Acta Math. Univ. Comenianae
Vol. LXVIII, 2(1999), pp. 249–252

CUBIC GRAPHS WITHOUT A PETERSEN MINOR HAVE NOWHERE–ZERO 5–FLOWS

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Abstract. We show that every bridgeless cubic graph without a Petersen minor has a nowhere-zero 5-flow. This approximates the known 4-flow conjecture of Tutte.

A graph has a **nowhere-zero k-flow** if its edges can be oriented and assigned nonzero elements of the group $\mathbb{Z}_k$ so that the sum of the incoming values equals the sum of the outcoming ones for every vertex of the graph. An equivalent definition we get using any Abelian group of order k or integers $\pm 1, \dots, \pm(k-1)$, as follows from Tutte [**13**], [**14**], [**15**] (see also [**5**], [**16**]). It is also known that graphs with bridges (1-edge-cuts) have no nowhere-zero k-flows for any $k \geq 2$, and that if a graph has a nowhere-zero k-flow, then it has a nowhere-zero $(k+1)$-flow.

There are three celebrated conjectures dealing with nowhere-zero flows in bridgeless graphs, all due to Tutte. The first is the 5-flow conjecture of [**13**], that every such graph admits a nowhere-zero 5-flow. The 3-flow conjecture states that if the graph does not contain a 3-edge cut, then it has a nowhere-zero 3-flow. Finally the 4-flow conjecture of [**15**] suggests that if the graph does not contain a subgraph contractible to the Petersen graph, then it has a nowhere-zero 4-flow. The last conjecture has also a second variant, where are considered only cubic (3-regular) graphs. It is not known, whether the second variant implies the first one.

We know the best possible approximations for the 5- and 3-flow conjectures, because Seymour [**12**] and Jaeger [**4**] proved that every bridgeless and 4-edge-connected graphs have nowhere-zero 6- and 4-flows, respectively. We show that a similar approximation holds also for the cubic variant of the 4-flow conjecture, i.e., that every bridgeless cubic graph without a Petersen minor has a nowhere-zero 5-flow. The proof is an easy corollary of two already known results. The first one is a strong structural theorem announced recently by Robertson, Seymour and Thomas [**10**] (mentioned also in [**16**, Theorem 3.7.16]).

Theorem 1. *Every cubic graph without a Petersen minor has girth at most* 5.

Received January 10, 1999.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 05C15.
Key words and phrases. Petersen minor, nowhere-zero k-flow, cubic graph.

Before introducing the second result we give some more notation. An edge cut is called **essential** if deleting its edges we get components so that at least two of them have edges. We consider graphs with multiple edges and loops. A class $\mathcal{G}$ of graphs is called **hereditary** if for any graph $G \in \mathcal{G}$ it contains all graphs homeomorphic with G and all subgraphs of G. The following theorem was in fact proved in [**2**] (see also [**16**], [**6**], [**7**]). We sketch its proof because [**2**] has not been published.

Theorem 2. *Let $\mathcal{G}$ be a hereditary class of graphs and G be a bridgeless graph from $\mathcal{G}$ having no nowhere-zero 5-flow and with the smallest possible order. Then G has girth at least 7.*

Proof. Primarily G has no essential 2- or 3-edge cuts, otherwise we get a smaller graph with this property as shown in [**2**] and also in [**5**, p. 91], [**16**, p. 41], [**11**]. Similarly G cannot have a vertex of degree 2 and without loss of generality we can suppose it has no loop. Suppose C is a cycle of G with the smallest possible order c (≥ 2).

Let $c \leq 6$ and M be the edge set of a maximal matching in C, $m = |M|$. Then $1 \leq m \leq 3$. $G - M$ must be a connected graph, otherwise M is an essential edge cut of G.

Suppose e is a bridge of $G - M$ and $G_1, \ldots, G_k$ are the components of $G - M - e$. We can check that then $k = 2$, $m = 3$, $c = 6$, e is an edge of C, and G_1, G_2 have no bridges (otherwise G has a bridge or an essential 2- or 3-edge-cut). Set $E' = M \cup \{e\}$ in this case. If $G - M$ is bridgeless, then set $E' = M$.

$G - E'$ is a bridgeless graph from $\mathcal{G}$, and from minimality of G it follows that $G - E'$ has a nowhere-zero 5-flow. Take an orientation of G so that the edges of C form an oriented cycle and a nowhere-zero $\mathbb{Z}_5$-flow by this orientation. Let a be a nonzero element of $\mathbb{Z}_5$ different from the values of the edges from $C - E'$ (a exists because $|C - E'| \leq 3$). Then assign each edge of E' value $-a$ and add to the edges from $C - E'$ value $-a$. We get a nowhere-zero 5-flow in G – a contradiction. Therefore $c \geq 7$ as claimed. □

Theorem 3. *Every bridgeless cubic graph without a Petersen minor has a nowhere-zero 5-flow.*

Proof. The class of all graphs homeomorphic with cubic graphs without a Petersen minor is hereditary. Thus, by Theorem 2, the smallest cubic graph without a Petersen minor has girth at least 7, what is a contradiction by Theorem 1. □

We can prove that the minimal graph from a hereditary class with no nowhere-zero 4-flow must have girth at least 5. This cannot be improved to girth 6, as follows from the following example. Let $\mathcal{G}'$ be the class of all graphs homeomorphic with the subgraphs of Petersen graph. Then $\mathcal{G}'$ is hereditary. We can also check that every graph from $\mathcal{G}'$ is homeomorphic with a cubic graph with girth at most 5. This implies that every graph from $\mathcal{G}'$ (including the Petersen graph) has a nowhere-zero 5-flow. But it is well known that Petersen graph has no nowhere-zero 4-flow.

Another approximation of the 4-flow conjecture was obtained by Huck [**3**] who shows that every bridgeless cubic graph without a Petersen minor has a 5-cycle double covering. His proof is based on Theorem 1 and results similar to Theorem 2.

Recently Robertson, Sanders, Seymour and Thomas [**9**] have announced that every bridgeless cubic graph without a Petersen minor has a nowhere-zero 4-flow. They prove this result extending the ideas of the proof of the four color theorem [**8**] (which improves the proof of Appel and Haken [**1**]). By [**9**], this proof is even more complicated than that form [**8**] and needs to check plenty of cases by computer. Finally note that if we can extend Theorem 1 for all graphs, then Theorem 3 holds for all graphs as well.

Acknowledgment. This paper was finished during the Alexander von Humboldt Fellowship in Germany. The author would like to thank to the AvH, Freie University and Professor Aigner for the hospitality. The research was partially supported by Ch. 77 foundation.

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