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A SURVEY ON NAMBU–POISSON BRACKETS

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Abstract. The paper provides a survey of known results on geometric aspects related to Nambu-Poisson brackets.

1. Introduction

In 1973, Nambu [**30**] studied a dynamical system which was defined as a Hamiltonian system with respect to a ternary Poisson bracket. A few other papers on this bracket have followed at the time [**3**], [**27**]. A few years ago, Takhtajan [**34**] reconsidered the subject, proposed a general, algebraic definition of a **Nambu-Poisson bracket of order** n, and gave the basic characteristic properties of this operation. The Nambu-Poisson bracket is an intriguing operation, in spite of its rather restrictive character, which follows from the fact conjectured in [**34**], and proven by several authors [**41**] (cited by [**10**], and much older than [**34**]), [**14**], [**1**], [**28**], [**18**], [**23**] namely, that, locally and with respect to well chosen coordinates, any Nambu-Poisson bracket is just a Jacobian determinant as in [**30**]. In particular, the deformation quantization of the Nambu-Poisson bracket leads to interesting mathematical developments [**10**], [**9**]. On the other hand, the bracket inspired some generalizations of Lie-algebraic constructions (anticipated in [**13**]) [**6**], [**35**], [**8**], [**14**], [**2**], [**25**].

The aim of this paper is to give a survey of the subject from the point of view of geometry. In the next section, we review the basics, and present the geometric structure of Nambu-Poisson manifolds. Another section will be devoted to Nambu-Poisson-Lie groups. Finally, while we do not intend to review quantization theories, we formulate some related questions in the last section.

The paper does not contain new results. Everything in the paper is in the C^∞ category. Information on the usual Poisson manifolds may be found in [**37**], for instance. More general Nambu-Jacobi brackets were also studied [**19**], [**23**], [**16**] but, we will not discuss this subject here.

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2. Nambu-Poisson Brackets

Let M^m be an m-dimensional differentiable manifold, and $\mathcal{F}(M)$ its algebra of real valued C^∞-functions. A **Nambu-Poisson bracket or structure** of order n, $3 \le n \le m$ (this condition is always imposed in the paper) is an internal n-ary operation on $\mathcal{F}(M)$, denoted by $\{\ \}$, which satisfies the following axioms:

(i) $\{\ \}$ is $\mathbf{R}$-multilinear and totally skew-symmetric;

(ii) the **Leibniz rule**:

$$\{f_1, \dots, f_{n-1}, gh\} = \{f_1, \dots, f_{n-1}, g\}h + g\{f_1, \dots, f_{n-1}, h\};$$

(iii) the **fundamental identity**:

$$\{f_1, \dots, f_{n-1}, \{g_1, \dots, g_n\}\} = \sum_{k=1}^{n} \{g_1, \dots, g_{k-1}, \{f_1, \dots, f_{n-1}, g_k\}, g_{k+1}, \dots, g_n\}.$$

A manifold endowed with a Nambu-Poisson bracket is a **Nambu-Poisson manifold**. Remember that if we use the same definition for $n = 2$, we get a **Poisson bracket**.

By (ii), $\{\ \}$ acts on each factor as a vector field, whence it must be of the form

$$\{f_1, \dots, f_n\} = P(df_1, \dots, df_n), \tag{2.1}$$

where P is a field of n-vectors on M. If such a field defines a Nambu-Poisson bracket, it is called a **Nambu-Poisson tensor (field)**. P defines a bundle mapping

$$\sharp_P : \underbrace{T^*M \times \ldots \times T^*M}_{n-1 \text{ times}} \longrightarrow TM \tag{2.2}$$

given by

$$\langle \beta, \sharp_P(\alpha_1, \dots, \alpha_{n-1}) \rangle = P(\alpha_1, \dots, \alpha_{n-1}, \beta) \tag{2.3}$$

where all the arguments are covectors.

In what follows, we denote an n-sequence of functions or forms, say $f_1, \dots, f_n$, by $f_{(n)}$, and, if an index k is missing, by $f_{(n,\hat{k})}$.

The next basic notion is that of the P-**Hamiltonian vector field** of $(n-1)$ functions defined by

$$X_{f_{(n-1)}} = \sharp_P(df_{(n-1)}). \tag{2.4}$$

Then, the fundamental identity (iii) means that **the Hamiltonian vector fields are derivations of the Nambu-Poisson bracket**.

Another interpretation of (iii) is

$$(L_{X_{f_{(n-1)}}} P)(dg_1, \dots, dg_n) = 0, \tag{2.5}$$

where L is the Lie derivative, i.e., **the Hamiltonian vector fields are infinitesimal automorphisms of the Nambu-Poisson tensor**.

The fundamental identity also implies

$$\begin{aligned} X_{f_{(n-1)}} X_{g_{(n-1)}} h = & \sum_{k=1}^{n-1} \big\{ g_1, \dots, g_{k-1}, X_{f_{(n-1)}} g_k, g_{k+1}, \dots, g_{n-1}, h \big\} \\ & + X_{g_{(n-1)}} X_{f_{(n-1)}} h, \end{aligned}$$

whence

$$[X_{f_{(n-1)}}, X_{g_{(n-1)}}] = \sum_{k=1}^{n-1} X_{(g_1, \dots, g_{k-1}, X_{f_{(n-1)}} g_k, g_{k+1}, \dots, g_{n-1})}. \tag{2.6}$$

Therefore, **the set $\mathcal{H}(P)$ of all the real, finite, linear combinations of Hamiltonian vector fields is a Lie algebra**. (Notice that for $n \geq 3$ such a combination may not be a Hamiltonian vector field itself!)

The Nambu-Poisson tensor fields were characterized as follows by Takhtajan [**34**]

2.1. Theorem. *The n-vector field P is a Nambu-Poisson tensor of order n ($n \geq 3$) iff the natural components of P with respect to any local coordinate system x^a of M satisfy the equalities:*

$$\sum_{k=1}^{n} \left[P^{b_1 \dots b_{k-1} u b_{k+1} \dots b_n} P^{v a_2 \dots a_{n-1} b_k} + P^{b_1 \dots b_{k-1} v b_{k+1} \dots b_n} P^{u a_2 \dots a_{n-1} b_k} \right] = 0, \tag{2.7}$$

$$\sum_{u=1}^{m} \Big[P^{a_1 \dots a_{n-1} u} \partial_u P^{b_1 \dots b_n} - \sum_{k=1}^{n} P^{b_1 \dots b_{k-1} u b_{k+1} \dots b_n} \partial_u P^{a_1 \dots a_{n-1} b_k} \Big] = 0. \tag{2.8}$$

Furthermore, P is a Nambu-Poisson tensor field iff $P/_U$ is a Nambu-Poisson tensor field, for $U = \{x \in M \mid P_x \neq 0\}$.

Proof. Fix a point $p \in M$, and local coordinates x^i around p such that $x^i(p) = 0$. Then, with the Einstein summation convention, denote

$$P7 = \frac{1}{n!} P^{i_1 \ldots i_n} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_n}}, \tag{2.9}$$

$$\partial_u P = \frac{1}{n!} \frac{\partial P^{i_1 \ldots i_n}}{\partial x^u} \frac{\partial}{\partial x^{i_1}} \wedge \ldots \wedge \frac{\partial}{\partial x^{i_n}}, \tag{2.10}$$

$$\partial P = \partial_u P \otimes dx^u. \tag{2.11}$$

If the fundamental identity is expressed by means of (2.1), the terms which contain the second derivatives of the functions g cancel, and the identity becomes

$$\begin{aligned} &\sum_{u=1}^{m} P(df_{(n-1)}, dx^u)(\partial_u P)(dg_{(n)}) \\ &= \sum_{u=1}^{m} \sum_{k=1}^{n} \Big[P(dg_1, \ldots, dg_{k-1}, dx^u, dg_{k+1}, \ldots, dg_n)(\partial_u P)(df_{(n-1)}, dg_k) \\ &\quad + \sum_{h=1}^{n-1} P(dg_1, \ldots, dg_{k-1}, dx^s, dg_{k+1}, \ldots, dg_n) P(df_1, \ldots, df_{h-1}, \\ &\quad \frac{\partial^2 f_h}{\partial x^s \partial x^t} dx^t, df_{h+1}, \ldots, df_{n-1}, dg_k) \Big]. \end{aligned} \tag{2.12}$$

Now, (2.12) is always true at p if it is true in the following two cases:

a) $f_i = x^{a_i}$, $g_j = x^{b_j}$,

b) same as in a) with the exception of $f_1 = x^u x^v$.

Case a) yields (2.8), and case b) yields (2.7). Finally, the restriction $x^i(p) = 0$ may be removed by a translation of the coordinates.

The last assertion of the theorem is an obvious consequence of (2.7), (2.8). $\square$

Equality (2.7) is algebraic, and it is called the **quadratic identity**. This condition does not appear for the usual Poisson structures ($n = 2$). Equality (2.8) is called the **differential identity**, and it does not have a tensorial character. However, it is clear that if (2.7), (2.8) hold for one coordinate system at $p \in M$ the fundamental identity holds hence, (2.7), (2.8) will hold in any coordinate system.

The quadratic identity is rather intriguing. For this reason, we give several equivalent expressions below. First, (2.7) is equivalent with

$$\begin{aligned} \sum_{k=1}^{n} \big[&\{\varphi, f_1, \ldots, f_{n-2}, g_k\}\{\psi, g_1, \ldots, \hat{g}_k, \ldots, g_n\} \\ &+ \{\psi, f_1, \ldots, f_{n-2}, g_k\}\{\varphi, g_1, \ldots, \hat{g}_k, \ldots, g_n\} \big] \end{aligned} \tag{2.13}$$

for arbitrary functions. Indeed, using (2.1) we see that (2.7) implies (2.13), and on the other hand (2.13) reduces to (2.7) in the case of the coordinate functions.

Then, the expression (2.12) of the fundamental identity is the same as

$$
\begin{aligned}
(2.14)\qquad \left\langle \sharp_P(df_{(n-1)}), \partial P(dg_{(n)})\right\rangle &= \sum_{k=1}^{n}(-1)^{n-k}\Big[\left\langle \sharp_P(dg_{(n,\hat{k})}), \partial P(df_{(n-1)}, dg_k)\right\rangle \\
&+ \sum_{h=1}^{n-1}(-1)^{h+k}(\mathrm{Hess}\ f_h)(\sharp_P(dg_{(n,\hat{k})}), \sharp_P(df_{(n-1),\hat{h})}, dg_k))\Big],
\end{aligned}
$$

where all $f \in \mathcal{F}(M)$, and

$$
\mathrm{Hess}\, f := \frac{\partial^2 f}{\partial x^s \partial x^t} dx^s \otimes dx^t
$$

is the non invariant Hessian of f.

Moreover, if ∇ is an arbitrary torsionless connection on M, (2.14) is equivalent with the same relation where the partial derivatives in ∂P and in the Hessians are replaced by ∇-covariant derivatives. This yields a tensorial expression of the fundamental identity.

Formula (2.14) also yields another invariant expression of the quadratic identity if we proceed as follows. Notice that the quadratic identity holds iff (2.14) holds for functions which have a vanishing second derivatives at the point p, except for f_1, for which we ask the vanishing of the first derivatives, while $\mathrm{Hess}\, f_1 = T$ is an arbitrary 2-covariant symmetric tensor. Accordingly, the quadratic identity is equivalent to

$$
(2.15)\qquad \sum_{k=1}^{n}(-1)^{k+1}T(\sharp_P(\lambda_{(n,\hat{k})}), \sharp_P(\mu_{(n-1,\hat{1})}, \lambda_k)) = 0
$$

for any 2-covariant, symmetric tensor T, and any covectors λ, μ.

Finally, we indicate the following equivalent form of (2.7) noticed by P. Michor [**24**]

$$
(2.7')\qquad i(\alpha)P \wedge i(\Phi)i(\beta)P + i(\beta) \wedge i(\Phi)i(\alpha)P = 0,
$$

$\forall \alpha, \beta \in T^*M$ and $\forall \Phi \in \wedge^{n-2}T^*M$. Moreover, (2.7′) is the polarization of the, once more equivalent, condition

$$
(2.7'')\qquad i(\alpha)P \wedge i(\Phi)i(\alpha)P = 0.
$$

The geometric meaning of the quadratic identity will be shown in the forthcoming Theorem 2.4.

A mapping $\varphi\colon (M_1, P_1) \to (M_2, P_2)$ between two Nambu-Poisson manifolds of the same order n is a **Nambu-Poisson morphism** if the tensor fields P_1 and P_2 are φ-related or, equivalently, $\forall g_{(n)} \in \mathcal{F}(M_2)$, one has

$$\{g_1 \circ \varphi, \dots , g_n \circ \varphi\}_1 = \{g_1, \dots , g_n\}_2.$$

Moreover, if φ is a diffeomorphism, the two manifolds are said to be **equivalent Nambu-Poisson manifolds**. The notion of a Nambu-Poisson morphism also allows us to give the following definition: a submanifold N of the Nambu-Poisson manifold (M, P) is a Nambu-Poisson submanifold if N has a (necessarily unique) Nambu-Poisson tensor field Q of the same order as P such that the inclusion of (N, Q) in (M, P) is a Nambu-Poisson morphism. As in the Poisson case $n = 2$, Q exists iff, along N, P vanishes whenever evaluated on n 1-forms one of which, at least, belongs to the annihilator space $Ann(TN)$, and then $im\sharp_P$ is a tangent distribution of N e.g., [**37**].

By Theorem 2.1 P is a Nambu-Poisson tensor on the manifold M iff it is such on its nonvanishing subset. The following theorem [**14**], [**1**], [**28**], [**18**], [**23**] establishes the local canonical structure of the Nambu-Poisson brackets around nonvanishing points, up to equivalence.

2.2. Theorem. *P is a Nambu-Poisson tensor field of order n iff $\forall p \in M$ where $P_p \neq 0$ there are local coordinates (x^k, y^α) $(k = 1, \dots , n,\ \alpha = 1, \dots , m-n)$ around p such that*

$$P = \frac{\partial}{\partial x^1} \wedge \dots \wedge \frac{\partial}{\partial x^n} \tag{2.16}$$

on the corresponding coordinate neighborhood.

Proof. If (2.16) holds, we have $P^{1\dots n} = 1$, and the components of P which have other indices than a permutation of $(1, \dots , n)$ vanish. It is easy to see that (2.7), (2.8) hold in this case.

The following proof of the converse result belongs to Nakanishi [**28**], and is modeled on Weinstein's proof of the local structure theorem of Poisson manifolds (e.g., [**40**], [**37**]). Around p, take functions $x_{(n-1)}$ such that $X_{x_{(n-1)}} \neq 0$, then change to local coordinates $z_{(m)}$ where $X_{x_{(n-1)}} = \partial/\partial z_1$, and put $x_n = z_1$. Since

$$\{x_1, \dots , x_n\} = 1, \tag{2.17}$$

$x_{(n)}$ are functionally independent, and the vector fields $Y_k := (-1)^{n-k} X_{x_{(n,\hat{k})}}$, which satisfy $Y_k(x_h) = \delta_{kh}$, are linearly independent. Moreover, (2.6) shows that Y_k commute, and there exist local coordinates (s_k, y_α) $(k = 1, \dots , n,\ \alpha = 1, \dots , m - n)$ such that $Y_k = \partial/\partial s_k$ for all k. Furthermore, by looking at the corresponding Jacobian, we see that (x_k, y_α) also are local coordinates around p, and such that $Y_k = \partial/\partial x_k$, and all $\{x_{k_1}, \dots , x_{k_{n-1}}, y_\alpha\} = 0$.

The following trick is to evaluate in two ways the bracket

$$\frac{1}{2}(-1)^{k-1}\{x_1^2, x_2, \dots, x_{n-1}, \{x_2, \dots, x_k, x_n, y_{\alpha_1}, \dots, y_{\alpha_h}\}\}$$

where $k + h = n$. If we use first the fundamental identity and then the Leibniz rule we get $\{x_1, \dots, x_k, y_{\alpha_1}, \dots, y_{\alpha_h}\}$. If we use first the Leibniz rule and then the fundamental identity, we get 0. (Use (2.17) in both computations.) Similarly, we get the general result

$$P^{i_1\dots i_k\alpha_1\dots\alpha_h} = \{x_{i_1}, \dots, x_{i_k}, y_{\alpha_1}, \dots, y_{\alpha_h}\} = 0. \tag{2.18}$$

Finally, we must compute the components of P with Greek indices only. Of course, they vanish if $m < 2n$. If $m \geq 2n \geq 6$, these components are again given by using (2.17), (2.18) and a two-way computation of a Nambu bracket namely,

$$\begin{aligned} 0 &= \{x_1 y_{\alpha_1}, x_2, \dots, x_{n-1}, \{x_n, y_{\alpha_2}, \dots, y_{\alpha_n}\}\} \\ &= \{y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_n}\} = P^{\alpha_1\dots\alpha_n}. \end{aligned} \tag{2.19}$$

The results (2.17), (2.18), (2.19), with the notational change of writing the indices of the coordinates up as usual, imply (2.16). □

2.3. Remark. On the canonical coordinate neighborhood where (2.16) holds we have

$$\mathcal{D} := \operatorname{span}(im\, \sharp_P) = \operatorname{span}\{\partial/\partial x^k\}.$$

Hence, globally $\mathcal{D}$ is a foliation with singularities whose leaves are either points, called **singular points** of P, or n-dimensional submanifolds with a Nambu-Poisson bracket induced by P. (In other words, the computation of the latter is along the leaves of $\mathcal{D}$).

This remark extends well known results of Poisson geometry (e.g., [**37**]), and it was proven in [**14**] and [**18**]. In [**18**] the proof is by applying the Stefan-Sussmann-Frobenius theorem to $\mathcal{D}$, which is possible because $\mathcal{D}$ is also equal to $span\, \mathcal{H}(P)$. We call $\mathcal{D}$ the **canonical foliation** of the Nambu-Poisson structure P. The canonical foliation is regular i.e., all the leaves are n-dimensional, iff P never vanishes, and then we will say that P is a **regular Nambu-Poisson structure**.

The structure theorem 2.2 allows us to find the geometric meaning of the quadratic identity (2.7), which was conjectured in [**34**] and proven by many authors, independently [**41**], [**14**], [**1**], [**28**], [**23**], [**24**]. We say that an n-vector field is **decomposable** if, $\forall p \in M$, there are $V_1, \dots, V_n \in T_pM$ such that $P_p = V_1 \wedge \cdots \wedge V_n$. (This does not mean that such a decomposition holds for global vector fields on M.) Then, we have

2.4. Theorem. *The quadratic identity* (2.7) *is equivalent with the fact that the n-vector field P is decomposable.*

Proof. This is a pointwise, algebraic result, and it suffices to prove it in $\mathbf{R}^m$. If P is decomposable, we use a vector basis which has $V_1, \ldots, V_n$ as its first vectors, and a straightforward inspection of (2.7) shows that this condition holds.

Conversely, if P is given at a point, and it satisfies the quadratic identity, we may extend it to a tensor field with constant components on $\mathbf{R}^m$. The latter then obviously also satisfies the differential identity (2.8), and is a Nambu-Poisson tensor field on $\mathbf{R}^m$. Thus, P is decomposable by Theorem 2.2. □

Purely algebraic proofs were given in [**1**] and [**24**]. In particular, in [**24**] the result is proven by using the classical Plücker decomposability conditions (e.g., [**33**, p. 42]). Namely, one first proves a lemma which tells that an n-vector P ($n \geq 3$) is decomposable iff $i(\alpha)P$ is decomposable for all the covectors α. Then, ($2.7''$) is exactly the Plücker condition for $i(\alpha)P$.

Another immediate consequence of Theorem 2.2 is [**18**]

2.5. Corollary. *A Nambu-Poisson tensor field P of an even order $n = 2s$ satisfies the condition $[P, P] = 0$, where the operation is the Schouten-Nijenhuis bracket.*

This corollary suggests the study of **generalized Poisson structures** [**2**], [**18**] defined by a $(2s)$-vector field P such that

$$[P, P] = 0. \tag{2.20}$$

The canonical expression (2.16) provides the basic example of a Nambu-Poisson bracket, which was considered in Nambu's original paper [**30**] for $n = 3$. Namely, (2.16) means that we have

$$\{f_1, \ldots, f_n\} = \frac{\partial(f_1, \ldots, f_n)}{\partial(x^1, \ldots, x^n)}. \tag{2.21}$$

This example may be extended to a description of all the regular Nambu-Poisson structures [**14**], [**18**].

2.6. Theorem. *A regular Nambu-Poisson structure of order n on a differentiable manifold M^m is the same thing as a regular n-dimensional foliation S of M, and a bracket operation defined by the formula*

$$d_S f_1 \wedge \ldots \wedge d_S f_n = \{f_1, \ldots, f_n\}\omega, \tag{2.22}$$

where ω is an S-leafwise volume form, and d_S is differentiation along the leaves of S.

Proof. First, let M^m be a differentiable manifold endowed with a regular n-dimensional foliation S, and an S-leafwise volume form ω. (E.g., see [**26**] for foliation theory.) Then, the bracket defined by (2.22) is a regular Nambu-Poisson bracket. Indeed, if $x^{(n)}$ are local coordinates along the leaves of S, and if

$$\omega = \varphi d_S x^1 \wedge \ldots \wedge d_S x^n,$$

we get the local expression

$$\{f_1, \ldots, f_n\} = \frac{1}{\varphi}\frac{\partial(f_1, \ldots, f_n)}{\partial(x^1, \ldots, x^n)}. \tag{2.23}$$

Then, the change of the local coordinates

$$\tilde{x}^1 = \int \varphi dx^1,\ \tilde{x}^2 = x^2, \ldots, x^n = x^n$$

leads to (2.21) in the new coordinates $\tilde{x}^{(n)}$.

In particular, notice from the proof above that any formula of the type (2.23) defines a regular Nambu-Poisson bracket.

Now, conversely, if P is a regular Nambu-Poisson structure on M, we take S to be the canonical foliation of P, and choose the leafwise volume form ω such that $i(P)\omega = 1$. Then, we see that (2.22) holds by applying to it the operator $i(P)$. Clearly, the chosen volume form is the only possible one. □

Following is a number of other interesting facts relevant to Nambu-Poisson structures.

2.7. Remarks. (i) [**15**]. A decomposable n-vector field P is a Nambu-Poisson tensor iff the distribution $\mathcal{D} = \mathrm{span}\,(im\sharp_P)$ is involutive on the set of the non singular points of P.

(ii) [**34**]. If we have a Nambu-Poisson bracket of order $n > 2$, and keep p of its arguments fixed, we get a Nambu-Poisson bracket of order $n - p$ (a Poisson bracket if $n - p = 2$), and, conversely [**15**], if the result of an arbitrary fixed choice of p arguments ($p = 1, \ldots, n-2$) always yields a Nambu-Poisson tensor, P is a Nambu-Poisson tensor.

(iii) If (M_a, P_a) are Nambu-Poisson manifolds of order $n_a \geq 3$ ($a = 1, 2$), then $(M_1 \times M_2, P_1 \wedge P_2)$ is a Nambu-Poisson manifold of order $n_1 + n_2$. iv). If P is a Nambu-Poisson tensor on a manifold M, so is fP for any function $f \in C^\infty(M)$. In particular, this implies that (2.7) is equivalent to

$$P^{i_1 \ldots i_{n-1}k} P^{j_1 \ldots j_n} = \sum_{h=1}^{n} P^{j_1 \ldots j_{h-1} k j_{h+1} \ldots j_n} P^{i_1 \ldots in-1 j_h}. \tag{2.7$'$}$$

Concerning the first remark, we already know that $\mathcal{D}$ is involutive whenever P is Nambu-Poisson (Remark 2.3). On the other hand, since for $P = V_1 \wedge \cdots \wedge V_n$,

$\mathcal{D} = \operatorname{span}\{V_1, \ldots, V_n\}$, if $\mathcal{D}$ is involutive, we have $P = (\partial/\partial x_1) \wedge \cdots \wedge (\partial/\partial x_n)$ in some well chosen local coordinates on a neighborhood of $x \in M$ where $P_x \neq 0$ (Frobenious Theorem). Then, the corresponding bracket takes the form (2.23), and it is a Nambu-Poisson bracket.

The direct part of the second remark follows by checking the axioms. For the converse, it suffices to take $p = 1$, and check by a computation that if (2.7), (2.8) hold for $i(df)P$, $\forall f \in C^\infty(M)$, they also hold for P itself.

The third remark is an immediate consequence of (2.16).

The last remark follows by putting P under the form (2.16), and using the proof of Theorem 2.6. Then, (2.7′) is the coordinate expression of the fact that fP satisfies the fundamental identity $\forall f \in C^\infty(M)$. (It is obvious that (2.7′) implies (2.7).) For arbitrary functions (2.7′) yields

$$\begin{aligned}(2.7'')\qquad &\{f_1, \ldots, f_{n-1}, f\}\{g_1, \ldots, g_n\} \\ &= \sum_{h=1}^{n} \{g_1, \ldots, g_{h-1}, f, g_{h+1}, \ldots, g_n\}\{f_1, \ldots, f_{n-1}, g_h\}.\end{aligned}$$

The structure theorem 2.2 was used by Dufour and Zung [**11**], and by Nakanishi [**29**] in order to characterize Nambu-Poisson manifolds by means of differential forms, which are better suited for calculus than the multivectors. Namely, if ω is a volume form on the manifold M^m, for every n-vector P there exists a corresponding $(m-n)$-form $\varpi := i(P)\omega$, and the result proven in [**11**] is that P is a Nambu-Poisson tensor iff

$$(2.24)\qquad (i(A)\varpi) \wedge \varpi = 0, \qquad (i(A)\varpi) \wedge d\varpi = 0,$$

for any $(m-n-1)$-vector A. In [**11**], a differential form ϖ which satisfies (2.24) is called a **Nambu co-form**. In [**29**] it is shown that ϖ is a Nambu co-form iff it is decomposable and integrable i.e., $d\varpi = \theta \wedge \varpi$ for some 1-form θ.

On $\mathbf{R}^m$, any constant, decomposable n-vector field $k^{i_1 \cdots i_n}$ is a Nambu-Poisson tensor, since it satisfies both the quadratic and the differential identities. If we use Remark 2.7(ii) for this Nambu-Poisson tensor k, and keep as a fixed function $(1/2)\sum_{j=1}^m (x^j)^2$, we get a new Nambu-Poisson tensor, of order $n-1$, with the natural components

$$(2.25)\qquad P^{i_1 \cdots i_{n-1}} = \sum_{j=1}^{m} k^{i_1 \cdots i_{n-1} j} x^j.$$

A Nambu-Poisson structure defined on $\mathbf{R}^m$ by a tensor whose natural components are linear functions of x^j is called a **linear Nambu-Poisson structure**, and (2.25) gives the basic example [**6**]. Linear Nambu-Poisson structures are a generalization of the Lie-Poisson structures of Lie coalgebras (e.g., [**37**]). Accordingly, a definition and study of n**-Lie algebras** is suggested [**13**], [**34**], [**35**], [**8**], [**14**], [**25**], [**23**]. More precisely, a n-Lie algebra (called **Fillipov algebra** in [**15**]) is a vector space endowed with an internal, n-ary, skew symmetric bracket which satisfies the fundamental identity of a Nambu-Poisson bracket. (Different notions of n-Lie algebras were studied in [**17**] and [**25**].) By looking at brackets of linear functions, it easily follows that a linear Nambu-Poisson structure of order n on $\mathbf{R}^m$ induces a n-Lie algebra structure on the dual of $\mathbf{R}^m$ [**34**]. The converse may not be true since the structure constants of a general n-Lie algebra may form a non decomposable n-vector.

For instance, if $m = n + 1$ we may take k in (2.25) to be the canonical volume tensor of $\mathbf{R}^{n+1}$, and we get the linear Nambu-Poisson structure of order n discussed in [**6**]. The corresponding n-Lie algebra is the vector space $\mathbf{R}^{n+1}$ endowed with the operation of a **vector product** of n vectors (the determinant which has the coordinates of the vectors, and the canonical, positive, orthonormal basis as its columns [**5**]). Another definition of this operation, denoted by $\times$, is

$$v_1 \times \ldots \times v_n = *(v_1 \wedge \ldots \wedge v_n), \tag{2.26}$$

where $*$ is the Hodge star operator of the canonical Euclidean metric of $\mathbf{R}^{n+1}$. It is also easy to see that the canonical foliation of the linear Nambu-Poisson structure of $\mathbf{R}^{n+1}$ defined above has the origin as a 0-dimensional leaf, and the spheres with center at the origin as n-dimensional leaves. (For $n = 2$, this is the dual of the Lie algebra $o(3)$ with its well known Lie-Poisson structure.)

Of course, we may replace $\mathbf{R}^m$ by any vector space , with linear coordinates, in the definition of a linear Nambu-Poisson structure. Then, as in the case of Poisson structures [**40**], we notice that, if (M, P) is a Nambu-Poisson manifold, and if $p \in M$ is a **singular point** of P (i.e., $P(p) = 0$), the linear part of the Taylor development of P at p defines a linear Nambu-Poisson structure on T_pM, and a corresponding n-Lie algebra structure on T_p^*M, which are independent of the choice of the local coordinates at p. This linear Nambu-Poisson structure of T_pM should be regarded as the **linear approximation** of P at p, and P is **linearizable** at p if P is equivalent with its linear approximation on some neighbourhood of p.

The linear Nambu-Poisson tensors are completely determined by Dufour and Zung in [**11**] (see also [**23**] and [**15**]), and the result is

2.8. Theorem. *For any linear Nambu-Poisson structure P of order n on the linear space V^m there exists a basis of V such that the tensor P is of one of the following types.*

Type I:

$$P = \sum_{j=1}^{r+1} \pm x_j \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{j-1}} \wedge \frac{\partial}{\partial x_{j+1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{n+1}}$$
$$+ \sum_{j=1}^{s} \pm x_{n+j+1} \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{r+j}} \wedge \frac{\partial}{\partial x_{r+j+2}} \wedge \cdots \wedge \frac{\partial}{\partial x_{n+1}},$$

with $-1 \le r \le n$, $0 \le s \le \min(m-n-1, n-r)$;

Type II:

$$P = \frac{\partial}{\partial x_1} \wedge \cdots \wedge \frac{\partial}{\partial x_{n-1}} \wedge \Big(\sum_{i,j=n}^{m} a^i_j x_i \frac{\partial}{\partial x_j} \Big).$$

In the proof of Theorem 2.8 an essential role is played by the following results of linear geometry (Lemma 3.2 and Theorem 3.1 of [**11**], Lemma 1 of [**15**])

2.9. Lemma. *(i) Let* P_1, P_2 *be decomposable* n*-vectors of a linear space* V *such that* $P_1 + P_2$ *is also decomposable and let* D_1, D_2 *be the subspaces spaned by the factors of* P_1, P_2*, respectively. Then* $\dim(P_1 \cap P_2) \ge n-1$.

(ii) Let P_α*, where* α *runs in a set* A*, be an arbitrary family of decomposable* n*-vectors of a linear space* V *such that every sum* $P_{\alpha_1} + P_{\alpha_2}$ *is also decomposable, and let* D_α *be the subspaces spaned by the factors of* P_α*, respectively. Then either* $\dim(\cap_{\alpha\in A} D_\alpha) \ge n-1$ *or* $\dim(\sum_{\alpha\in A} D_\alpha) = n+1$.

Based on Theorem 2.8, Dufour and Zung prove several linearization theorems, and we refer the reader to [**11**] for these theorems.

3. Nambu-Poisson-Lie Groups

Nambu-Poisson-Lie groups as defined below were discussed in [**38**] and, independently, in [**15**], where a complete description of the multiplicative Nambu-Poisson tensor fields on a Lie group is given. In this section we reproduce the relevant part of our preprint [**38**], and refer the reader to [**15**] for general structural results.

Since Poisson-Lie groups play an important role in Poisson geometry (e.g., [**37**]), we are motivated to discuss similarly defined Nambu-Poisson-Lie groups. These cannot be defined by the demand that the multiplication is a Nambu-Poisson morphism since the direct sum of Nambu-Poisson tensors is not Nambu-Poisson (it is not decomposable). But, it makes sense to say that a Nambu-Poisson tensor P endows the Lie group G with the structure of a **Nambu-Poisson-Lie group** if P is a **multiplicative tensor field** i.e. (e.g., [**37**]), $\forall g_1, g_2 \in G$, one has

$$P_{g_1 g_2} = L_{g_1^*} P_{g_2} + R_{g_2^*} P_{g_1}, \tag{3.1}$$

where L and R denote left and right translations in G, respectively.

The multiplicativity of P implies $P_e = 0$, where e is the unit of G. Moreover, if G is connected, P is multiplicative iff $P_e = 0$, and the Lie derivative $L_X P$ is a left (right) invariant tensor field whenever X is left (right) invariant (e.g., [**37**]). As an immediate consequence it follows that the Nambu-Poisson-Lie group structures on the additive Lie group $\mathbf{R}^m$ are exactly the linear Nambu-Poisson structures of $\mathbf{R}^m$.

From (3.1), it follows that the set

$$G_0 := \{g \in G \mid P_g = 0\}$$

is a closed subgroup. Indeed, (3.1) shows that if $g_1.g_2 \in G_0$, the product $g_1 g_2 \in G_0$. Furthermore, if $g \in G_0$, then

$$0 = P_e = P_{gg^{-1}} = L_{g^*} P_{g^{-1}},$$

hence $g^{-1} \in G_0$.

In order to give another characterization of Nambu-Poisson-Lie groups, we generalize a bracket of 1-forms, which plays a fundamental role in Poisson geometry (e.g., [**37**]), to Nambu-Poisson manifolds.

The natural extension of the bracket of 1-forms to Nambu-Poisson structures of order n on M^m is defined as follows

$$\begin{aligned} \{\alpha_1, \dots, \alpha_n\} &= d(P(\alpha_{(n)})) + \sum_{k=1}^{n} (-1)^{n+k} i(\sharp_P(\alpha_{(n,\hat{k})})) d\alpha_k \\ &= \sum_{k=1}^{n} (-1)^{n+k} L_{\sharp_P(\alpha_{n,\hat{k}})} \alpha_k - (n-1) d(P(\alpha_{(n)})), \end{aligned} \tag{3.2}$$

where α_k $(k = 1, \dots, n)$ are 1-forms on M. The equality of the two expressions of the new bracket follows by using the classical relation $L_X = di(X) + i(X)d$. The bracket (3.2) will be called the **Nambu-Poisson form-bracket**, and we have

3.1. Proposition. *The Nambu-Poisson form-bracket satisfies the following properties:*

(i) *the form-bracket is totally skew-symmetric;*

(ii) $\forall f_{(n)} \in \mathcal{F}(M)$, *one has*

$$\{df_1, \dots, df_n\} = d\{f_1, \dots, f_n\}; \tag{3.3}$$

(iii) *for any* 1*-forms* $\alpha_{(n)}$, *and* $\forall f \in \mathcal{F}(M)$ *one has*

$$\{f\alpha_1, \alpha_2, \dots, \alpha_n\} = f\{\alpha_1, \alpha_2, \dots, \alpha_n\} + P(df, \alpha_2, \dots, \alpha_n)\alpha_1. \tag{3.4}$$

(iv) $\forall f_{(n-1)} \in \mathcal{F}(M)$ *and for any* 1*-form* α *one has*

$$\{df_1, \dots, df_{n-1}, \alpha\} = L_{X_{f_{(n-1)}}} \alpha. \tag{3.5}$$

Proof. (i) is obvious. (ii) and (iii) follow from the first expression of (3.2).

(iv) is a consequence of the first expression (3.2) and of the commutativity of d and L. □

Of course, in view of the skew symmetry formulas corresponding to (3.4), (3.5) may be used if the factor f and, respectively, the 1-form α appear at another factor of the bracket.

It would be nice if the form-bracket would also satisfy the fundamental identity of Nambu-Poisson brackets. This happens for $n = 2$ but, generally, we only have the following weaker result

3.2. Proposition. *The Hamiltonian vector fields act as derivations of the form-bracket by the Lie derivative operation.*

Proof. Suppose that the required property holds for the 1-forms $\alpha_{(n)}$ i.e.,

$$L_{X_{f_{(n-1)}}}\{\alpha_1, \alpha_2, \dots, \alpha_n\} = \sum_{k=1}^{n}\{\alpha_1, \alpha_2, \dots, \alpha_{k-1}, L_{X_{f_{(n-1)}}}\alpha_k, \dots, \alpha_n\}. \tag{3.6}$$

Then, a straightforward computation which uses (3.4) and (2.5) shows that $L_{X_{f_{(n-1)}}}$ also acts as a derivation of the bracket $\{f\alpha_1, \alpha_2, \dots, \alpha_n\}$, $\forall f \in \mathcal{F}(M)$.

This remark shows that the proposition is true if the result holds for a bracket of the form $\{dg_1, \dots, dg_n\}$, $\forall g_k \in \mathcal{F}(M)$. We see that this happens by applying (3.3), and the fundamental identity for functions, since we have

$$L_{X_{f_{(n-1)}}}\{dg_1, \dots, dg_n\} = L_{X_{f_{(n-1)}}}d\{g_1, \dots, g_n\} = dL_{X_{f_{(n-1)}}}\{g_1, \dots, g_n\}. \quad \square$$

The relation between (3.6) and the fundamental identity for 1-forms is given by (3.5). Moreover, since locally any closed form is an exact form, we see that the **fundamental identity**

$$\begin{aligned}\{\beta_1, \dots, \beta_{n-1}, \{\alpha_1, \dots, \alpha_n\}\} \\ = \sum_{k=1}^{n}\{\alpha_1, \dots, \alpha_{k-1}, \{\beta_1, \dots, \beta_{n-1}, \alpha_k\}, \alpha_{k+1}, \dots, \alpha_n\}\end{aligned} \tag{3.7}$$

holds whenever the 1**-forms** β **are closed**.

Another remark is that, since (3.5) expresses a Lie derivative, it defines a representation of the Lie algebra $\mathcal{H}(P)$ of the real, finite, linear combinations of Hamiltonian vector fields on the space $\wedge^1 M$ of the 1-forms on M, and Theorem 3.2 tells us that this representation is by derivations of the form-bracket.

Now, coming back to Nambu-Poisson-Lie groups, we can extend the following result of Dazord and Sondaz [**7**]

3.3. Theorem. *If G is a connected Lie group endowed with a Nambu-Poisson tensor field P which vanishes at the unit e of G, then (G, P) is a Nambu-Poisson-Lie group iff the P-bracket of any n left (right) invariant 1-forms of G is a left (right) invariant 1-form.*

Proof. The same proof as in the Poisson case (e.g., [**37**]) holds. Namely, the evaluation of the Lie derivative via (3.2) yields

$$(L_Y\{\alpha_1, \dots, \alpha_n\})(X) = Y((L_X P)(\alpha_{(n)})) \tag{3.8}$$

for any left invariant vector field X, right invariant vector field Y, and left invariant 1-forms $\alpha_{(n)}$. (Same if left and right are interchanged.) Hence, the condition of the theorem is equivalent with the fact that $L_X P$ is left invariant if X is left invariant. □

Some other basic properties of Poisson-Lie groups also have a straightforward generalization. First of all, since $P_e = 0$ for a Nambu-Poisson-Lie group G with unit e, and Nambu-Poisson tensor P, the linear approximation of P at e defines a linear Nambu-Poisson structure on the Lie algebra $\mathcal{G}$ of G, and a dual n-Lie algebra structure on the dual space $\mathcal{G}^*$. As for $n = 2$, a compatibility relation between the Lie bracket and the linear Nambu-Poisson structure of $\mathcal{G}$ exists.

First, following [**22**], let us consider the **intrinsic derivative** $\pi_e := d_e P : \mathcal{G} \to \wedge^n \mathcal{G}$ defined by

$$\pi_e(X)(\alpha_{(n)}) = (L_{\check{X}} P)_e(\alpha_{(n)}), \tag{3.9}$$

where $\alpha_{(n)} \in \mathcal{G}^*$, $X \in \mathcal{G}$, and $\check{X}$ is any vector field on $\mathcal{G}$ with the value X at e. Then we have

3.4. Theorem. *(i) The bracket of the dual n-Lie algebra structure of $\mathcal{G}^*$ is the dual of the mapping π_e, and it has each of the following expressions*

$$\begin{aligned}[\alpha_1, \dots, \alpha_n] &= d_e(P(\check{\alpha}_{(n)})) = \pi_e^*(\alpha_{(n)}) \\ &= \{\bar{\alpha}_1, \dots, \bar{\alpha}_n\}_e = \{\tilde{\alpha}_1, \dots, \tilde{\alpha}_n\}_e,\end{aligned} \tag{3.10}$$

where $\alpha_{(n)} \in \mathcal{G}^*$, $\check{\alpha}_{(n)}$ are 1-forms on G which are equal to $\alpha_{(n)}$ at e, and $\bar{\alpha}_{(n)}$, $\tilde{\alpha}_{(n)}$ are the left and right invariant 1-forms, respectively, defined by $\alpha_{(n)}$.

(ii) The mapping π_e is a $\wedge^n\mathcal{G}$-valued 1-cocycle of $\mathcal{G}$ with respect to the adjoint representation

$$\operatorname{ad} X(Y_1 \wedge \ldots \wedge Y_n) = \sum_{k=1}^{n} Y_1 \wedge \ldots Y_{k-1} \wedge [X, Y_k]_{\mathcal{G}} \wedge Y_{k+1} \wedge \ldots \wedge Y_n,$$

$(X, Y_{(n)} \in \mathcal{G})$.

Proof. The proofs are exactly the same as for $n = 2$; see [**22**] or Chapter 10 of [**37**]. We repeat them briefly here.

(i) By the definition of a dual mapping, and since $P_e = 0$, we have

$$\begin{aligned}\left\langle \pi_e^*(\alpha_{(n)}), X \right\rangle &= \pi_e(X)(\alpha_{(n)}) = (L_{\check{X}} P)_e(\alpha_{(n)}) = X(P(\check{\alpha}_{(n)})) \\ &= \left\langle d_e(P(\check{\alpha}_{(n)}), X \right\rangle,\end{aligned}$$

and this differential clearly is the n-Lie algebra structure of the linear approximation of P at e. This justifies the first two equality signs of (3.10). The remaining part of (3.10) follows from:

$$\begin{aligned}\{\bar{\alpha}_1, \dots, \bar{\alpha}_n\}_e(X) &\overset{(3.2)}{=} X(P(\bar{\alpha}_{(n)})) + \sum_{k=1}^{n} (-1)^{n+k} (d\bar{\alpha}_k)_e(\sharp_P(\alpha_{(n,\hat{k})}), X) \\ &= X(P(\bar{\alpha}_{(n)})) - \sum_{k=1}^{n} (-1)^{n+k} (L_{\tilde{X}} \bar{\alpha}_k)_e(\sharp_P(\alpha_{(n,\hat{k})})) \\ &\quad + \sum_{k=1}^{n} \sharp_P(\alpha_{(n,\hat{k})})_e(\bar{\alpha}_k(\tilde{X})) = X(P(\bar{\alpha}_{(n)}),\end{aligned}$$

where $\tilde{X}$ is the right invariant vector field defined by X, and we used the equalities $P_e = 0$, $L_{\tilde{X}} \bar{\alpha}_k = 0$.

(ii) The fact that π_e is a 1-cocycle means that we have

$$\text{ad } X(\pi_e(Y)) - \text{ad } Y(\pi_e(X)) - \pi_e([X, Y]_{\mathcal{G}}) = 0, \tag{3.11}$$

where $X, Y \in \mathcal{G}$. We always use the notation with bars and tildes for left and right invariant objects on Lie groups as we did above. Then, it follows that

$$\text{ad } X(\pi_e(Y)) = \frac{d}{ds}\Big/_{s=0} \text{Ad } \exp(sX)((L_{\bar{Y}} P)_e) = (L_{\bar{X}} L_{\bar{Y}} P)_e,$$

and (3.11) is a consequence of this result. □

Now we get the relation announced earlier:

3.5. Corollary. $\forall \alpha_{(n)} \in \mathcal{G}^*$ *and* $\forall X, Y \in \mathcal{G}$ *the following relation holds*

$$\begin{aligned}\left\langle [\alpha_1, \dots, \alpha_n], [X, Y]_{\mathcal{G}} \right\rangle = \sum_{k=1}^{n} \Big(&\left\langle [\alpha_1, \dots, \alpha_{k-1}, \text{coad}_X \alpha_k, \alpha_{k+1}, \dots, \alpha_n], Y \right\rangle \\ &- \left\langle [\alpha_1, \dots, \alpha_{k-1}, \text{coad}_Y \alpha_k, \alpha_{k+1}, \dots, \alpha_n], X \right\rangle \Big).\end{aligned} \tag{3.12}$$

Proof. The result is nothing but a reformulation of the cocycle condition (3.11). □

In agreement with Corollary 3.5, we will define a **Nambu-Poisson-Lie algebra** as a Lie algebra with a linear Nambu-Poisson structure which satisfies (3.12).

The question is: given a Nambu-Poisson-Lie algebra $\mathcal{G}$, is it possible to integrate it to a Nambu-Poisson-Lie group? In the updated version [**39**] of [**38**] we show that the general answer is no, even if the definition of a Nambu-Poisson-Lie algebra is changed by adding one more necessary condition which is implied by [**15**]. A corresponding negative example on the unitary Lie algebra $u(2)$ will be quoted later on.

But, some of the results known for $n = 2$ still hold. If G is connected and simply connected, for any 1-cocycle π_e as in Theorem 3.4(ii), there exists a unique multiplicative n-vector field P on G, called the **integral field** of π_e such that d_eP is the given cocycle. Indeed, for the given cocycle π_e,

$$\pi_g(X_g) := \mathrm{Ad}\, g(\pi_e(L_{g^{-1}*}X_g)) \qquad (g \in G,\ X_g \in T_gG)$$

defines a $\wedge^n\mathcal{G}$-valued 1-form π on G which satisfies the equivariance condition $L_g^*\pi = (\mathrm{Ad}\, g) \circ \pi$. This implies that $d\pi = 0$, and, since G is connected and simply connected, $\pi = dP$ for a unique n-vector field P on G, which can be seen to be multiplicative [**22**], [**37**]. If this field is Nambu-Poisson, we are done. But, this final part is more complicated than for $n = 2$ since it involves the quadratic identity (2.7), and the non-tensorial differential identity (2.8). We only have

3.6. Proposition. *If $\mathcal{G}$ is a Nambu-Poisson-Lie algebra of even order n, the integral field P of the dual cocycle π_e of the linear Nambu-Poisson structure of $\mathcal{G}$, on the connected, simply connected Lie group G which integrates $\mathcal{G}$, is a multiplicative generalized Poisson structure on G.*

Proof. The same proof as for $n = 2$ [**22**], [**37**] shows that the Schouten-Nijenhuis bracket $[P, P] = 0$. Indeed, since P is multiplicative, so is $[P, P]$ and, in particular, $[P, P]_e = 0$. Furthermore, since n is even, $P_e = 0$, and using the coordinate expression of the Schouten-Nijenhuis bracket [**37**], we have

$$\begin{aligned}(3.13)\qquad d_e[P, P](X) &= 2[P, L_{\tilde{X}}P]_e \\ &= \frac{2}{(2n-1)!n!(n-1)!}\delta^{k_1\ldots k_{2n-1}}_{i_1\ldots i_n j_2\ldots j_n}\frac{\partial P^{i_1\ldots i_n}}{\partial x^u}\left(\xi^v\frac{\partial P^{uj_2\ldots j_n}}{\partial x^v}\right)\frac{\partial}{x^{k_1}} \\ &\quad\wedge\cdots\wedge\frac{\partial}{\partial x^{k_{2n-1}}}\Big/_e,\end{aligned}$$

where $X = \xi^v(\partial/\partial x^v)/_e$. Now, $\xi^v(\partial P^{j_1\ldots j_n}/\partial x^v)$ are the coordinates of the n-vector $(d_eP)(X)$ of the linear approximation of P at e. Hence, the result of (3.13) is the algebraic Schouten-Nijenhuis bracket $[d_eP, d_eP]_{\mathcal{G}}$ (e.g., [**37**]), which is zero by Corollary 2.5. The conclusion is that $d_e[P, P] = 0$. But, a multiplicative tensor field with a vanishing intrinsic derivative at e is identically 0 [**22**], [**37**]. Hence, $[P, P] = 0$. □

Theorem 3.4 also allows us to get a result on subgroups just as in the Poisson case. A Lie subgroup H of a Nambu-Poisson-Lie group (G, P) will be called a

Nambu-Poisson-Lie subgroup if H has a (necessarily unique) multiplicative Nambu-Poisson tensor Q such that (H, Q) is a Nambu-Poisson submanifold of (G, P). If H is connected, it is a Nambu-Poisson-Lie subgroup of (G, P) iff $\mathrm{Ann}(\mathcal{H})$, where $\mathcal{H}$ is the Lie algebra of H, is an ideal in $(\mathcal{G}^*, [., \ldots, .])$. By this we mean that the bracket (3.10) is in $\mathrm{Ann}(\mathcal{H})$ whenever one of the arguments (at least) is in $\mathrm{Ann}(\mathcal{H})$. The proof is the same as for $n = 2$ e.g., [**37**].

Furthermore, if (H, Q) is a Nambu-Poisson-Lie subgroup of (G, P), the homogeneous space $M = G/H$ inherits a Nambu-Poisson structure S of the same order as P, Q such that the natural projection $p\colon (G, P) \to (M, S)$ is a Nambu-Poisson morphism. This holds since the brackets $\{f_1 \circ p, \ldots, f_n \circ p\}_P$ are constant along the fibers of p, which is easy to check using (3.1). (E.g., see Proposition 10.30 in [**37**] for the case $n = 2$.) Moreover, as a consequence of (3.1), the natural left action of G on M satisfies the multiplicativity condition

$$S_{g(x)} = \varphi_{g^*}(S_x) + \varphi^x_*(P_g), \tag{3.1$'$}$$

where $\varphi_g(x) = \varphi^x(g) = g(x)$ for $g \in G, x \in M$, and $\varphi_g\colon M \to M$, $\varphi^*\colon G \to M$. Accordingly, any action of a Nambu-Poisson-Lie group (G, P) on a Nambu-Poisson manifold (M, S) which satisfies (3.1$'$) will be called a **Nambu-Poisson action**. If G is connected, one has the same infinitesimal characteristic properties of Nambu-Poisson actions as in the Poisson case e.g., Proposition 10.27 in [**37**]. In particular, that $\forall X \in \mathcal{G}$, $L_{X_M} S = -[(d_e P)(X)]_M$, where e is the unit of G, and the index M denotes the infinitesimal action on M.

Now, we give a number of examples of non commutative Nambu-Poisson-Lie groups.

A first example is that of the 3-dimensional solvable Lie group

$$G_3 := \left\{ \begin{pmatrix} x & 0 & y \\ 0 & x & z \\ 0 & 0 & 1 \end{pmatrix}, \quad x, y, z \in \mathbf{R},\ x \neq 0 \right\}. \tag{3.14}$$

The left invariant forms of this group are dx/x, dy/x, dz/x, and if we look for a Nambu tensor of the form

$$P = f(x) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \tag{3.15}$$

such that $\{dx/x, dy/x, dz/x\}$ is left-invariant, and $f(1) = 0$, we see that $f = x(x^2 - 1)/2$ does the job. The corresponding Nambu-Poisson-Lie algebra is $\mathbf{R}^3$ with the linear Nambu structure $x^1(\partial/\partial x^1) \wedge (\partial/\partial x^2) \wedge (\partial/\partial x^3)$.

The next example is that of the **generalized Heisenberg group**

$$H(1, p) := \left\{ \begin{pmatrix} \mathrm{Id}_p & X & Z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \right\}, \tag{3.16}$$

where $X = {}^t(x_1 \dots x_p)$, $Z = {}^t(z_1 \dots z_p)$. The left invariant 1-forms of this group are

$$dx_1, \dots, dx_p, dy, dz_1 - x_1 dy, \dots, dz_p - x_p dy, \tag{3.17}$$

and

$$P = y\frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial y} \tag{3.18}$$

makes $H(1,p)$ into a Nambu-Poisson-Lie group. Indeed, it vanishes at the unit, and it follows easily that the brackets of the left invariant 1-forms are left invariant. The corresponding Nambu-Poisson-Lie algebra is $\mathbf{R}^{2p+1}$ with the same Nambu tensor (3.18).

A third example is that of the direct product $G = H(1,1) \times \mathbf{R}_+$, where $\mathbf{R}_+$ is the multiplicative group of the positive real numbers t. The left invariant 1-forms of the group are those given by (3.17), and dt/t. The tensor

$$P = t(\ln t)\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t} \tag{3.19}$$

makes G into a Nambu-Poisson-Lie group for the same reasons as in the previous examples. The corresponding Nambu-Poisson-Lie algebra is $\mathbf{R}^4$ with the linear Nambu structure

$$P = x_4\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4}. \tag{3.20}$$

We may notice that if (G_1, P) is a Nambu-Poisson-Lie group, and G_2 is any other Lie group, fP, where $f \in C^\infty(G_2$, is a Nambu-Poisson-Lie structure on $G_1 \times G_2$.

The next example is that of a Nambu-Poisson-Lie algebra. Consider the unitary Lie algebra $u(2)$ with the basis

$$X_1 = \frac{\sqrt{-1}}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad X_2 = \frac{\sqrt{-1}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$
$$X_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad X_4 = \frac{\sqrt{-1}}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, the linear Nambu tensor

$$P = x_1\frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \tag{3.21}$$

yields a Nambu-Poisson-Lie algebra structure. Indeed, straightforward computations show that (3.12) is satisfied. In a new version of [**38**], we show that this structure does not come from a Nambu-Poisson-Lie group. Namely, the structure theory of [**15**] implies that if (3.21) commes from a Nambu-Poisson-Lie group structure Λ of $U(2)$ then

$$R_{g^{-1}*}\Lambda = \theta(g)\frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^3} \wedge \frac{\partial}{\partial x^4} \qquad (\forall g \in U(2)),$$

where θ comes from an additive character of the circle subgroup S^1 of $U(2)$ hence, $\theta = 0$.

In principle, all the Nambu-Poisson-Lie algebras can be determined from the Dufour-Zung classification of the linear Nambu structures [**11**] by looking for Lie algebras structure constants which, together with the canonical structures of [**11**], satisfy the condition (3.12).

Proposition 3.6 might suggest looking for examples of Nambu-Poisson-Lie groups by first looking for $(2p)$-vector fields P on a Lie group G which are multiplicative, and satisfy the Schouten-Nijenhuis bracket condition $[P, P] = 0$. For this purpose, the technique of the **generalized Yang-Baxter equation**

$$(\operatorname{ad} X)[\mathbf{r}, \mathbf{r}]_{\mathcal{G}} = 0 \qquad (X \in \mathcal{G},\ \mathbf{r} \in \wedge^{2p}\mathcal{G}), \tag{3.22}$$

used for $n = 2$ (e.g., [**20**], [**37**]) may be extended. But, since the $2p$-vector field to be considered is $P = \bar{\mathbf{r}} - \tilde{\mathbf{r}}$ [**37**] (remember that bar and tilde denote the left and right invariant corresponding tensor field, respectively), it is not clear whether we can get a decomposable tensor P.

On the other hand, we should look for decomposable solutions of the **classical Yang-Baxter equation**

$$[\mathbf{r}, \mathbf{r}] = 0 \qquad (\mathbf{r} \in \wedge^{2p}\mathcal{G}) \tag{3.23}$$

for another reason too. Namely, The left (right) invariant field generated by such a solution could give us left (right) invariant Nambu-Poisson structures on the Lie group G. General questions on left invariant Nambu-Poisson structures on Lie groups are studied in [**29**].

We end this section by indicating the method used in [**15**] for the construction of the Nambu-Poisson-Lie groups. It consists in looking at the sum and the intersection of the subspaces $V_g \subseteq \mathcal{G}$ spaned by the factors of the decomposable n-vectors $R_{g^{-1}*}P_g$, $\forall g \in G$, and showing that these provide an ideal $\mathcal{H}$ of dimension n, $n-1$ or $n+1$ in $\mathcal{G}$. (The result follows from the multiplicativity of P and the use of Lemma 2.9.) Accordingly, the multiplicative n-vector fields on G are given by acting on the left invariant n-vector defined by $\mathcal{H}$ via multiplication by a function φ, wedge product by a vector field X, and interior product by a 1-form α, respectively, with well determined properties described in [**15**]. In particular, it turns out that the simple Lie groups do not admit multiplicative Nambu-Poisson tensors P of order $n \geq 3$, and, if $G = G_1 \times \cdots \times G_s$ is semisimple with simple factors G_i $(i = 1, \ldots, s)$, the only multiplicative Nambu-Poisson tensors on G are wedge products of "contravariant volumes" on a part of the factors with either multiplicative Poisson bivectors or multiplicative vector fields on other factors.

4. Questions on Quantization

The quantization of the Nambu-Poisson bracket was considered from the very first paper [**30**], and it was discussed by many authors [**3**], [**34**], [**6**], [**10**], [**9**], etc. This section is not a survey of the quoted references but, a preliminary discussion about possible approaches to geometric and deformation quantization of Nambu-Poisson brackets.

We consider the Kostant-Souriau geometric quantization [**21**], [**32**] first.

The prequantization of a symplectic manifold M is defined by a canonical lifting of the Hamiltonian vector field X_f of an **observable** i.e., a function $f \in \mathcal{F}(M)$, to the total space of a principal $\mathbf{C}^*$-bundle $p \colon L^* \to M$ ($\mathbf{C}^* = \mathbf{C} \backslash \{0\}$) or, equivalently, a circle bundle. The lifting is defined by introducing a Hermitian metric h, and a Hermitian connection ∇ on the associated complex line bundle L. Namely, ∇ decomposes the tangent bundle of L^* into a **horizontal** and a **vertical part**. The horizontal component of the prequantization lift $\hat{f}$ of X_f will be the ∇-horizontal lift of X_f, and the vertical component of $\hat{f}$ will be the infinitesimal right translation defined on the fibers of L^* by the values of $2\pi\sqrt{-1}f$ along the trajectory of X_f which starts at the base point of the fiber. (The factor $2\pi\sqrt{-1}$ is explained by technical reasons.) It is shown [**21**] that $\hat{f}$ is determined by the conditions

$$p_*(\hat{f}) = X_f, \quad \alpha(\hat{f}) = -2\pi\sqrt{-1}f, \tag{4.1}$$

where α is the connection form of ∇ on L^*.

Furthermore [**21**], $\hat{f}$ can be reinterpreted as a linear operator on the space $\Gamma(L)$ of the global cross sections of L given by

$$\hat{f}(\sigma) = \nabla_{X_f}\sigma + 2\pi\sqrt{-1}f\sigma \qquad (\sigma \in \Gamma(L)), \tag{4.2}$$

and the operator $\hat{f}$ of (4.2) is called the **prequantization** of f. Iff the curvature form Ω of ∇ satisfies the condition

$$\Omega(X_f, X_g) = -2\pi\sqrt{-1}\omega(X_f, X_g) \qquad (f, g \in \mathcal{F}(M)), \tag{4.3}$$

where ω is the symplectic form of M, the prequantization operators satisfy the celebrated Dirac commutation condition

$$\widehat{\{f, g\}} = [\hat{f}, \hat{g}] := \hat{f} \circ \hat{g} - \hat{g} \circ \hat{f}. \tag{4.4}$$

Then, if L is tensorized by the line bundle D of half-densities (or half-forms) on M, the Hermitian metric h yields a pre-Hilbert scalar product on the space $\Gamma_c(L\otimes D)$ of the cross sections with compact support of the tensor product $L\otimes D$, by integration along M. The new **prequantization operators** $\check{f} := \hat{f}\otimes Id + Id\otimes L_{X_f}$ are anti-Hermitian with respect to this product (e.g., [**42**] or the brief survey [**36**]).

In the case of a Nambu-Poisson manifold (M^m, P) of order n, a Hamiltonian vector field is defined by $n-1$ functions $f_{(n-1)} \in \mathcal{F}(M)$. Since $\sharp_P$ as defined by (2.3) is multilinear, rather than linear, we may follow [**8**], [**28**], and introduce the non associative, non commutative, real algebra $\mathcal{O}(M) = \wedge^{n-1}_{\mathbf{R}}\mathcal{F}$ with the product

$$f_{(n-1)} \times_{\mathcal{O}} g_{(n-1)} = \sum_{k=1}^{n-1} g_1\wedge\ldots\wedge g_{k-1}\wedge X_{f_{(n-1)}}g_k\wedge g_{k+1}\wedge\ldots\wedge g_{n-1}. \tag{4.5}$$

Then, taking the Hamiltonian vector field extends to a $\mathbf{R}$-linear mapping $\mathrm{ham} : \mathcal{O}(M)\to\mathcal{H}(P)$ (also denoted by $\mathrm{ham}\,(A) = X_A$, $A\in\mathcal{O}(M)$), where the Lie algebra $\mathcal{H}(P)$ is that defined in Section 2. Furthermore, in view of (2.6), we have

$$\mathrm{ham}\,(f_{(n-1)} \times_{\mathcal{O}} g_{(n-1)}) = [\mathrm{ham}\,(f_{(n-1)}), \mathrm{ham}\,(g_{(n-1)})], \tag{4.6}$$

and from (4.6) we get

$$X_{(f_{(n-1)}\times_{\mathcal{O}} g_{(n-1)} + g_{(n-1)}\times_{\mathcal{O}} f_{(n-1)})} = 0.$$

Accordingly, if we agree to say that $A\in\mathcal{O}(M)$ is a **Casimir "function"** of P if $X_A = 0$, it follows that the bracket (4.5) induces a bracket on $\mathcal{S}(M) := \mathcal{O}(M)/\{\text{Casimir "functions"}\}$ which makes $\mathcal{S}(M)$ into a Lie algebra isomorphic to $\mathcal{H}(P)$ [**28**].

Since $\times_{\mathcal{O}}$ is not skew symmetric, we consider the bracket

$$[A,B]_{\mathcal{O}} := \frac{1}{2}(A\times_{\mathcal{O}} B - B\times_{\mathcal{O}} A),$$

and we will say that $(\mathcal{O}(M), [\ \ ]_{\mathcal{O}})$ is the **algebra of the multi-observables** of the Nambu-Poisson manifold (M,P). It may be seen as a central extension of $\mathcal{H}(P)$ by the Casimir "functions" of P. For $n=2$, this is just the Poisson algebra $(\mathcal{F}(M), \{\})$, and the described construction generalizes the situation which exists in symplectic and Poisson geometry.

In spite of the fact that $(\mathcal{O}(M), [\ \ ]_{\mathcal{O}})$ is not a Lie algebra for $n>2$, it is handy to use the terminology of Lie algebra theory whenever the definitions there naturally extend to our situation. In particular, via the mapping ham, $\mathcal{O}(M)$ has a representation on $\mathcal{F}(M)$, and we may speak of $\mathcal{F}(M)$-valued cochains and their coboundary ∂, where ∂^2 is not necessarily 0.

Now, any 1-cochain $Q\colon \mathcal{O}(M) \to \mathcal{F}(M)$ allows us to define the prequantization of the observable $A \in \mathcal{O}(M)$ as being the operator

$$\hat{A}(\sigma) := \nabla_{X_A}\sigma + 2\pi\sqrt{-1}Q(A)\sigma, \tag{4.7}$$

where ∇ and σ are as in formula (4.2).

The geometric meaning of $\hat{A}$ on the principal bundle L^* is similar to that of $\hat{f}$ of (4.1), and we get

4.1. Theorem. *Let (M, P) be a Nambu-Poisson manifold, and let Q be a 1-cochain of $\mathcal{O}(M)$. Then, if ∂Q satisfies the condition*

$$\begin{aligned}(\partial Q)(A, B) &:= X_A(Q(B)) - X_B(Q(A)) - Q([A, B]_{\mathcal{O}}) \\ &= -2\pi\sqrt{-1}\lambda(X_A, X_B) \quad (\forall A, B \in \mathcal{O}(M)),\end{aligned} \tag{4.8}$$

for some closed 2-form λ which represents an integral cohomology class of M, then there exists a complex line bundle L on M, endowed with a Hermitian metric and connection, such that the operators (4.7) *satisfy the Dirac commutation condition*

$$\widehat{[A, B]}_{\mathcal{O}} = [\hat{A}, \hat{B}] := \hat{A} \circ \hat{B} - \hat{B} \circ \hat{A}. \tag{4.9}$$

Conversely, if such a bundle exists, Q satisfies the condition (4.8).

Proof. For an arbitrary L and ∇ as at the beginning of this section, formula (4.7) leads to the following commutation relation:

$$\begin{aligned}\widehat{[A, B]}_{\mathcal{O}} = \hat{A} \circ \hat{B} - \hat{B} \circ \hat{A} &+ 2\pi\sqrt{-1}((\partial Q)(A, B) \\ &+ \frac{1}{2\pi\sqrt{-1}}\Omega(X_A, X_B)),\end{aligned} \tag{4.10}$$

where Ω is the curvature 2-form of ∇. Hence, if (4.9) holds, we have (4.8) for $\lambda = \Omega$. This is the last assertion of the theorem. The first part follows from (4.10) again. Indeed, if we have (4.8) with the integral form λ, it is well known that there exists a bundle L with a Hermitian connection ∇ such that $2\pi\sqrt{-1}\lambda$ is the curvature of ∇ (e.g., [**21**]). Using these L and ∇, we get the desired result. □

4.2. Remark. If P is regular on an open, dense subset N of M, it is enough to quantize the restriction of the multi-observables to N. Thus, we might concentrate on the study of the quantization of regular Nambu-Poisson manifolds M, which have the simple structure described in Theorem 2.6. (For $n = 2$, this structure is not so simple, however.) Then, for geometric quantization, it suffices to use only connections and forms along the leaves of the canonical foliation S of P, and replace Theorem 4.1 by the S-leafwise version of the same theorem.

For the clarification of this remark, see the case of the Poisson manifolds in [**37**].

4.3. Remark. The prequantization operators (4.7) act on the complex, linear space $\Gamma(L)$. But, it is again possible to tensorize by the halfdensities, and get anti-Hermitian operators on a pre-Hilbert space $\Gamma(L \otimes D)$ as described earlier for the classical case.

A cochain Q which satisfies the hypothesis of Theorem 4.1, or of its leafwise version, will be called a **quantifier** of the Nambu-Poisson manifold (M, P). The prequantization problem reduces to that of finding good quantifiers but, we have no method to find them. For $n = 2$, the tautological quantifier $Q(f) = f$ $(f \in \mathcal{F})$ leads to the classical geometric quantization. For $n \geq 2$,

$$Q(A) = \alpha(X_A) \qquad (A \in \mathcal{O}(M)), \tag{4.11}$$

where α is a 1-form on M, defines a quantifier. For it, we have $(\partial Q)(A, B) = d\alpha(X_A, X_B)$, and we may use the trivial bundle L with the connection defined by the connection form $-2\pi\sqrt{-1}\alpha$ as a prequantization bundle. This yields $\hat{A} = X_A$, which is a trivial quantization, while what we need is a non trivial quantization.

It is to be noted that if Q is a 1-cocycle i.e., $\partial Q = 0$, we obtain a prequantization which satisfies the Dirac condition on the trivial complex line bundle over M.

Following is an exaple of a 1-cochain on $\mathcal{O}(M)$ which shows the basic difficulty in finding a quantifier. Namely, let $Y_1, \dots, Y_{n-2}$ be arbitrary vector fields on M, and put

$$Q(f_{(n-1)}) = \det(f_{(n-1)}, Y_1 f_{(n-1)}, \dots, Y_{n-2} f_{(n-1)}), \tag{4.12}$$

where the $(n-1)$-dimensional vectors included are the columns of the determinant. Then, the properties of a determinant show that Q extends to a well defined 1-cochain of $\mathcal{O}$, and we get

$$\begin{aligned}(\partial Q)(f_{(n-1)}, g_{(n-1)}) = \sum_{k=1}^{n-1} (-1)^k [&\det(U_g; f_{(n-1,\hat{k})}, Y_1 f_{(n-1,\hat{k})}, \dots, Y_{n-2} f_{(n-1,\hat{k})}) \\ &- \det(U_f; g_{(n-1,\hat{k})}, Y_1 g_{(n-1,\hat{k})}, \dots, Y_{n-2} g_{(n-1,\hat{k})})] + Q([f_{(n-1)}, g_{(n-1)}]_{\mathcal{O}}),\end{aligned}$$

where, U_f is the operation of adding at the top of each column of the remaining matrix 0 on the first column, and $[X_{f_{(n-1)}}, Y_{k-1}]g_k$ on the k^{th} column, and U_g is similar but with the roles of f and g interchanged. The 1-cochain Q of (4.12) is a generalization of the tautological quantifier of the Poisson case but, it is not a quantifier for $n \geq 3$ since $(\partial Q)(f_{(n-1)}, g_{(n-1)})$ depends on the functions and not just on the corresponding Hamiltonian vector fields.

One possible way to avoid this difficulty is restrict prequantization to a subalgeba of $\mathcal{O}(M)$, in the spirit of the second step, quantization, in classical geometric quantization theory. For instance, let $\mathcal{S}$ be the subalgebra of the elements

$A \in \mathcal{O}(M)$ such that $[Y_i, X_A] = 0$, $\forall i = 1, \dots, n-2$, and let $\mathcal{C}$ be an Abelian subalgebra of $\mathcal{S}$. Then, the expression of ∂Q given above shows that the restriction of Q to $\mathcal{C}$ is a cocycle on this latter subalgebra, and $Q/_{\mathcal{C}}$ allows us to do geometric prequantization on the trivial complex line bundle. A second way out of the mentioned difficulty would be to conveniently change the definition of the bracket $[\,]_{\mathcal{O}}$.

Now, let us refer to deformation quantization. It was shown by Dito, Flato, Sternheimer and Takhtajan [**9**], [**10**] that the deformation quantization of Nambu-Poisson brackets of order $n \geq 3$ should be done via a preliminary **Abelian deformation** of the usual product of functions. Again, it suffices to study only regular Nambu-Poisson brackets i.e., brackets defined by a Jacobian determinant (see Section 2). The basic remark [**10**] is that a Jacobian determinant defines a Nambu-Poisson bracket because the usual product of functions satisfies the following properties: a) associativity, b) commutativity, c) distributivity, d) the Leibniz rule of derivation. Hence, any deformation of the usual product which continues to satisfy a), b), c), d) allows us to define a deformed Jacobian which is a Nambu-Poisson bracket on the deformed algebra of C^∞-functions on M. (In [**31**] the authors claim the non-existence of a Nambu-Poisson deformation quantization on $C^\infty(M)$ itself.)

In a different formulation, let (M, P) be a regular Nambu-Poisson manifold of order $n \geq 3$, which has the bracket defined by formula (2.22). Assume that there exists an embedding of complex, linear spaces

$$\iota : \mathcal{F}(M, \mathbf{C}) \to \mathcal{A}_\nu := \mathcal{F}(M, \mathbf{C})[[\nu]], \tag{4.13}$$

where $\mathcal{F}(M, \mathbf{C})$ is the algebra of complex valued, differentiable functions on M, ν is a parameter, and $\mathcal{A}_\nu$ is the linear space of formal power series, endowed with a product $*_\nu$ which makes it an associative, commutative algebra. The product $*_\nu$ of $\mathcal{A}_\nu$ is called an **Abelian product deformation**, and $\forall f, g \in \mathcal{F}(M, \mathbf{C})$ one defines the **star product** $f *_\nu g := \iota(f) *_\nu \iota(g)$. Assume also that the Lie algebra $\chi(M)$ of the vector fields on M has a representation ρ by derivations of $(\mathcal{A}_\nu, *_\nu)$ Then, we can define $\mathcal{A}_\nu$-valued forms, and their $*_\nu$-exterior product and ρ-exterior differential d_ρ by extending the classical definitions. Now, if we write (2.22) for these new operations, we get

$$d_{\rho S}(\iota f_1) \wedge \ldots \wedge d_{\rho S}(\iota f_n) = \{f_1, \dots, f_n\}_\nu(\iota\omega), \tag{4.14}$$

where $\iota\omega$ is defined by

$$(\iota\omega)(X_1, \dots, X_n) = \iota(\omega(X_1, \dots, X_n)).$$

The bracket $\{f_1, \dots, f_n\}_\nu$ is the **quantum deformation** of $\{f_1, \dots, f_n\}$, and it satisfies the properties of a Nambu-Poisson bracket (i.e., (i), (ii), (iii) of Section 2).

In [**9**], [**10**], the authors propose a construction of an Abelian product $*_\nu$, which leads to a quantum deformation of a Nambu-Poisson bracket called **Zariski quantization**. For this theory we refer the reader to the quoted original papers.

Here, we modify a construction used in symplectic deformation quantization [**4**], [**12**] in order to get a deformation of the Nambu-Poisson bracket if the algebra $\mathcal{F}(M, \mathbf{C})$ is embedded into a larger algebra $\tilde{\mathcal{F}}(M, \mathbf{C})$ first, and the space $\tilde{\mathcal{A}}_\nu := \tilde{\mathcal{F}}(M, \mathbf{C})[[\nu]]$ of formal power series is used. This construction is not an answer to the deformation quantization problem since the obtained $*_\nu$-product of functions is a power series with coefficients which may not be functions. It is an example of a general, commutative, product deformation process, associated with a fixed Riemannian metric g on the Nambu-Poisson manifold (M, P).

Let us introduce the associative, commutative algebra

$$\tilde{\mathcal{F}}(M, \mathbf{C}) = \oplus_{i=0}^{\infty} \Gamma \odot^i T_c^* M, \tag{4.15}$$

where $\Gamma \odot^i T_c^* M$ is the space of symmetric, i-covariant, complex tensor fields, any particular element of $\tilde{\mathcal{F}}(M, \mathbf{C})$ consists of a finite sum of terms, and the product in the algebra (4.15) is the symmetric tensor product $\odot$. As in [**4**], [**12**], we define a **Weyl-Moyal product** of power series

$$a_u = \sum_{k=0}^{\infty} \sum_i \nu^k a^i_{(u)k} \in \tilde{\mathcal{A}}_\nu = \tilde{\mathcal{F}}(M, \mathbf{C})[[\nu]] \quad (u = 1, 2) \tag{4.16}$$

by the formula

$$a_1 *_\nu a_2 = \sum_{p=0}^{\infty} \frac{\nu^p}{p!} (\partial^p a_1, \partial^p a_2)_g, \tag{4.17}$$

where the **algebraic derivative** ∂ is defined on each term of the series (4.16) as the operator $\partial \colon \odot^i T_c^* M \to \mathrm{Hom}\,(\odot^{i-1} T_c^* M, T_c^* M)$ given by

$$(\partial t)(X_1, \ldots, X_{i-1})(Y) := t(Y, X_1, \ldots, X_{i-1}),$$

$t \in \odot^i T_c^* M$, and all the arguments are tangent vectors. Of course, ∂^p is the iteration of ∂. Finally, $(\ ,\ )_g$ is the scalar product induced by g. (In the symplectic case, there was a symplectic scalar product instead.) Here, the symmetry of g ensures that formula (4.17) defines the structure of an associative, commutative algebra on $\tilde{\mathcal{A}}_\nu$.

Furthermore, the action of any vector field X on M as a directional derivative of functions extends to $\tilde{\mathcal{A}}_\nu$ by means of the covariant derivative ∇_X of the tensor fields with respect to the Riemannian connection of g. This action is a representation ρ by derivations. Accordingly, the Nambu-Poisson bracket P gets deformed to a Nambu-Poisson bracket on $\tilde{\mathcal{A}}_\nu$.

Now, we have to consider an embedding $\iota\colon \mathcal{F}(M,\mathbf{C}) \to \tilde{\mathcal{A}}_\nu$ e.g., the **gradiental deformation**

$$\iota(f) = f + \sum_{i=1}^{\infty} \frac{\nu^i}{i!}(\odot^i df) \quad (f \in \mathcal{F}(M,\mathbf{C})), \tag{4.18}$$

then put

$$f *_\nu k := (\iota f) *_\nu (\iota k) \qquad (f,k \in \mathcal{F}(M,\mathbf{C})),$$

as given by (4.17).

Then, we might look at the "semi-classical approximation" i.e., take only the term $i = 1$ in (4.18). This yields

$$f *_\nu k = fk + \nu(fdk + kdf) + \nu^2(df \odot dk) + \nu^3(df,dk)_g. \tag{4.19}$$

The result is a polynomial deformation of the product which has symmetric tensor fields as coefficients. This product is commutative, and associative, since it is a restriction of $*_\nu$. Then, if we define $\{f_1,\dots,f_n\}_\nu$ by formula (4.14) interpreted on $\tilde{\mathcal{A}}_\nu$, we get a polynomial deformation of the P-bracket of functions $\{f_1,\dots,f_n\}$, with symmetric tensor fields as coefficients, which satisfies all the axioms of a Nambu-Poisson bracket.

If, instead of (4.19), we take the star product

$$f *_\lambda k := fk + \lambda(df,dk)_g, \tag{4.20}$$

where $\lambda = \nu^3$ is the new deformation parameter, this product is also commutative, but it is associative only in the semi-classical approximation i.e., up to terms in λ^k with $k \geq 2$. Indeed, (4.20) implies

$$\begin{aligned}(f *_\lambda k) *_\lambda l = fkl &+ \lambda[f(dk,dl)_g + k(dl,df)_g \\ &+ l(df,dk)_g] + \lambda^2(dl, d(df,dk)_g)_g,\end{aligned} \tag{4.21}$$

which justifies the previous assertion.

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