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EXISTENCE OF CONSERVATION LAWS IN NILPOTENT CASE

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Abstract. Using the Spencer-Goldschmidt version of the Cartan-Kähler theorem, we prove the local existence of conservation laws for analytical quasi-linear systems of two independent variables in the nilpotent and 2-cyclic case.

Introduction

A conservation law for a (1-1) tensor field h on a manifold M, $\dim M = n$, is a 1-form θ which satisfies $d\theta = 0$ and $dh^*\theta = 0$, where h^* is the transpose of h: $h^*\theta := \theta \circ h$. Conservation laws arise, for example, in the following classical problem. Consider a system of n quasi-linear equations in two independent variables:

$$(*) \qquad \frac{\partial x^i}{\partial u} + h^i_j(x)\frac{\partial x^j}{\partial v} = 0 \quad (i, j = 1, \dots, n).$$

If $\theta := \lambda_i(x)dx^i$ is a conservation law with respect to the (1-1) tensor field h defined by the matrix h^i_j, there exist locally two functions f and g so that $\theta = df$ and $h^*\theta = dg$, (i.e. $\lambda_i = \frac{\partial f}{\partial x^i}$ and $h^i_j\lambda_i = \frac{\partial g}{\partial x^j}$), and we have

$$0 = \lambda_i\frac{\partial x^i}{\partial u} + \lambda_i h^i_j(x)\frac{\partial x^j}{\partial v} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial u} + \frac{\partial g}{\partial x^j}\frac{\partial x^j}{\partial v} = 0.$$

Then for any solution $x^i(u, v)$ of the system $(*)$, we have

$$\frac{\partial f(x(u, v))}{\partial u} + \frac{\partial g(x(u, v))}{\partial v} = 0,$$

and it contains a conservation law in the sense of Lax ([**6**]).

Locally, giving a conservation law is equivalent to giving a function f such that $(dh^*d)(f) = 0$. Thus the study of the local existence of conservation laws is equivalent (in an analytic context) to the study of the formal integrability of the differential operator dh^*d.

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This problem has already been studied by Osborn, who, using Cartan's theory of exterior differential systems, showed the existence of conservation laws when h has constant coefficients in a suitable coordinate system ([**7**]).

In a paper published in 1964, Osborn ([**8**]) proved the formal Integrability of the operator dh^*d in the case when h is cyclic and if there exists a generator v^1 such that $v^1, \dots, h^{n-1}v^1$ commutes in the sense of the square bracket.

Using the theory presented by Spencer and Goldschmidt ([**4**], [**9**]), we improve in ([**2**]) the case when h is cyclic, by getting rid of the supplementary condition given by Osborn. Recently, we show in ([**5**]) the following theorem:

Theorem. *Suppose that h is nilpotent of order p, $(p \geq 2)$, analytic and such that $[h,h] = 0$. Fix $x_0 \in M$. Then there exists a neighborhood U of x_0 such that any $x \in U$ admits a "complete system" of conservation laws (i.e. every $\omega_0 \in T_x^*(M)$ can be prolonged in a germ of conservation laws) if and only if $\ker h, \ker h^2, \dots, \ker h^{p-1}$ are involutive.*

In this case the operator dh^*d is completely integrable ([**5**]).

The main result of the present paper, whose essential ideas were given in ([**5**]), can be expressed as following theorem:

Theorem. *Suppose that h is nilpotent of order p, $(p \geq 2)$, analytic, $[h,h] = 0$ and such that $\dim(\mathrm{Im}\ h^{p-1}) \geq \dim(\ker h) - 1$. Fix $x_0 \in M$. Then there exists a neighborhood U of x_0 such that any $x \in U$ admits a "complete system" of conservation laws.*

Corollary. *Suppose that h is nilpotent of order p, $(p \geq 2)$, analytic, $[h,h] = 0$ and such that h is 2-cyclic, (1-1) form. Fix $x_0 \in M$. Then there exists a neighborhood U of x_0 such that any $x \in U$ admits a "complete system" of conservation laws.*

1. Algebraic preliminaries

Using Frölicher-Nijenhuis formalism ([**3**]), we know that for any point $x \in M$ and for any (1-1) tensor field h there exists a neighborhood U of x such that h decomposes TU as a direct sum of the cyclic subspaces V_i, $i = 1, \dots, s$ stable for h, (i.e. the restriction of h to V_i is cyclic) ([**1**], [**7**], [**8**]). Let q_i designate the dimension of V_i at x and at any point in U. We suppose that V_i, $i = 1, \dots, s$ are arranged in such a way that $q_1 \geq q_2 \geq \cdots \geq q_s$. In this and following section we designe by v_i^1 a generator of V_i (for $i = 1, \dots, s$) and denote $v_i^{\alpha_i} := h^{\alpha_i - 1}v_i^1$, $\alpha_i = 1, \dots, q_i$. The vectors $\{(v_1^{\alpha_1})_{\alpha_1=1,\dots,q_1}, \dots, (v_s^{\alpha_s})_{\alpha_s=1,\dots,q_s}\} \equiv \{v_i^{\alpha_i}\}_{\substack{i=1,\dots,s \\ \alpha_i=1,\dots,q_i}}$, form a basis of TU which called "adapted" to the decomposition into cyclic subspaces. By convention, we write $v_i^{\beta} = 0$ for $\beta > q_i$.

Proposition 1.1. *If h is nilpotent of order p and $r \in \{1, \dots, p\}$, we have:*

1. $\ker h^r$ *is generated by* $\{v_i^{\alpha_i+q_i-r}\}_{\substack{i=1,\dots,s\\ \alpha_i=1,\dots,q_i}}$
2. $\mathrm{Im}\ h^r$ *is generated by* $\{v_i^{\alpha_i+r}\}_{\substack{i=1,\dots,s\\ \alpha_i=1,\dots,q_i}}$
3. $\dim \ker h = s$.

Proof. Conformally to the introduction of this section we can write the following table ([**5**]), which explain the relation between the elements of the set $\{v_i^{\alpha_i}\}_{\substack{i=1,\dots,s\\ \alpha_i=1,\dots,q_i}}$. In fact:

dimension of V_i	cyclic subspaces	sequence defined by h
q_1	V_1	$v_1^1 \xrightarrow{h} v_1^2 \xrightarrow{h} \dots v_1^\alpha \xrightarrow{h} \dots \xrightarrow{h} v_1^{q_1} \xrightarrow{h} 0$
$\vdots$	$\vdots$	$\vdots$
q_i	V_i	$v_i^1 \xrightarrow{h} v_i^2 \xrightarrow{h} \dots \xrightarrow{h} v_i^{q_i} \xrightarrow{h} 0$
$\vdots$	$\vdots$	$\vdots$
q_s	V_s	$v_s^1 \xrightarrow{h} \dots \xrightarrow{h} v_s^{q_s} \xrightarrow{h} 0$

We prove this proposition by simple application of this table ([**5**]). □

Definition 1. We call a Nijenhuis-manifold (M, h) every C^∞ manifold M equipped with a $(1-1)$ tensor field h such that $[h, h] = 0$. $[h, h]$ being the Nijenhuis square bracket of h defined by:

$$\frac{1}{2}[h,h](X,Y) := [hX, hY] + h^2[X,Y] - h[hX,Y] - h[X,hY] \qquad \forall X, Y \in TM.$$

Proposition 1.2. *On the Nijenhuis-manifold (M, h) we have:*

$$h^\alpha[X,Y] = -\sum_{j=1}^{\alpha-1}[h^{\alpha-j}X, h^jY] + \sum_{j=0}^{\alpha-1} h[h^{\alpha-j-1}X, h^jY]$$

$\forall \alpha = 1, \dots$ *and* $\forall X, Y \in TM$.

Proof. It is easy to prove by induction the proposition, which holds when $[h, h] = 0$. In fact, it is true for $\alpha = 2$. Suppose it is true up the order $\alpha - 1$. Then

$\forall X, Y \in TM$; $\forall \alpha = 1, 2, \dots$ we have:

$$h^\alpha[X,Y] = hh^{\alpha-1}[X,Y] = -\sum_{j=1}^{\alpha-2} h[h^{\alpha-j-1}X, h^jY] + \sum_{j=0}^{\alpha-2} h^2[h^{\alpha-j-2}X, h^jY]$$

$$= -\sum_{j=1}^{\alpha-2} h[h^{\alpha-j-1}X, h^jY] - \sum_{j=1}^{\alpha-1}[h^{\alpha-j}X, h^jY] + \sum_{j=0}^{\alpha-1} h[h^{\alpha-j-1}X, h^jY]$$

$$+ \sum_{j=1}^{\alpha-1} h[h^{\alpha-j-1}X, h^jY] - h[X, h^{\alpha-1}Y]$$

$$= -\sum_{j=1}^{\alpha-1}[h^{\alpha-j}X, h^jY] + \sum_{j=0}^{\alpha-1} h[h^{\alpha-j-1}X, h^jY].$$

□

2. Complete Integrability of dh^*d in the Nilpotent Case

Suppose, in this section, that (M,h) is a Nijenhuis-manifold, h is nilpotent and decomposes TM in s cyclic subspaces. Using the notations of section 1 we have:

Proposition 2.3. *The subspaces* $\ker h^r$; $r = 1, \dots p-1$ *are involutive if and only if* $\forall i, j = 1, \dots, s$ *such that* $j \geq i$, *we have;* $[v_i^\alpha, v_j^\beta] \in \ker h^{q_i}$ *for* $\alpha = 1, \dots, q_i$, $\beta = 1, \dots, q_j$.

Proof. The condition is sufficient. Let $r \in \{1, \dots, p-1\}$. $\ker h^r$ is involutive, In fact: Let $X := h^{q_i - r'}(v_i^1)$, $Y := h^{q_j - r''}(v_j^1)$ where $r'' \geq r$ and $r' \geq r$, be two elements of $\ker h^r$. We suppose that i, j are arranged in such a way $i \leq j$. If $r'' \geq r' \geq r$ we have:

$$0 = h^{q_i}[v_i^1, h^{q_j - r''}(v_j^1)]$$

$$= -\sum_{u=1}^{q_i-1}[h^{q_i-u}(v_i^1), h^{q_j-r''+u}(v_j^1)] + \sum_{u=0}^{q_i-1} h[h^{q_i-u-1}(v_i^1), h^{q_j-r''+u}(v_j^1)]$$

$$= -\sum_{u=1}^{r''-1}[h^{q_i-u}(v_i^1), h^{q_j-r''+u}(v_j^1)] + \sum_{u=0}^{r''-1} h[h^{q_i-u-1}(v_i^1), h^{q_j-r''+u}(v_j^1)]$$

$$= -\sum_{u=1}^{r'-1}[h^{r'-u}h^{q_i-r'}(v_i^1), h^u h^{q_j-r''}(v_j^1)] + \sum_{u=0}^{r'-1} h[h^{r'-u-1}h^{q_i-r'}(v_i^1), h^u h^{q_j-r''}(v_j^1)]$$

$$= h^{r'}[X,Y].$$

We deduce that $[X,Y] \in \ker h^{r'}$ and consequently $[X,Y] \in \ker h^r$ because $r' \leq r$. Similarly, if $r' \geq r'' \geq r$, then

$$0 = h^{q_i}[v_i^{r''-r'+1}, v_j^{q_j-r''+1})] = h^{q_i}[h^{r''-r'}v_i^1, h^{q_j-r''}v_j^1)] = h^{r''}[X,Y].$$

Consequently $[X,Y] \in \ker h^r$. Therefore $\ker h^r$ is involutive for every natural integer r. Conversely, let $v_i^\alpha \in \ker h^{q_i}$, $v_j^\beta \in \ker h^{q_j}$, $i \leq j$. We deduce that $\ker h^{q_j} \subseteq \ker h^{q_i}$ since $q_i \geq q_j$ but $\ker h^{q_i}$ is involutive, then $[v_i^\alpha, v_j^\beta] \in \ker h^{q_i}$.

Theorem 2.1. *Suppose that h is nilpotent of order $p, (p \geq 2)$, analytic, $[h,h] = 0$ and such that $\dim(\mathrm{Im}\ h^{p-1}) \geq \dim(\ker h) - 1$. Fix $x_0 \in M$. Then there exists a neighborhood U of x_0 such that any $x \in U$ admits a "complete system" of conservation laws.*

Proof. $\dim(\mathrm{Im}\ h^{p-1}) \geq \dim(\ker h) - 1$ implies that $\dim V_i = q_i = p$ for $i = 1,\dots,s-1$. In the other hand, all the cyclic subspaces but the last are of the same dimension. In this case the order of nilpotence of h is equal to p, which implies that the square bracket of two arbitrary vector fields, at point x_0 is an element of $\ker h^p_{x_0} = T_{x_0}M$. Then, $\forall i,j = 1,\dots,s$ such that $j \geq i$, we have: $[v_i^\alpha, v_j^\beta] \in \ker h^{q_i}$ for $\alpha = 1,\dots,q_i$, $\beta = 1,\dots,q_j$. In particular case, if $j = i = s$ the two vectors v_s^α, v_s^β are in the cyclic subspace V_s, so the bracket of the two vectors is an element of V_s then $[v_s^\alpha, v_s^\beta] \in \ker h^{q_s}$. This allows us to apply the previous proposition and say that the operator dh^*d is completely integrable. $\square$

Corollary 2.1. *If h is nilpotent of order p, $(p \geq 2)$, analytic, $[h,h] = 0$ and such that h is 2-cyclic, then the operator dh^*d is completely integrable.*

Proof. It's particular case of the previous theorem. In fact $s - 1 = 1$ and $\dim V_1 = p$. $\square$

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