi_Z(N)$ by

$$\lambda'(z) = \lambda(x), \text{ where } x \text{ is any element of } N \text{ such that } \pi_Z(x) = z,$$

is well defined. In fact, one may show that λ' is a morphism from $\pi_Z(N)$ to $\underset{\sim}{\mathbf{M}}$ by using an argument similar to the one above and the fact that λ preserves the (partial) operations and relations in $G^{M_\infty} \cup H^{M_\infty} \cup R^{M_\infty}$.

We now have a substructure $\pi_Z(N)$ of $\underset{\sim}{\mathbf{M}}^Z$ and a morphism $\lambda' \colon \pi_Z(N) \to \underset{\sim}{\mathbf{M}}$, so, by invoking (IC) for M, our induced λ' extends to a Z-ary term function $t^{\mathbf{M}} \colon \mathbf{M}^Z \to \mathbf{M}$. To complete the proof, it remains to show that the n-ary term function

$$s^{\mathbf{M}^*_\infty} = (t * v_{i_1} * \cdots * v_{i_k})^{\mathbf{M}^*_\infty},$$

where $Z = \{i_1, \ldots, i_k\}$, extends λ.

We have already shown that for all $x \in X$, one has $\lambda(x) = \infty$ if and only if $x_i = \infty$ for some $i \in Z$. It is easy to see that for all $x \in {M_\infty}^n$,

$$s^{\mathbf{M}^*_\infty}(x) = \infty \iff x_i = \infty \text{ for some } i \in Z,$$

since s involves all variables with indices in Z. Hence

$$\lambda(x) = \infty \iff s^{\mathbf{M}^*_\infty}(x) = \infty \text{ for all } x \in X.$$

Finally, let $x \in N$. Since $x_i \in M$ for all $i \in Z$, we have

$$\begin{aligned}\lambda(x) &= \lambda'(\pi_Z(x)) = t^{\mathbf{M}}(\pi_Z(x)) = t^{\mathbf{M}}(x) = t^{\mathbf{M}^*_\infty}(x)\\ &= t^{\mathbf{M}^*_\infty}(x) * x_{i_1} * \cdots * x_{i_k} = (t * v_{i_1} * \cdots * v_{i_k})^{\mathbf{M}^*_\infty}(x) = s^{\mathbf{M}^*_\infty}(x),\end{aligned}$$

as required. □

The following proposition shows the necessity of the relation M in the structure

$$\underset{\sim}{\mathbf{M}}_\infty := \langle M_\infty; G^{M_\infty} \cup \{*\} \cup \{\infty\}, H^{M_\infty}, R^{M_\infty} \cup \{M\}, \tau\rangle,$$

provided no nullaries appear in G^M. We will later provide examples where, in the presence of nullaries, the unary relation M may be avoided.

Proposition 2.4. *Suppose G^M contains no nullaries. Then*

$$G^{M_\infty} \cup \{*\} \cup \{\infty\} \cup H^{M_\infty} \cup R^{M_\infty}$$

does not yield a duality on the algebra $\mathbf{M}^* \in \boldsymbol{\mathcal{A}} = \mathbb{ISP}\mathbf{M}^*_\infty$.

Proof. We will show that the constant map $\widehat{\infty} \colon D(\mathbf{M}^*) \to \{\infty\}$, although in $ED(\mathbf{M}^*)$, cannot be an evaluation at any $a \in M$.

Let $x \in D(\mathbf{M}^*) = \boldsymbol{\mathcal{A}}(\mathbf{M}^*, \mathbf{M}^*_\infty)$ and note that for all $a, b \in M$, we have $x(a) = x(a * b) = x(a) * x(b)$. This shows that if $x(b) = \infty$ for some $b \in M$, then $x(a) = \infty$ for all $a \in M$, hence $D(\mathbf{M}^*)$ consists precisely of the endomorphisms of $\mathbf{M}^*$ together with the constant map $\mathbf{M}^* \to \{\infty\}$.

Clearly, $\widehat{\infty} \in ED(\mathbf{M}^*)$, but any endomorphism of $\mathbf{M}^*$ evaluated at any $a \in M$ will differ from ∞. $\square$

According to Lemma 2.2, we may replace $\mathbf{M}^*_\infty$ with $\mathbf{M}_\infty$ in Theorem 2.3, provided $\mathbf{M}$ has a left-zero term. As an example of when this is not the case, let $\mathbf{2}$ be the two-element meet-semilattice $\langle\{0,1\};\cdot\rangle$. Then the structure $\underset{\sim}{\mathbf{2}} = \langle\{0,1\};\cdot,0,1;\tau\rangle$ dualises $\mathbf{2}$ and indeed satisfies (IC) (see [**2**]). Observe that $\mathbf{2}$ cannot have a left-zero term, since the identity $u \approx v$ is satisfied in a semilattice if and only if the terms u and v contain precisely the same variables. Hence, when applying Theorem 2.3, we obtain a duality for the algebra $\mathbf{2}^*_\infty = \langle\{0,1,\infty\};\cdot,*\rangle$. Since $*$ is a homomorphism $(\mathbf{2}^*_\infty)^2 \to \mathbf{2}^*_\infty$, we have the medial law

$$(x \cdot y) * (w \cdot z) \approx (x * w) \cdot (y * z)$$

true in $\mathbf{2}^*_\infty$. Using this together with the idempotence of the operations $\cdot$ and $*$, we obtain the distributive laws

$$\begin{aligned} x * (y \cdot z) &\approx (x * y) \cdot (x * z) \\ (x \cdot y) * z &\approx (x \cdot z) * (y \cdot z), \end{aligned}$$

hence $\mathbf{2}^*_\infty$ is a semiring. Such semirings were considered in [**16**] and [**15**]

Note that there is a duality for the three-element chain $\mathbf{2}_\infty$ arising from Proposition 2.1 given by the structure $\underset{\sim}{\mathbf{2}}_\infty = \langle\{0,1,\infty\};\mathrm{End}\,(\mathbf{2}_\infty)\cup\{\cdot\};\tau\rangle$ (see also [**6**]).

3. Płonka Sums and Regularisations

In this section, we consider a fixed **plural type** F. That is, F has an operation symbol of arity at least 2 and no nullaries.

An algebra $\mathbf{M}$ of type F is called **regular** if it satisfies only regular identities (that is, identities in which the same variables appear on both sides), and **irregular** otherwise. For a variety $\boldsymbol{\mathcal{V}}$ of F-algebras, the **regularisation** $\overline{\boldsymbol{\mathcal{V}}}$ of $\boldsymbol{\mathcal{V}}$ is defined to be the variety of algebras satisfying all regular identities that are satisfied in $\boldsymbol{\mathcal{V}}$. An alternative description of $\overline{\boldsymbol{\mathcal{V}}}$ may be given as follows.

Given a semilattice $\mathbf{S} = \langle S;\cdot\rangle$, define the F-algebra $\mathbf{S}_F := \langle S; F^{\mathbf{S}_F}\rangle$ having fundamental operations given by $f^{\mathbf{S}_F}(x_1,\dots,x_n) = x_1\cdots x_n$ for all $x_1,\dots,x_n \in S$ and each n-ary $f \in F$. Denote by $\boldsymbol{\mathcal{SL}}_F$ the class consisting of the $\mathbf{S}_F$ where $\mathbf{S}$ ranges over all semilattices. Evidently, $\boldsymbol{\mathcal{SL}}_F$ is precisely the variety of F-algebras satisfying **all** regular identities of type F. Indeed, as before, any $\mathbf{S}_F \in \boldsymbol{\mathcal{SL}}_F$ satisfies an identity $u \approx v$ of type F if and only if the term functions $u^{\mathbf{S}_F}$ and $v^{\mathbf{S}_F}$ are the same product of variables in $\mathbf{S}$, that is, if and only if $u \approx v$ is a regular identity. We now have

$$\overline{\boldsymbol{\mathcal{V}}} = \mathbb{HSP}(\boldsymbol{\mathcal{V}} \cup \boldsymbol{\mathcal{SL}}_F).$$

Also, note that $\mathcal{SL}_F$ is term equivalent to the variety of semilattices since in any $\mathbf{S}_F$ we may recover the original $\cdot$ operation in $\mathbf{S}$ by: $x \cdot y = f^{\mathbf{S}_F}(x, y, \ldots, y)$ where $f \in F$ is any operation symbol of arity greater than 1. Accordingly, we call $\mathcal{SL}_F$ the variety of **semilattices of type** F.

Recall that a semilattice $\mathbf{S} = \langle S; \cdot \rangle$ may be regarded as a (small) category $(\mathbf{S})$ with object set S and homsets given by

$$(\mathbf{S})(s,t) = \begin{cases} \{s \to t\} & \text{if } s \leq t, \\ \varnothing & \text{otherwise,} \end{cases}$$

for all $s, t \in S$ (where $s \leq t \iff s \cdot t = s$). Note that $(\mathbf{S})$ has products of any two of its objects, the product of s and t in $(\mathbf{S})$ being $s \cdot t$.

Definition 3. Let $\mathbf{S} = \langle S; \cdot \rangle$ be a semilattice and let $\mathcal{V}$ be a variety of algebras of type F. Let Q be a contravariant functor from $(\mathbf{S})$ to $\mathcal{V}$. We write $\mathbf{A}_s := Q(s)$ for each $s \in S$, and $\varphi_{t,s}\colon = Q(s \to t)$ for each $s \leq t$ in S, the **fibre map** from $\mathbf{A}_t$ to $\mathbf{A}_s$. Then the **Płonka sum of** Q is the F-algebra $\mathbf{A}$ with universe $A = \dot{\bigcup}\{A_s \mid s \in S\}$ and fundamental operations given, for each n-ary $f \in F$, by

$$f^{\mathbf{A}}(x_1, \ldots, x_n) = f^{\mathbf{A}_s}(\varphi_{s_1,s}(x_1), \ldots, \varphi_{s_n,s}(x_n))$$

for all $x_1, \ldots, x_n \in A$, where $x_i \in A_{s_i}$ and $s = s_1 \cdots s_n$. We will sometimes say $\mathbf{A}$ is the Płonka sum of the system of **fibres** $(\mathbf{A}_s \mid s \in S)$ with fibre maps $(\varphi_{t,s} \mid s \leq t)$ over the **semilattice replica S**.

For example, the algebra $\mathbf{A}_\infty$ in Definition 2.1 is the Płonka sum of the functor Q from the category corresponding to the two-element meet-semilattice $\mathbf{2} := \langle \{0, 1\}; \cdot \rangle$, where $Q(1) = \mathbf{A}$ and $Q(0)$ is the trivial F-algebra $\langle \{\infty\}; F \rangle$. The only non-identity fibre map $Q(0 \to 1)\colon \mathbf{A} \to \{\infty\}$ is constant. Note that $\mathbf{1}_\infty$, where $\mathbf{1}$ is the trivial F-algebra $\langle \{1\}; F \rangle$, is isomorphic to $\mathbf{2}_F$.

An algebra $\mathbf{M}$ is called **strongly irregular** if it has a left-zero term $*$ (cf. Lemma 2.2). For example, any algebra with an underlying lattice structure is strongly irregular (we may take the term $x * y$ to be $x \vee (x \wedge y)$). Similarly, an algebra with an underlying group structure is strongly irregular (we may take the term $x * y$ to be $xy^{-1}y$). More generally, any non-trivial algebra in a congruence-modular variety is strongly irregular since amongst the terms of such an algebra is a 4-ary Day term d (which is not a projection) satisfying $d(x, y, y, x) \approx x$, and we may take $x * y$ to be $d(x, y, y, x)$.

A variety $\mathcal{V}$ of algebras will be called **strongly irregular** if there is a binary term $*$ which is a left-zero term on every algebra in $\mathcal{V}$. It turns out that a strongly irregular variety $\mathcal{V}$ has a basis for its identities consisting of some regular identities together with the single identity $x * y \approx x$ (see [**17**] or [**11**]). The regularisation of $\mathcal{V}$ then has a very concrete description:

Theorem 3.1. ([**9**], [**10**], [**11**], [**17**], [**12**]) *Let $\mathcal{V}$ be a strongly irregular variety of F-algebras defined by a set Σ of regular identities and a single identity of the form $x * y \approx x$. Then the following classes coincide:*

(1) *The regularisation $\overline{\mathcal{V}}$ of $\mathcal{V}$;*
(2) *The class of Płonka sums of algebras in $\mathcal{V}$;*
(3) *The variety of F-algebras defined by the identities Σ and the following identities (for each n-ary $f \in F$):*

$$
\begin{gathered}
x * x \approx x, \\
(x * y) * z \approx x * (y * z), \\
x * y * z \approx x * z * y, \\
f(x_1, \dots, x_n) * y \approx f(x_1 * y, \dots, x_n * y), \\
y * f(x_1, \dots, x_n) \approx y * x_1 * \dots * x_n.
\end{gathered}
$$

Observe that the first three identities of (3) above tell us that for an algebra $\mathbf{A}$ in $\overline{\mathcal{V}}$, the term reduct $\langle A; * \rangle$ is an idempotent left normal semigroup.

In the presence of strong irregularity, there is also a characterisation of the subdirectly irreducibles in $\overline{\mathcal{V}}$:

Theorem 3.2. ([**8**]) *Let $\mathcal{V}$ be a strongly irregular variety. The subdirectly irreducible members of $\overline{\mathcal{V}}$ are the algebras $\mathbf{A}$ and $\mathbf{A}_\infty$, where $\mathbf{A}$ ranges over all subdirectly irreducible members of $\mathcal{V}$, and the algebra $\mathbf{1}_\infty$, where $\mathbf{1}$ is a trivial algebra in $\mathcal{V}$.*

From the point of view of natural duality theory, this gives a corollary indicating that a 2-sorted duality may be necessary if we are seeking to lift a given natural duality for a strongly irregular variety $\mathcal{V}$ to a duality covering the whole regularisation $\overline{\mathcal{V}}$. We need not have $\overline{\mathcal{V}} = \mathbb{ISP}\mathbf{M}_\infty$ even when $\mathcal{V} = \mathbf{M}$ for some $\mathbf{M}$. A detailed treatment of multi-sorted dualities may be found in Chapter 7 of [**2**].

Corollary 3.3. *Let $\mathcal{V}$ be a strongly irregular variety with $\mathcal{V} = \mathbb{ISP}\mathbf{M}$ for some algebra $\mathbf{M}$. Then $\overline{\mathcal{V}} = \mathbb{ISP}(\mathbf{M}_\infty, \mathbf{1}_\infty)$, and further, $\overline{\mathcal{V}} = \mathbb{ISP}\mathbf{M}_\infty$ if and only if $\mathbf{M}$ has a one element subalgebra.*

Let $\mathbf{A} \in \overline{\mathcal{V}}$ where $\mathcal{V}$ is strongly irregular and satisfies the identity $x * y \approx x$. Suppose (according to Theorem 3.1) that $\mathbf{A}$ is a Płonka sum of fibres $(\mathbf{A}_s \mid s \in S)$ over its semilattice replica $\mathbf{S}$ with fibre maps $(\varphi_{s,t} \mid s \geq t)$. We define the **canonical homomorphism** $\mu_\mathbf{A} \colon \mathbf{A} \to \mathbf{S}_F$ by $\mu_\mathbf{A}(A_s) = s$. The kernel Φ of $\mu_\mathbf{A}$ (whose congruence classes are exactly the fibres of $\mathbf{A}$) may in fact be recovered via

$$(x, y) \in \Phi \iff x * y = x \text{ and } y * x = y.$$

Note that the quotient algebra $\mathbf{A}/\Phi$ is isomorphic to $\mathbf{S}_F$. For convenience, we will often identify $\mu_{\mathbf{A}}$ with the semigroup homomorphism from $\langle A;*\rangle$ to $\mathbf{S} = \langle S;\cdot\rangle$ having the same definition.

Observe that for $s \geq t$ in S, for any $a \in A_s$ and $b \in A_t$ we have $a *^{\mathbf{A}} b = \varphi_{s,t}(a) *^{\mathbf{A}_t} \varphi_{t,t}(b) = \varphi_{s,t}(a) *^{\mathbf{A}_t} b = \varphi_{s,t}(a)$. Hence we may recover the fibre map $\varphi_{s,t}$ by

$$\varphi_{s,t}(a) = a * b \text{ where } b \in A_t \text{ is arbitrary}$$

for all $a \in A_s$.

From the operation $*$, we may define a partial order on A by setting

$$x \leq_* y :\iff x = y * x.$$

Under this ordering, each fibre of $\mathbf{A}$ is an anti-chain. Further, for all $x, y, w, z \in A$, if $x \leq_* y$ then $x * w \leq_* y * w$ and $z * x \leq_* z * y$, that is, $\langle A;*\rangle$ forms a partially ordered semigroup under $\leq_*$. Since $\mu_{\mathbf{A}} : \langle A;*\rangle \to \mathbf{S}$ is a homomorphism, it follows immediately that $\mu_{\mathbf{A}}$ is order preserving, that is, $x \leq_* y$ in $\mathbf{A}$ implies $\mu_{\mathbf{A}}(x) \leq \mu_{\mathbf{A}}(y)$ in $\mathbf{S}$.

Suppose $\mathbf{A}, \mathbf{B} \in \overline{\boldsymbol{\mathcal{V}}}$ with semilattice replicas $\mathbf{S}$ and $\mathbf{T}$ respectively and let $u\colon A \to B$ be a map. We may attempt to "define" $\Gamma_u\colon S \to T$, the **replica map of** u, by

$$\Gamma_u(s) = \mu_{\mathbf{B}}(u(x)) \text{ where } x \in \mu_{\mathbf{A}}^{-1}(s)$$

for all $s \in S$, although, as it stands, Γ_u need not be well defined.

The following lemma characterises homomorphisms in $\overline{\boldsymbol{\mathcal{V}}}$. They are, in a sense, "Płonka sums" of $\boldsymbol{\mathcal{V}}$-homomorphisms over a replica map.

Lemma 3.4. *Let $\mathbf{A}, \mathbf{B} \in \overline{\boldsymbol{\mathcal{V}}}$ (where $\boldsymbol{\mathcal{V}}$ is strongly irregular and satisfies $x * y \approx x$). Suppose $\mathbf{A}$ has semilattice replica $\mathbf{S}$, fibres $(\mathbf{A}_s \mid s \in S)$ and fibre maps $(\varphi_{s,t} \mid s \geq t)$. Suppose $\mathbf{B}$ has semilattice replica $\mathbf{T}$, fibres $(\mathbf{B}_v \mid v \in T)$ and fibre maps $(\phi_{v,w} \mid v \geq w)$.*

Let $u\colon A \to B$ be a map such that for each $s \in S$, the restriction

$$u{\restriction}_{A_s}\colon \mathbf{A}_s \to \mathbf{B}$$

is a homomorphism. Then the following are equivalent:

(1) *u is a homomorphism $\mathbf{A} \to \mathbf{B}$;*
(2) *The replica map Γ of u is a well-defined semilattice homomorphism $\mathbf{S} \to \mathbf{T}$ and u preserves the order $\leq_*$;*
(3) *u preserves $*$;*
(4) *Γ is a well-defined semilattice homomorphism $\mathbf{S} \to \mathbf{T}$ and for each $s \geq t$ in $\mathbf{S}$, we have*

$$u(\varphi_{s,t}(a)) = \phi_{\Gamma(s),\Gamma(t)}(u(a))$$

for all $a \in A_s$.

Proof. (1) $\implies$ (2): This is a routine check after noting that a homomorphism must preserve the term function $*$. (Hence we also have (3) $\implies$ (2)).

(2) $\iff$ (3): Let $u\colon A \to B$ be an order-preserving map such that the replica map Γ is a well-defined semilattice homomorphism. We obtain

$$\Gamma(\mu_{\mathbf{A}}(x)) = \mu_{\mathbf{B}}(u(x))$$

for all $x \in A$, since certainly $x \in \mu_{\mathbf{A}}^{-1}(\mu_{\mathbf{A}}(x))$. Let $x, y \in A$. Using the above together with the fact that $\mu_{\mathbf{A}}$ and $\mu_{\mathbf{B}}$ are semigroup homomorphisms, we have

$$\mu_{\mathbf{B}}(u(x * y)) = \mu_{\mathbf{B}}(u(x) * u(y)),$$

showing that $u(x * y)$ and $u(x) * u(y)$ are in the same fibre of $\mathbf{B}$.

Note that $x * y \leq_* x$ hence $u(x * y) \leq_* u(x)$ since u is order preserving. Then $u(x * y) * u(x) * u(y) \leq_* u(x) * u(y)$, from which, by definition we obtain

$$u(x * y) * [u(x) * u(y)] = [u(x) * u(y)] * u(x * y).$$

Hence $u(x * y) = u(x) * u(y)$, since $u(x * y)$ and $u(x) * u(y)$ lie in the same fibre, showing (3). (Note that the proof of (2) $\iff$ (3) is independent of our assumption that the restriction of u to each fibre be a homomorphism.)

(3) $\implies$ (4): Again, preservation of $*$ ensures that the replica map Γ is a well-defined semilattice homomorphism. Let $s \geq t$ in $\mathbf{S}$ and $a \in A_s$. Let $b \in A_t$ be arbitrary. Then $u(a) \in B_{\Gamma(s)}$ and $u(b) \in B_{\Gamma(t)}$ with $\Gamma(s) \geq \Gamma(t)$ in $\mathbf{T}$. Therefore

$$u(\varphi_{s,t}(a)) = u(a * b) = u(a) * u(b) = \phi_{\Gamma(s),\Gamma(t)}(u(a)).$$

(4) $\implies$ (1): We must show that for each n-ary $f \in F$ and each $a_1, \dots, a_n \in A$, we have

$$u(f^{\mathbf{A}}(a_1, \dots, a_n)) = f^{\mathbf{B}}(u(a_1), \dots, u(a_n)).$$

By using the Płonka sum description of the fundamental operations of $\mathbf{A}$ and $\mathbf{B}$ given in Definition 3.1 and the assumption that the replica map Γ is a well-defined semilattice homomorphism, this condition becomes

$$\begin{aligned} &u(f^{\mathbf{A}_s}(\varphi_{s_1,s}(a_1), \dots, \varphi_{s_n,s}(a_n)) = \\ &\qquad f^{\mathbf{B}_{\Gamma(s)}}(\phi_{\Gamma(s_1),\Gamma(s)}(u(a_1)), \dots, \phi_{\Gamma(s_n),\Gamma(s)}(u(a_n))) \end{aligned}$$

where $a_i \in A_{s_i}$ and $s = s_1 \cdots s_n$ in $\mathbf{S}$. This is easily verified since, by assumption, we have $u{\restriction}_{A_s}\colon \mathbf{A}_s \to \mathbf{B}_{\Gamma(s)}$ a homomorphism and $u(\varphi_{s_i,s}(a_i)) = \phi_{\Gamma(s_i),\Gamma(s)}(u(a_i))$ for each i. $\square$

Starting from a dualisable, strongly irregular algebra $\mathbf{M}$, we may use the general theory of Płonka sums to produce a version of Theorem 2.3 that preserves the type of $\mathbf{M}_\infty$ while weakening the assumptions on the duality for $\mathbf{M}$. Since $\mathbf{M}$ is strongly irregular, by Lemma 2.2, $\mathbf{M}_\infty$ is term equivalent to the algebra $\mathbf{M}_\infty^*$ of the previous section. Letting $\boldsymbol{\mathcal{V}} := \mathbb{HSP}\mathbf{M}$, we have $\mathbf{M}_\infty \in \overline{\boldsymbol{\mathcal{V}}}$. Algebras in $\mathbb{ISP}\mathbf{M}_\infty$ (which is then a subclass of $\overline{\boldsymbol{\mathcal{V}}}$) are Płonka sums of $\boldsymbol{\mathcal{V}}$-algebras by Theorem 3.1. The following lemma tightens this description and allows us to bypass (IC).

Lemma 3.5. *Algebras in $\mathbb{ISP}\mathbf{M}_\infty$ are Płonka sums of $\mathbb{ISP}\mathbf{M}$-algebras.*

Proof. Let $\mathbf{A} \in \mathbb{ISP}\mathbf{M}_\infty$. Then, as remarked above, $\mathbf{A}$ is a Płonka sum of $\boldsymbol{\mathcal{V}}$-algebras where $\boldsymbol{\mathcal{V}} = \mathbb{HSP}\mathbf{M}$. Let $\mathbf{A}_s$ be a non-trivial fibre of $\mathbf{A}$ and let $a \neq b$ in A_s. By the Separation Theorem, there is a homomorphism $u\colon \mathbf{A} \to \mathbf{M}_\infty$ such that $u(a) \neq u(b)$. Since u preserves $*$ (which is a term function), by Lemma 3.4 the replica map Γ_u is well defined and therefore $u(A_s)$ is a subset of a single fibre of $\mathbf{M}_\infty$. But then $u(A_s) \subseteq M$ (since the only other fibre, $\{\infty\}$, is trivial and we have $u(a) \neq u(b)$). The restriction of u to A_s is then a homomorphism $\mathbf{A}_s \to \mathbf{M}$ that separates a and b and hence (again by the Separation Theorem) $\mathbf{A}_s \in \mathbb{ISP}\mathbf{M}$. $\square$

Using Definition 2.3 we obtain a set of algebraic (partial) operations and relations $G^{M_\infty} \cup H^{M_\infty} \cup R^{M_\infty}$ on $\mathbf{M}_\infty$ from any set $G^M \cup H^M \cup R^M$ of algebraic (partial) operations and relations on $\mathbf{M}$.

Theorem 3.6. *Let $\mathbf{M}$ be a strongly irregular finite algebra dualised by the structure $\underset{\sim}{\mathrm{M}} = \langle M; G^M, H^M, R^M; \tau\rangle$ of finite type. Then*

$$\underset{\sim}{\mathrm{M}}_\infty := \langle M_\infty; G^{M_\infty} \cup \{*\} \cup \{\infty\}, H^{M_\infty}, R^M \cup \{M\}; \tau\rangle$$

dualises $\mathbf{M}_\infty$.

Proof. Let $\mathbf{A}$ be a finite algebra in $\mathbb{ISP}\mathbf{M}_\infty$ and let $\lambda\colon D(\mathbf{A}) \to \underset{\sim}{\mathrm{M}}_\infty$ be a morphism. We aim to extend λ to an A-ary term function ${\mathbf{M}_\infty}^A \to \mathbf{M}_\infty$, the result will then follow by the First Duality Theorem and the Duality Compactness Theorem.

In the first case, suppose that λ is the constant map onto $\{\infty\}$. Since λ preserves the unary relation M, we must have for each $x \in D(A)$, an $a \in A$ such that $x(a) = \infty$. Then, letting $A = \{a_1, \dots, a_m\}$, we see that the term function $(v_1 * \cdots * v_m)^{\mathbf{M}_\infty}$ extends λ since, for all $x \in D(A)$, we have $(v_1 * \cdots * v_m)^{\mathbf{M}_\infty}(x) = x(a_1) * \cdots * x(a_m) = \infty = \lambda(x)$.

Alternatively, suppose the set of homomorphisms

$$N := \{x \in D(A) \mid \lambda(x) \neq \infty\}$$

is non-empty. We enumerate $N = \{x_1, \dots, x_l\}$ and define

$$x^N := x_1 * \cdots * x_l,$$

that is, $x^N(a) = x_1(a) * \cdots * x_l(a)$ for all $a \in A$. Then $x^N \in D(A)$ and, indeed, $x^N \in N$, for otherwise

$$\lambda(x^N) = \infty \implies \lambda(x_1) * \cdots * \lambda(x_l) = \infty \iff \lambda(x_i) = \infty \text{ for some } i,$$

a contradiction.

Let
$$Z := \{a \in A \mid x(a) \neq \infty \text{ for all } x \in N\}.$$

We claim that Z is non-empty. Certainly $x^N \neq \underline{\infty}$ (for otherwise $\lambda(x^N) = \lambda(\underline{\infty}) = \infty$ since λ preserves the constant ∞), that is, $x^N(a) \neq \infty$ for some $a \in A$. Now, suppose that there exists an $x \in N$ such that $x(a) = \infty$. Then $x^N(a) = x_1(a) * \cdots * x(a) * \cdots * x_l(a) = \infty$, a contradiction. Therefore $a \in Z$ and we have in fact shown that $Z \supseteq \{a \in A \mid x^N(a) \neq \infty\}$. Conversely, if $a \in Z$ then $x^N(a) = x_1(a) * \cdots * x_l(a) = x_1(a)$ (since all of $x_1(a), \ldots, x_l(a)$ are in M), hence

$$Z = \{a \in A \mid x^N(a) \neq \infty\}.$$

Observe that $x \in N \implies x^N * x = x^N$. Conversely, let $x \in D(A)$ with $x^N * x = x^N$ and suppose that $\lambda(x) = \infty$. Then $\lambda(x^N) = \lambda(x^N * x) = \lambda(x^N) * \lambda(x) = \lambda(x^N) * \infty = \infty$, a contradiction. Therefore

$$x \in N \iff x^N * x = x^N.$$

But, $x^N * x = x^N$ if and only if, for each $a \in A$, if $x^N(a) \neq \infty$ then $x(a) \neq \infty$. That is,
$$N = \{x \in D(A) \mid x(a) \neq \infty \text{ for all } a \in Z\},$$

from which we obtain

$$\lambda(x) = \infty \iff \text{there exists } a \in Z \text{ such that } x(a) = \infty.$$

Suppose now that $\mathbf{A}$ is a Płonka sum with semilattice replica $\mathbf{S}$, fibres $(\mathbf{A}_s \mid s \in S)$ and fibre maps $(\varphi_{s,t} \mid s \geq t)$. According to Lemma 3.5, each $\mathbf{A}_s$ is in $\mathbb{ISPM}$.

Since $x^N : \mathbf{A} \to \mathbf{M}_\infty$ is a homomorphism and therefore preserves $*$, the replica map $\Gamma_{x^N} : \mathbf{S} \to \mathbf{2}$ is a well-defined semilattice homomorphism by Lemma 3.4. (Recall that $\mathbf{2}$ denotes the two element meet semilattice on $\{0, 1\}$, the semilattice replica of $\mathbf{M}_\infty$). We then have, for each $s \in S$,

$$\Gamma_{x^N}(s) = \begin{cases} 1 & \text{if } s \geq \sigma, \\ 0 & \text{if } s \ngeq \sigma, \end{cases}$$

for some $\sigma \in S$. Hence, for each $a \in A$ with $a \in A_s$ we obtain

$$\begin{cases} x^N(a) \neq \infty & \text{if } s \geq \sigma \\ x^N(a) = \infty & \text{if } s \ngeq \sigma, \end{cases}$$

showing that
$$Z = \dot{\bigcup}\{A_s \mid s \geq \sigma\}.$$

Consider the least fibre $\mathbf{A}_\sigma$ of Z and let $\pi_{A_\sigma}\colon \mathbf{M}_\infty{}^A \to \mathbf{M}_\infty{}^{A_\sigma}$ be restriction to A_σ. For each $x \in N$, the map $\pi_{A_\sigma}(x)$ is a homomorphism $\mathbf{A}_\sigma \to \mathbf{M}$, that is, $\pi_{A_\sigma}(N)$ is a subset of the first dual of $\mathbf{A}_\sigma$ with respect to $\boldsymbol{\mathcal{A}} := \mathbb{ISP}\mathbf{M}$, in symbols,

$$\pi_{A_\sigma}(N) \subseteq D_{\mathbf{M}}(\mathbf{A}_\sigma) := \boldsymbol{\mathcal{A}}(\mathbf{A}_\sigma, \mathbf{M}).$$

We claim that in fact

$$\pi_{A_\sigma}(N) = D_{\mathbf{M}}(\mathbf{A}_\sigma).$$

Let $y \in D_{\mathbf{M}}(\mathbf{A}_\sigma)$, that is, let $y\colon \mathbf{A}_\sigma \to \mathbf{M}$ be a homomorphism. We must show that y is the restriction to A_σ of some homomorphism $\mathbf{A} \to \mathbf{M}_\infty$ in N. Define $\overline{y}\colon A \to M_\infty$ by

$$\overline{y}(a) = \begin{cases} y(\varphi_{s,\sigma}(a)) & \text{if } a \in A_s \subseteq Z, \\ \infty & \text{otherwise.} \end{cases}$$

It may be verified that $\overline{y}$ satisfies condition (2) of Lemma 3.4 and is therefore a homomorphism $\mathbf{A} \to \mathbf{M}_\infty$. Noting that $\varphi_{\sigma,\sigma}$ is the identity $A_\sigma \to A_\sigma$, we have $\pi_{A_\sigma}(\overline{y}) = y$. If $a \in Z$, then a is in some fibre $\mathbf{A}_s$ of $\mathbf{A}$ with $s \geq \sigma$ and we have $\overline{y}(a) = y(\varphi_{s,\sigma}(a)) \in M$, that is, $\overline{y}(a) \neq \infty$ for all $a \in Z$. Therefore $\overline{y} \in N$, establishing the claim.

We will now show that given a homomorphism in N, its values on Z are completely determined by its values on A_σ, and this also determines its image in $\underset{\sim}{\mathbf{M}}_\infty$ under λ. Let $x, y \in N$ agree on A_σ, that is $\pi_{A_\sigma}(x) = \pi_{A_\sigma}(y)$. We claim that x and y then agree on Z. Let $a \in Z$, say $a \in A_s$ where $s \geq \sigma$. Using the Płonka sum description of the fundamental operations (and therefore term functions) of $\mathbf{A}$, we have

$$a *^{\mathbf{A}} \varphi_{s,\sigma}(a) = \varphi_{s,\sigma}(a) *^{\mathbf{A}_\sigma} \varphi_{s,\sigma}(a) = \varphi_{s,\sigma}(a).$$

Hence, by definition, $\varphi_{s,\sigma}(a) \leq_* a$ in $\mathbf{A}$. Since x is order-preserving by Lemma 3.4, we then have $x(\varphi_{s,\sigma}(a)) \leq_* x(a)$ in $\mathbf{M}_\infty$. But $x(a)$ and $x(\varphi_{s,\sigma}(a))$ lie in the same fibre of $\mathbf{M}_\infty$, namely $\mathbf{M}$, and $\leq_*$ is the antichain order on M, so we obtain $x(a) = x(\varphi_{s,\sigma}(a))$ and similarly $y(a) = y(\varphi_{s,\sigma}(a))$. Hence for all $a \in Z$ we have

$$x(a) = x(\varphi_{s,\sigma}(a)) = y(\varphi_{s,\sigma}(a)) = y(a),$$

showing that x and y agree on Z.

Since $*$ is a left-zero operation on $\mathbf{M}$ and x^N takes the value ∞ on the complement of Z in A, we also have

$$x * x^N = y * x^N.$$

Therefore

$$\lambda(x) = \lambda(x) * \lambda(x^N) = \lambda(x * x^N) = \lambda(y * x^N) = \lambda(y) * \lambda(x^N) = \lambda(y).$$

According to the above, we may unambiguously define the map $\lambda' \colon \pi_{A_\sigma}(N) \to M$ by

$$\lambda'(z) = \lambda(x) \text{ for any } x \in N \text{ such that } \pi_{A_\sigma}(x) = z,$$

for all $z \in \pi_{A_\sigma}(N) = D_{\mathbf{M}}(\mathbf{A}_\sigma)$.

We now show that λ' is a morphism $D_{\mathbf{M}}(\mathbf{A}_\sigma) \to \utilde{\mathbf{M}}$ with respect to the original $\utilde{\mathbf{M}}$ structure.

Let $r \in R$ be n-ary and let $z_1, \ldots, z_n \in D_{\mathbf{M}}(\mathbf{A}_\sigma) = \pi_{A_\sigma}(N)$ be such that $(z_1, \ldots, z_n) \in r^{D_{\mathbf{M}}(\mathbf{A}_\sigma)}$. That is, $(z_1(a), \ldots, z_n(a)) \in r^M$ for all $a \in A_\sigma$. Let $x_1, \ldots, x_n \in N$ with $\pi_{A_\sigma}(x_1) = z_1, \ldots, \pi_{A_\sigma}(x_n) = z_n$. By the above arguments we have $\pi_{A_\sigma}(x_1 * x^N) = z_1, \ldots, \pi_{A_\sigma}(x_n * x^N) = z_n$. Noting that each $x_i * x^N$ takes the value ∞ outside of Z, we have, for all $a \in A_s \subseteq A$,

$$\begin{aligned}
(x_1 * x^N(a), \ldots, x_n * x^N(a)) &= (x_1(a) * x^N(a), \ldots, x_n(a) * x^N(a)) \\
&= \begin{cases} (x_1(\varphi_{s,\sigma}(a)), \ldots, x_n(\varphi_{s,\sigma}(a))) & \text{if } a \in Z, \\ (\infty, \ldots, \infty) & \text{if } a \notin Z \end{cases} \\
&= \begin{cases} (z_1(a), \ldots, z_n(a)) & \text{if } a \in Z, \\ (\infty, \ldots, \infty) & \text{if } a \notin Z. \end{cases}
\end{aligned}$$

Then, using the definition of r^{M_∞}, we have $(x_1 * x^N, \ldots, x_n * x^N) \in r^{D(\mathbf{A})}$, and since λ preserves r^{M_∞}, we have $(\lambda(x_1 * x^N), \ldots, \lambda(x_n * x^N)) \in r^{M_\infty}$. Hence

$$(\lambda'(z_1), \ldots, \lambda'(z_n)) = (\lambda(x_1 * x^N), \ldots, \lambda(x_n * x^N)) \in r^{M_\infty},$$

but $(\lambda'(z_1), \ldots, \lambda'(z_n)) \in M^n$, therefore $(\lambda'(z_1), \ldots, \lambda'(z_n)) \in r^M$, that is λ' preserves r^M. Similar arguments show that λ' preserves the (partial) operations in $G^M \cup H^M$, hence λ' is a morphism $D_{\mathbf{M}}(\mathbf{A}_\sigma) \to \utilde{\mathbf{M}}$.

Since $\utilde{\mathbf{M}}$ yields a duality on $\mathbf{M}$, by the First Duality Theorem λ' extends to an A_σ-ary term function $\mathbf{M}^{A_\sigma} \to \mathbf{M}$, say $t^{\mathbf{M}}$. Consider the A-ary term

$$s = t * \prod_{a \in Z} v_a$$

where v_a is the variable term corresponding to the ath projection term function π_a and $\prod_{a \in Z}$ is the $|Z|$-fold $*$ product.

For $x \in D(\mathbf{A})$ with $x \notin N$, we have some $a \in Z$ such that $x(a) = \infty$, hence $s^{\mathbf{M}_\infty}(x) = t^{\mathbf{M}_\infty}(x) * \infty = \infty = \lambda(x)$. Alternatively, if $x \in N$, we have

$$\begin{aligned}
\lambda(x) &= \lambda'(\pi_{A_\sigma}(x)) = t^{\mathbf{M}}(\pi_{A_\sigma}(x)) = t^{\mathbf{M}}(x) \\
&= t^{\mathbf{M}_\infty}(x) = t^{\mathbf{M}_\infty}(x) * \prod_{a \in Z} x(a) = s^{\mathbf{M}_\infty}(x).
\end{aligned}$$

Hence the A-ary term function $s^{\mathbf{M}_\infty}$ extends λ, completing the proof. $\square$

We conclude this section with several examples. Let $\mathbf{D} = \langle \{0,1\}; \vee, \wedge \rangle$ be the two-element distributive lattice. The extended operations, $\vee$ and $\wedge$, on the set $D_\infty := \{0, 1, \infty\}$ are just the join in the three-element chain $0 < 1 < \infty$ and meet in the three-element chain $\infty < 0 < 1$. The variety (= quasi-variety) generated by $\mathbf{D}_\infty$ is precisely the variety of distributive bisemilattices (see [**7**]). Priestley duality for distributive lattices states that $\underset{\sim}{\mathbf{D}} = \langle \{0,1\}; 0, 1, \leq, \tau \rangle$ dualises $\mathbf{D}$. Thus, by Theorem 3.6, $\underset{\sim}{\mathbf{D}}_\infty := \langle \{0, 1, \infty\}; *, 0, 1, \infty, \leq, \tau \rangle$ dualises $\mathbf{D}_\infty$, where $\leq$ is the order on $\{0, 1, \infty\}$ whose only non-trivial comparability is $0 < 1$. Note that here we may avoid the unary relation $D = \{0,1\}$ since any map preserving $\leq$ and the constants $0, 1$ must preserve D. The dualising structure for $\mathbf{D}_\infty$ given by Gierz and Romanowska in [**7**] uses a different order, namely the order $\leq_\wedge$, with $\infty < 0 < 1$, associated with the the meet operation on $\mathbf{D}_\infty$. It could be argued that the order $\leq$ which arises from Theorem 3.6 is more natural than $\leq_\wedge$ as it is symmetric in its relationship to the operations $\vee$ and $\wedge$ on $\mathbf{D}_\infty$.

If $\mathbf{S} = \langle S; \cdot \rangle$ is a finite semigroup which possesses a left-zero term and is dualised by a structure of finite type, then Theorem 3.6 shows that the semigroup $\mathbf{S}_\infty$ is also dualised by a structure of finite type. In particular, every finite group, regarded as a semigroup, has a left-zero term. Since finite abelian groups are dualisable (see Davey [**6**]), it follows that a semigroup obtained by adding a new zero to a finite abelian group is dualisable (see also [**13**] and [**14**]). Similarly, semigroups obtained by adding a new zero to certain non-abelian groups, for example dihedral groups of order $2n$ with n odd (see Davey and Quackenbush [**4**]), are dualisable.

Perhaps the simplest strongly irregular variety is the variety $\mathcal{LZ}$ of **left-zero** semigroups (algebras with one binary operation $*$ satisfying the identity $x*y \approx x$). This variety is term equivalent to the variety of non-empty sets and we have $\mathcal{LZ} = \mathbb{ISP}\mathbf{L}$ where $\mathbf{L} = \langle \{0,1\}; * \rangle$ is the uniquely determined two-element left zero semigroup on $\{0,1\}$. The structure

$$\underset{\sim}{\mathbf{L}} = \langle \{0,1\}; \vee, \wedge, ', 0, 1; \tau \rangle,$$

where $\langle \{0,1\}; \vee, \wedge, ', 0, 1 \rangle$ is the two element Boolean algebra yields a (strong) duality on $\mathcal{LZ}$. Theorem 3.6 gives a duality for the left normal idempotent semigroup $\mathbf{L}_\infty$ via the structure

$$\underset{\sim}{\mathbf{L}}_\infty = \langle L_\infty; \vee, \wedge, ', *, 0, 1, \infty; \tau \rangle$$

where $\vee$, $\wedge$ on L_∞ are the distributive bisemilattice operations as in the first example and $'$ is given by:

$'$	0	1	∞
	1	0	∞

We may again avoid the unary relation L due to the presence of the constants $0, 1$ and either of the chain orders arising from $\vee$ or $\wedge$. Further, since $\mathbb{ISP}\mathbf{L} = \mathcal{LZ}$ and $\mathbf{L}$ has a one element subalgebra, by Corollary 3.3 we obtain $\mathbb{ISP}\mathbf{L}_\infty =$

$\overline{\mathfrak{LZ}}$. By Theorem 3.1, $\overline{\mathfrak{LZ}}$ is precisely the variety of left normal idempotent semigroups. This duality is closely related to the regularised Lindenbaum-Tarski duality discussed in [**13**] and [**14**].

4. Quasi-regularised Sets

It is natural to ask what may be said about the dualisability of other kinds of Płonka sums with dualisable fibres. Here, we consider a two-fibred Płonka sum where we attach the left-zero band $\mathbf{L}$ to the bottom of the semilattice and attach a trivial fibre to the top and give a "bare-hands" proof that the resulting semigroup is dualisable.

Consider $\mathbf{L}'$ given by the following table:

$*$	0	1	$1'$
0	0	0	0
1	1	1	1
$1'$	1	1	$1'$

$\mathbf{L}' \in \overline{\mathfrak{LZ}}$ is the Płonka sum of the functor Q from the categorical two-element meet semilattice $(\mathbf{2})$, where $Q(0) = \mathbf{L}$ while $Q(1)$ is the trivial idempotent semigroup $\langle\{1'\}; *\rangle$ and the fibre map $\phi_{1,0} = Q(0 \to 1)\colon \{1'\} \to \mathbf{L}$ distinguishes the element 1. (If $\phi_{1,0}$ instead distinguishes 0, the semigroup obtained is isomorphic to $\mathbf{L}'$.)

The dualising structure for $\mathbf{L}'$ will contain a modification of $\underset{\sim}{\mathbf{L}}$ consistent with the Płonka sum construction, just as we obtained $\underset{\sim}{\mathbf{L}}_\infty$. However, in contrast with $\underset{\sim}{\mathbf{L}}_\infty$, we will need to add an essentially "new" binary relation. The modified operations $\vee'$, $\wedge'$ and $''$ on L' obtained from $\vee$, $\wedge$ and $'$ on L are given by the following tables:

$\vee'$	0	1	$1'$
0	0	1	1
1	1	1	1
$1'$	1	1	$1'$

$\wedge'$	0	1	$1'$
0	0	0	0
1	0	1	1
$1'$	0	1	$1'$

$''$	0	1	$1'$
	1	0	0

A routine check ensures that these operations are algebraic, as is the binary relation

$$\nearrow := \{(0,1), (1,1), (1,1')\},$$

which, as a subalgebra of $(\mathbf{L}')^2$, is isomorphic to $\mathbf{L}'$. We also include the semigroup operation $*$ and the unary relation L, hence

$$\underset{\sim}{\mathbf{L}'} := \langle L'; *, \vee', \wedge', '', 0, 1, 1'; \nearrow, L; \tau\rangle$$

will be our dualising structure.

Theorem 4.1. $\underset{\sim}{\mathbf{L}}'$ *dualises* $\mathbf{L}'$.

Proof. Since $\underset{\sim}{\mathbf{L}}'$ is of finite type, it suffices to show that (IC) holds by the Duality Compactness Theorem. To this end, let $n \in \mathbb{N}$, let $\mathbf{X}$ be a (closed) substructure of $(\underset{\sim}{\mathbf{L}}')^n$, and let $\alpha\colon \mathbf{X} \to \underset{\sim}{\mathbf{L}}'$ be a morphism.

Let $T := \{1, 1'\}$ and note that $\langle T; *\rangle$ is a meet-semilattice with $1 < 1'$.

Observe that L^n, with the operations $\vee'$, $\wedge'$, $''$ suitably restricted together with the constants $\underline{0}, \underline{1}$, forms a Boolean algebra having $X \cap L^n$ as a subalgebra. Since α preserves the unary relation L, the restriction of α to $X \cap L^n$ is a Boolean algebra homomorphism onto the two-element Boolean algebra on $\{0, 1\}$. Therefore

$$\alpha{\restriction}_{X\cap L^n}(x) = \begin{cases} 1 & \text{if } x \geq \beta, \\ 0 & \text{if } x \ngeq \beta, \end{cases}$$

for all $x \in X \cap L^n$ and some atom β of $X \cap L^n$. Here, $\leq$ is the usual order on $\{0, 1\}$ extended pointwise.

We may similarly characterise α restricted to the (non-empty) semilattice $\langle X \cap T^n; *\rangle \leq \langle T^n; *\rangle$. Since it is a semilattice homomorphism onto the two element meet semilattice $\langle T; *\rangle$, we have

$$\alpha{\restriction}_{X\cap T^n}(x) = \begin{cases} 1' & \text{if } x \geq \sigma, \\ 1 & \text{if } x \ngeq \sigma, \end{cases}$$

for all $x \in X \cap T^n$ and some $\sigma \in X \cap T^n$ with $\sigma \neq \underline{1}$ (since α preserves the nullary 1).

Our first claim is that the set of indices

$$\{i \in \{1, \ldots, n\} \mid \beta_i = 1 \text{ and } \sigma_i = 1'\}$$

is non-empty. Suppose to the contrary that for all $i \in \{1, \ldots, n\}$, whenever $\beta_i = 1$, we have $\sigma_i = 1$. Then $\sigma_i = 1$ whenever ${\beta''}_i = 0$, from which we obtain $\beta'' \nearrow \sigma$. But then $\alpha(\beta)'' \nearrow \alpha(\sigma)$ since α preserves $\nearrow$ and $''$, which gives $0 \nearrow 1'$, a contradiction.

Fix a $j \in \{i \in \{1, \ldots, n\} \mid \beta_i = 1 \text{ and } \sigma_i = 1'\}$ and let

$$K := \{i \in \{1, \ldots, n\} \mid \sigma_i = 1' \text{ and } i \neq j\}.$$

Now, let t be the term function $(\mathbf{L}')^n \to \mathbf{L}'$ given by

$$t(x) := x_j * (\prod_{i \in K} x_i)$$

for all $x \in (L')^n$. We will show that t extends α.

Clearly, t satisfies the following for all $x \in (L')^n$:

$$\begin{aligned} t(x) = 0 &\iff x_j = 0; \\ t(x) = 1 &\iff x_j \neq 0 \text{ and } x_i \neq 1' \text{ for some } i \in K \cup \{j\}; \\ t(x) = 1' &\iff x_i = 1' \text{ for all } i \in K \cup \{j\}. \end{aligned}$$

Let $x \in X$. We observe that $\underline{1'} * x \in X \cap T^n$ and hence

$$\begin{aligned} \alpha(x) = 1' &\iff 1' * \alpha(x) = 1' \\ &\iff \alpha(\underline{1'} * x) = 1' \\ &\iff \underline{1'} * x \geq \sigma \\ &\iff (\underline{1'} * x)_i = 1' \text{ for all } i \in K \cup \{j\} \\ &\iff x_i = 1' \text{ for all } i \in K \cup \{j\} \\ &\iff t(x) = 1'. \end{aligned}$$

Since $x * \underline{0} \in X \cap L^n$, we have

$$\begin{aligned} \alpha(x) = 0 &\iff \alpha(x) * 0 = 0 \\ &\iff \alpha(x * \underline{0}) = 0 \\ &\iff x * \underline{0} \ngeq \beta \\ &\iff (x * \underline{0})_j = 0 \text{ (see note below)} \\ &\iff x_j = 0 \\ &\iff t(x) = 0. \end{aligned}$$

To see that $x * \underline{0} \ngeq \beta$ implies $(x * \underline{0})_j = 0$ in the above argument, suppose that $x * \underline{0} \ngeq \beta$ but $(x * \underline{0})_j = 1$. Then $(x * \underline{0})_i = 0$ and $\beta_i = 1$ for some $i \neq j$. But then, since $\beta_j = 1$, in $X \cap L^n$ we have $\underline{0} < (x * \underline{0}) \wedge \beta < \beta$, contradicting the fact that β is an atom. $\square$

We close by showing that $\mathbb{ISP}\mathbf{L}'$ is in fact the **quasi-regularisation** of the variety $\boldsymbol{\mathcal{LZ}}$. The quasi-regularisation of a variety $\boldsymbol{\mathcal{V}}$ of type F, as introduced by Bergman and Romanowska in [**1**], is defined to be the quasi-variety generated by $\boldsymbol{\mathcal{V}} \cup \boldsymbol{\mathcal{SL}}_F$, in symbols: $\mathbb{Q}(\boldsymbol{\mathcal{V}} \cup \boldsymbol{\mathcal{SL}}_F)$. If $\boldsymbol{\mathcal{V}}$ is strongly irregular, it is shown in [**1**] that the quasi-regularisation of $\boldsymbol{\mathcal{V}}$ is always a proper subclass of its regularisation $\overline{\boldsymbol{\mathcal{V}}}$. The following Theorem summarises the characterisation of $\mathbb{Q}(\boldsymbol{\mathcal{V}} \cup \boldsymbol{\mathcal{SL}}_F)$ given there.

Theorem 4.2. ([**1**]) *Let* $\mathbf{A}$ *be an algebra in the regularisation of a strongly irregular variety* $\boldsymbol{\mathcal{V}}$ *(in which the identity* $x * y \approx x$ *is satisfied). Assume that* $\mathbf{A}$ *is the Płonka sum of subalgebras* $(\mathbf{A}_s \mid s \in S)$ *over the semilattice* $\mathbf{S}$*, with fibre maps* $\varphi_{s,t}$*. The following are equivalent.*

(1) $\mathbf{A} \in \mathbb{ISP}(\boldsymbol{\mathcal{V}} \cup \boldsymbol{\mathcal{SL}}_F)$;
(2) *For every $s \geq t$ in S, the homomorphism $\varphi_{s,t}$ is injective;*
(3) $\mathbf{A}$ *satisfies any of the three equivalent quasi-identities given below:*

(a) $$\begin{pmatrix} x * y \approx x \\ y * x \approx y \\ x * z \approx z * x \approx z \\ y * z \approx z * y \approx z \end{pmatrix} \implies x \approx y;$$

(b) $$\begin{pmatrix} x * y \approx x \\ y * x \approx y \\ x * z \approx z \\ y * z \approx z \end{pmatrix} \implies x \approx y;$$

(c) $x * z \approx y * z \implies x * y \approx y * x.$

(4) $\mathbf{A} \in \mathbb{Q}(\boldsymbol{\mathcal{V}} \cup \boldsymbol{\mathcal{SL}}_F)$.

Proof. We show here that the quasi-identities in (3) are equivalent, and refer the reader to [**1**] for a proof of the equivalence of (1) to (4), where the quasi-identity (a) is denoted q_*.

Let $\mathbf{A} \in \overline{\boldsymbol{\mathcal{V}}}$ have semilattice replica $\mathbf{S}$, canonical homomorphism $\mu\colon \mathbf{A} \to \mathbf{S}$ and fibre maps $(\varphi_{t,s} \mid s \leq t)$.

Let $\mathbf{A}$ satisfy (a). To show that $\mathbf{A}$ satisfies (b), it will suffice to show that if $x, z \in A$ and $x * z = z$, then $z * x = z$. If $x * z = z$, then by definition we have $z \leq_* x$ and consequently $\mu(z) \leq \mu(x)$ in $\mathbf{S}$. But then $z * x = z * \varphi_{\mu(x),\mu(z)}(x) = z$.

Now, let $\mathbf{A}$ satisfy (b) and let $x, y, z \in A$ with $x * z = y * z = w$, say. Clearly $x * w = y * w = w$. We have $(x * y) * (y * x) = x * y$ and $(y * x) * (x * y) = y * x$. Also, $(x * y) * w = x * w = w$ and similarly $(y * x) * w = w$, hence $x * y = y * x$ by (b), showing that $\mathbf{A}$ satisfies (c).

Let $\mathbf{A}$ satisfy (c) and let $x, y, z \in A$ satisfy the antecedent of (a). Using (c), from $x * z = y * z = z$ we obtain $x * y = y * x$, but $x * y = x$ and $y * x = y$ shows that x and y are in the same fibre, from which it follows that $x = y$. $\square$

We will denote the quasi-regularisation of $\boldsymbol{\mathcal{LZ}}$ by $\overline{\boldsymbol{\mathcal{LZ}}}_q$. To show that $\overline{\boldsymbol{\mathcal{LZ}}}_q = \mathbb{ISP}\mathbf{L}'$, we will rely on Lemma 3.4, characterising the homomorphisms in $\overline{\boldsymbol{\mathcal{LZ}}}$. Interestingly, a $\overline{\boldsymbol{\mathcal{LZ}}}_q$-semigroup will be seen to be not only a disjoint union of left-zero semigroups, but simultaneously a disjoint union of semilattices!

Theorem 4.3. $\overline{\boldsymbol{\mathcal{LZ}}}_q = \mathbb{ISP}\mathbf{L}'$.

Proof. We need only show $\overline{\boldsymbol{\mathcal{LZ}}}_q \subseteq \mathbb{ISP}\mathbf{L}'$, the reverse inclusion being given by Theorem 4.2. Let $\mathbf{A} \in \overline{\boldsymbol{\mathcal{LZ}}}_q$ have semilattice replica $\mathbf{S}$ and canonical homomorphism $\mu\colon \mathbf{A} \to \mathbf{S}$. Let $a, b \in A$ with $a \neq b$. We must find a homomorphism $\mathbf{A} \to \mathbf{L}'$ that separates a and b. Firstly, observe that the semilattice replica of $\mathbf{L}'$ is the two element meet semilattice on $\{0, 1\}$, and we have $1 <_* 1'$.

case 1: a and b lie in different fibres.

Without loss of generality, assume $\mu(a) \not\geq \mu(b)$. Define the map $\alpha\colon A \to L'$ by

$$\alpha(x) = \begin{cases} 1' & \text{if } \mu(x) \geq \mu(b), \\ 1 & \text{otherwise} \end{cases}$$

for all $x \in A$. Then α separates a and b.

It is clear that $\Gamma\colon S \to \{0,1\}$ is well defined, and indeed, is the characteristic function of the upset of $\mu(b)$, and hence a semilattice homomorphism.

Let $x \leq_* y$ in $\mathbf{A}$. Then $\mu(x) \leq \mu(y)$ and consequently $\alpha(x) \leq_* \alpha(y)$. Hence by Lemma 3.4, α is a homomorphism.

case 2: a and b lie in the same fibre.

Consider the relation

$$\Theta := \{(x,y) \in A^2 \mid x * y = y * x\}$$

which we claim is an equivalence on A. Clearly Θ is reflexive and symmetric, to see that it is transitive let $x, y, z \in A$ such that $x * y = y * x$ and $y * z = z * y$. We have

$$\begin{aligned} x * (x * y * z) &= x * y * z \\ &= y * x * z \\ &= y * z * x \\ &= z * y * x \\ &= z * z * y * x \\ &= z * (x * y * z) \end{aligned}$$

hence $x * z = z * x$ by the quasi-identity (c) of Theorem 4.2.

Let $\alpha\colon A \to L'$ be the map sending the Θ-class of b to 1 and everything else to 0. Then Γ is the constant map $S \mapsto \{0\}$, hence a semilattice homomorphism. Also, if $x \leq_* y$ in A, then $x * y = y * x * y = y * x$, showing $\alpha(x) = \alpha(y)$ and hence α is order preserving. Therefore α is a homomorphism by Lemma 3.4. Further, α separates a and b, for otherwise, if $\alpha(a) = \alpha(b) = 1$, we have $a * b = b * a$, from which we obtain the contradiction $a = b$. □

References

1. Bergman C. and Romanowska A., *Subquasivarieties of regularized varieties*, Algebra Universalis **36**(4) (1996), 536–563.

2. Clark D. M. and Davey B. A., *Natural dualities for the working algebraist*, CUP, Cambridge, 1998.

3. Clark D. M., Davey B. A. and Pitkethly J. G., *Binary homomorphisms and natural dualities*, Preprint.

4. Davey B. A. and Quackenbush R. W., *Natural dualities for dihedral varieties*, J. Austral. Math. Soc. Ser. A **61**(2) (1996), 216–228.

5. Davey B. A., *Duality theory on ten dollars a day*, Algebras and orders (Montreal, PQ, 1991), Kluwer Acad. Publ., Dordrecht, 1993, pp. 71–111.

6. ———, *Dualisability in general and endodualisability in particular*, Logic and algebra (Pontignano, 1994), Dekker, New York, 1996, pp. 437–455.

7. Gierz G. and Romanowska A., *Duality for distributive bisemilattices*, J. Austral. Math. Soc. Ser. A **51**(2) (1991), 247–275.

8. Lakser H., Padmanabhan R. and Platt C. R., *Subdirect decomposition of Płonka sums*, Duke Math. J. **39** (1972), 485–488.

9. Płonka J., *On a method of construction of abstract algebras*, Fund. Math. **61** (1967), 183–189.

10. ———, *On equational classes of abstract algebras defined by regular equations*, Fund. Math. **64** (1969), 241–247.

11. Płonka J. and Romanowska A., *Semilattice sums*, Universal algebra and quasigroup theory (Jadwisin, 1989), Heldermann, Berlin, 1992, pp. 123–158.

12. Romanowska A. B. and Smith J. D. H., *Modal theory: an algebraic approach to order, geometry, and convexity*, Heldermann Verlag, Berlin, 1985.

13. ———, *Semilattice-based dualities*, Special issue on Priestley duality, Studia Logica **56**(1-2) (1996), 225–261.

14. ———, *Duality for semilattice representations*, J. Pure Appl. Algebra **115**(3) (1997), 289–308.

15. Romanowska A., *Free idempotent distributive semirings with a semilattice reduct*, Math. Japon. **27**(4) (1982), 467–481.

16. ———, *Idempotent distributive semirings with a semilattice reduct*, Math. Japon. **27**(4) (1982), 483–493.

17. ———, *On regular and regularised varieties*, Algebra Universalis **23**(3) (1986), 215–241.

B. A. Davey, Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia; *e-mail*: B.Davey@latrobe.edu.au

B. J. Knox, Department of Mathematics, La Trobe University, Bundoora, Victoria 3083, Australia; *e-mail*: B.Knox@latrobe.edu.au