Acta Math. Univ. Comenianae
Vol. LXIX, 1(2000), pp. 59–70

COMPARING A CAYLEY DIGRAPH WITH ITS REVERSE

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Abstract. A Cayley digraph $G = C(\Gamma, X)$ for a group Γ and a generating set X is the digraph with vertex set $V(G) = \Gamma$ and arcs (g, gx) where $g \in \Gamma$ and $x \in X$. The **reverse** of $C(\Gamma, X)$ is the Cayley digraph $G^{-1} = C(\Gamma, X^{-1})$ where $X^{-1} = \{x^{-1}; x \in X\}$. We are interested in sufficient conditions for a Cayley digraph not to be isomorphic to its reverse and focus on Cayley digraphs of metacyclic groups with small generating sets.

1. Introduction

Let Γ be a finite group and let X be a generating set for Γ; we assume that the unit element of the group is not in X. The **Cayley digraph** $G = C(\Gamma, X)$ is a digraph with vertex set $V(G) = \Gamma$ and arc set $D(G) = \{(g, gx); g \in \Gamma, x \in X\}$. If, in addition, the set X is closed under taking inverses (i.e., if $x \in X$ implies $x^{-1} \in X$) then the two arcs (g, gx) and (gx, g) are usually identified to form a single undirected edge, which turns the digraph into an (undirected) Cayley graph.

Cayley digraphs are automatically vertex-transitive. In fact, for each fixed $h \in \Gamma$ the mapping $\theta_h \colon V(G) \to V(G)$ given by $\theta_h(g) = hg$ is an automorphism of the Cayley digraph $C(\Gamma, X)$. Thus, the (full) automorphism group of a Cayley digraph contains a subgroup acting regularly on its vertex set. The converse is true as well: By the (digraph modification of) Sabidussi's Theorem [**7**], if the automorphism group of digraph contains a subgroup acting regularly on its vertex set, then the digraph is necessarily a Cayley digraph.

In both theory and applications, Cayley graphs and digraphs play an increasingly important role [**4**]. Among the variety of research directions here, in the last few years there has been a considerable activity in the study of isomorphism of Cayley (di)graphs. The basic motivation is provided by following simple observation: If $C(\Gamma, X)$ and $C(\Gamma, Y)$ are two Cayley digraphs for the same group and if there is an automorphism σ of the group Γ such that $\sigma(X) = Y$, then σ naturally extends to an isomorphism of the two Cayley digraphs. The important **Cayley-Isomorphism** problem is to characterize the groups Γ for which **every** isomorphism of two Cayley digraphs $C(\Gamma, X)$ and $C(\Gamma, Y)$ is induced by a group

Received December 18, 1998.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 05C25.
Key words and phrases. Cayley digraph, digraph isomorphism.

automorphism in the above sense. For substantial results in this area we refer to [**1**], [**5**], [**6**].

Cayley digraphs have been successfully used in constructing large digraphs of given (comparatively small) degree and diameter [**3**]. In a detailed study of these digraphs [**2**] it was observed that some of the large Cayley digraphs G described in [**3**] have the following interesting property: The digraph G^{-1} obtained from G by reversing the direction of each arc is isomorphic to G. For obvious reasons we call the digraph G^{-1} the **reverse** of G. In the Cayley digraph setting, the reverse of $G = C(\Gamma, X)$ is the digraph $G^{-1} = C(\Gamma, X^{-1})$ where $X^{-1} = \{x^{-1}; x \in X\}$. The interesting question thus is: Which Cayley digraphs are isomorphic to their reverse? Does the isomorphism occur frequently or is it rather rare? In which Cayley digraphs is such an isomorphism induced by a group automorphism of Γ which takes X to X^{-1}?

The answers to the above questions are easy in the case of Cayley digraphs of Abelian groups. Indeed, if A is an Abelian group then the map $\phi\colon A \to A$ such that $\phi(a) = a^{-1}$ for each element $a \in A$ is a group automorphism which induces a digraph isomorphism $C(A, X) \cong C(A, X^{-1})$. In this note we therefore focus on metacyclic groups Γ (which are semidirect products of cyclic groups, thus, in some sense, "simplest" nonabelian groups) and show that there are only few groups for which $G = C(\Gamma, X)$ is isomorphic to $G^{-1} = C(\Gamma, X^{-1})$ for each generating set X.

2. Metacyclic Groups

Throughout this paper we use the following notation. For an arc (a, b) where $b = ax$ we shall often use the alternative notation $a \xrightarrow{x} b$ to emphasize that one can pass from the vertex a to the vertex b along an arc using the generator x. We reserve the symbol e for the unit element of a group.

Metacyclic groups are defined as semidirect products of cyclic groups and they have the following standard presentation:

$$Z_n \rtimes_k Z_m = \langle a, b \mid a^n = b^m = e, ba = a^k b\rangle,$$

where $\gcd(n, k) = 1$, $1 < k < n$ and $k^m \equiv 1 \pmod{n}$.

Note that for each $i \pmod{n}$ and $j \pmod{m}$ we have

$$b^j a^i = a^{ik^j} b^j. \tag{1}$$

Equivalently, metacyclic groups are split extensions of Z_n by Z_m under the group homomorphism $\theta\colon Z_m \to \mathrm{Aut}(Z_n)$ which sends each $i \in Z_m$ to the automorphism $\theta_i \in \mathrm{Aut}(Z_n)$ where $\theta_i(s) = k^i s, s \in Z_n$. In this setting the group multiplication is given by $(r, i)(s, j) = (r + \theta_i(s), i + j)$.

Let $x \in X$ be a generator of order t in Γ. In what follows, the cycle $(g, gx, \ldots, gx^{t-1})$ in the digraph $C(\Gamma, X)$ will be denoted simply by $C_x(g)$; the corresponding cycle $(g, gx^{-1}, \ldots, gx^{1-t})$ in the reverse digraph will be denoted by $C^{-1}_{x^{-1}}(g)$.

3. Reverses of Cayley digraphs of Valence 2

In this section we focus on comparing Cayley digraphs $C(\Gamma, X)$ with $C(\Gamma, X^{-1})$ such that $|X| = 2$.

Lemma 1. *Let $G = C(\Gamma, X)$ be a Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_m$ ($n \neq m$) and the generating set $X = \{a, b\}$. If ϕ is an isomorphism $\phi\colon G \to G^{-1}$ such that $\phi(e) = e$, then $\phi(a^i b^j) = a^{-i} b^{-j}$.*

Proof. We consider only the case $n > m$; the argument for $n < m$ is similar. An easy inspection shows that for every vertex g of the digraph G (G^{-1}) there is only one cycle of length m passing through g, namely $C_b(g)$ ($C^{-1}_{b^{-1}}(g)$). As $\phi(e) = e$, ϕ must map the cycle $C_b(e)$ onto the cycle $C^{-1}_{b^{-1}}(e)$. Therefore $\phi(b^i) = b^{-i}$. Now, the only possibility for the image of a is $\phi(a) = a^{-1}$. From this it follows that $\phi(ab^i) = a^{-1}b^{-i}$ and the only possibility for a^2 is $\phi(a^2) = a^{-2}$. Continuing in this manner we eventually obtain that $\phi(a^i b^j) = a^{-i} b^{-j}$ and the result follows. □

Lemma 2. *Let $G = C(\Gamma, X)$ be a Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_m$ ($n \neq m$) and the generating set $X = \{a, b\}$. If m is odd, or if m is even and $k^2 \not\equiv 1 \pmod n$, then $G \ncong G^{-1}$.*

Proof. Assume the contrary and let $\phi\colon G \to G^{-1}$ be a digraph isomorphism. Because every Cayley digraph is vertex-transitive, without loss of generality we may assume that $\phi(e) = e$. According to Lemma 1, this isomorphism must have the form $\phi(a^i b^j) = a^{-i} b^{-j}$. Then, for an arbitrary vertex $a^i b^j$ of G, the two arcs emanating from this vertex are

$$a^i b^j \xrightarrow{a} a^i b^j a = a^{i+k^j} b^j,$$

and

$$a^i b^j \xrightarrow{b} a^i b^j b = a^i b^{j+1}.$$

Similarly, the two arcs of G^{-1} emanating from the vertex $a^{-i} b^{-j}$ are

$$a^{-i} b^{-j} \xrightarrow{a^{-1}} a^{-i} b^{-j} a^{-1} = a^{-i-k^{-j}} b^{-j},$$

and

$$a^{-i} b^{-j} \xrightarrow{b^{-1}} a^{-i} b^{-j} b^{-1} = a^{-i} b^{-j-1}.$$

Since $\phi(a^i b^j) = a^{-i} b^{-j}$, it follows that

$$\{\phi(a^{i+k^j} b^j), \phi(a^i b^{j+1})\} = \{a^{-i-k^j} b^{-j}, a^{-i} b^{-j-1}\}.$$

On the other hand, evaluating ϕ turns the preceding equality into the following one:

$$\{a^{-i-k^j} b^{-j}, a^{-i} b^{-j-1}\} = \{a^{-i-k^{-j}} b^{-j}, a^{-i} b^{-j-1}\}.$$

Now, the two sets above are equal if and only if

$$a^{-i-k^j}b^{-j} = a^{-i-k^{-j}}b^{-j}.$$

Thus,

$$-k^j \equiv -k^{-j} \pmod{n}$$

for each j. This implies that $k^{2j} \equiv 1 \pmod{n}$, and for $j = 1$ we have $k^2 \equiv 1 \pmod{n}$. If m is even we already have a contradiction with our assumptions. If $m = 2r+1$ then our last congruence implies that $k^m = k^{2r+1} \equiv k \pmod{n}$. But by our description of the semidirect product we have $k^m \equiv 1 \pmod{n}$; since $k \equiv 1 \pmod{n}$, we arrive at a contradiction again. □

Lemma 3. *Let G be a Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_n$ and the generating set $X = \{a, b\}$. If n is odd, or if n is even and $k^2 \not\equiv 1 \pmod{n}$, then $G \not\cong G^{-1}$.*

Proof. Assume the contrary, again, and let $\phi\colon G \to G^{-1}$ be a digraph isomorphism; we may assume that $\phi(e) = e$. In this case, through any vertex g there are only two vertex-disjoint (up to the vertex g) cycles of length n, namely

$$C_a(g) \quad \text{and} \quad C_b(g)$$

in the digraph G and

$$C^{-1}_{a^{-1}}(g) \quad \text{and} \quad C^{-1}_{b^{-1}}(g)$$

in the digraph G^{-1}. It is easy to verify that a possible isomorphism $\phi(e) = e$ must satisfy one of the following conditions:

a) $\phi(a^i b^j) = a^{-i}b^{-j}$ or
b) $\phi(a^i b^j) = b^{-i}a^{-j}$.

In the case a) we have a situation similar to the case when $n \neq m$. In the case b) let us consider the two arcs

$$a^i b^j \xrightarrow{a} a^i b^j a = a^{i+k^j}b^j,$$

and

$$a^i b^j \xrightarrow{b} a^i b^j b = a^i b^{j+1}$$

in the digraph G. For G^{-1} it then holds that

$$\phi(a^i b^j) = b^{-i}a^{-j} \xrightarrow{a^{-1}} a^{(-j-1)k^{-i}}b^{-i},$$

and

$$\phi(a^i b^j) = b^{-i}a^{-j} \xrightarrow{b^{-1}} a^{-jk^{-i}}b^{-i-1}.$$

Now, using b) and the relations (1) we obtain

$$\phi(a^{i+k^j}b^j) = a^{-jk^{(-i-k^j)}}b^{-i-k^j},$$

and

$$\phi(a^i b^{j+1}) = a^{(-j-1)k^{-i}}b^{-i}.$$

Since ϕ is assumed to be a digraph isomorphism, we have

$$\{\phi(a^{i+k^j}b^j), \phi(a^i b^{j+1})\} = \{a^{-i-k^j}b^{-j}, a^{-i}b^{-j-1}\};$$

therefore (see the proof of Lemma 2)

$$\{a^{(-j-1)k^{-i}}b^{-i}, a^{-jk^{-i}}b^{-i-1}\} = \{a^{-jk^{(-i-k^j)}}b^{-i-k^j}, a^{(-j-1)k^{-i}}b^{-i}\};$$

in particular,

$$a^{-jk^{-i}}b^{-i-1} = a^{-jk^{(-i-k^j)}}b^{-i-k^j}$$

for all i, j. This implies that $k^j \equiv 1 \pmod{n}$ for all j. Setting $j = 1$ we obtain $k^j = k^1 = k \equiv 1 \pmod{n}$, a contradiction. $\square$

Note that in the case $k^2 \equiv 1 \pmod{n}$ in Lemma 2 and 3, the mapping $\phi\colon G \to G^{-1}$ such that $\phi(a^i b^j) = a^{-i}b^{-j}$ is a digraph isomorphism.

4. Reverses of Cayley Digraphs of Valence 4

In the next part of this paper we will consider the case when m is even and $k^2 \equiv 1 \pmod{n}$ only. The reason for continuing with the case of four generators will be clear from the conclusion of this section. The following three Lemmas will serve as auxiliary results for proving Lemma 7.

Lemma 4. *Let G be a Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_m$ $(n \neq m)$ with the generating set $X = \{a, b, ba^{i_1}, ba^{i_2}\}$, where i_1, $i_2 \neq 0$, $i_1 \not\equiv i_2 \pmod{n}$, and let $n \geq 6$. Let there exist an isomorphism $\phi\colon G \to G^{-1}$ such that $\phi(e) = e$. Then, fore some p, we have $\{b^{-1}a^p, b^{-1}a^{-i_1+p}, b^{-1}a^{-i_2+p}\} = \{b^{-1}, b^{-1}a^{-i_1k}, b^{-1}a^{-i_2k}\}$.*

Proof. i) Let $n > m$. Because the generators b, ba^{i_1}, ba^{i_2} all have order m, each vertex $g \in G$ is contained in exactly three cycles of length m (which is the shortest cycle length in G), namely

$$C_{y_i}(g) = (g, gy_i, gy_i{}^2, \ldots, gy_i{}^{m-1}, gy_i{}^m = g),$$

where $y_1 = b, y_2 = ba^{i_1}, y_3 = ba^{i_2}$. The situation is similar in the digraph G^{-1}, with y_i^{-1} in place of y_i. Because isomorphism preserves cycle lengths, all arcs of the form (g, gy_i) must be mapped by ϕ onto arcs of the form $(g', g'y_j^{-1})$. Since $\phi(e) = e$, the isomorphism ϕ has to map a^i to a^{-i}, and this implies that $\phi(b^j a^i) = b^{-j} a^{-i+p_j}$, where $0 \le p_j \le n-1$.

Except for the vertex a, the neighbourhood of the vertex e in the digraph G is the set of vertices $\{b, ba^{i_1}, ba^{i_2}\}$, and it is mapped onto the set $\{b^{-1}a^p, b^{-1}a^{-i_1+p}, b^{-1}a^{-i_2+p}\}$ by the isomorphism ϕ.

Similarly, except for the vertex a^{-1}, the neighbourhood of the vertex e in the digraph G^{-1} is the set of vertices $\{b^{-1}, (ba^{i_1})^{-1} = b^{-1}a^{-i_1 k}, (ba^{i_2})^{-1} = b^{-1}a^{-i_2 k}\}$.

ii) Let $n < m$. By similar arguments as above, the isomorphism ϕ must satisfy $\phi(a^i) = a^{-i}$ and $\phi(b^j a^i) = b^{-j} a^{-i+p_j}$. □

Lemma 5. *Let G be a Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_n$ with generating set $X = \{a, b, ba^{i_1}, ba^{i_2}\}$, where $i_1,\ i_2 \neq 0$, $i_1 \not\equiv i_2 \pmod n$, and let $n \ge 6$. If $\phi\colon G \to G^{-1}$ is an isomorphism such that $\phi(e) = e$ then $\phi(a^i) = a^{-i}$.*

Proof. Considering the three cycles $C_b(e)$, $C_{ba^{i_1}}(e)$, $C_{ba^{i_2}}(e)$, we have eight cases:

1) $|C_b(e) \cap C_{ba^{i_1}}(e)| = 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| = 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| = 1$,
2) $|C_b(e) \cap C_{ba^{i_1}}(e)| = 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| = 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| > 1$,
3) $|C_b(e) \cap C_{ba^{i_1}}(e)| = 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| > 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| = 1$,
4) $|C_b(e) \cap C_{ba^{i_1}}(e)| > 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| = 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| = 1$,
5) $|C_b(e) \cap C_{ba^{i_1}}(e)| = 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| > 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| > 1$,
6) $|C_b(e) \cap C_{ba^{i_1}}(e)| > 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| = 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| > 1$,
7) $|C_b(e) \cap C_{ba^{i_1}}(e)| > 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| > 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| = 1$,
8) $|C_b(e) \cap C_{ba^{i_1}}(e)| > 1 \wedge |C_b(e) \cap C_{ba^{i_2}}(e)| > 1 \wedge |C_{ba^{i_1}}(e) \cap C_{ba^{i_2}}(e)| > 1$.

The cases 5), 6), 7), 8) are quite easy. For example, in the case 5) the cycle $C_a(e)$ is vertex-disjoint (except vertex e) from the cycles $C_b(e), C_{ba^{i_1}}(e)$ and $C_{ba^{i_2}}(e)$. The image of the cycle $C_a(e)$ has the same property. But $|C_x(e) \cap C_y(e)| \ge 2$, where $x, y \in \{b, ba^{i_1}, ba^{i_2}\}$ for some $x \neq y$. So $\phi(a^i) = a^{-i}$.

The remaining cases are 1) and 2). (The cases 3) and 4) are similar to the case 2).) In the case 1), the set $\{a, b, ba^{i_1}, ba^{i_2}\}$ is equal to the set $\{a, b', b'a^{i'_1}, b'a^{i'_2}\}$ for $b' = ba^{i_1}$ and some i'; without loss of generality we may assume the contrary and let $\phi(a^i) = b^{-i}$. There are three arcs emanating from e and terminating in the cycle $C^{-1}_{a^{-1}}(b^{-1}) = (b^{-1}, b^{-1}a^{-1}, b^{-1}a^{-2}, \ldots, b^{-1}a)$ in the digraph G^{-1}. Similarly there are three arcs emanating from e and terminating in the cycle $C_y(a) = (a, ay, ay^2, \ldots, ay^{-1})$, where $y \in \{b, ba^{i_1}, ba^{i_2}\}$ in the digraph $\phi(G)$. These three arcs must terminate in the vertices a, ba^{i_1}, ba^{i_2}. But the cycle $(a, ay, ay^2, \ldots, ay^{-1})$ has the form $(a, ba^{r_1}, b^2a^{r_2}, \ldots, b^{n-1}a^{r_{n-1}}, a)$ and hence it is impossible to find there two elements of the form $b^1 a^{r_j}$.

As regards the case 2), the considerations are similar to the case 1). □

Lemma 6. *Let G be the Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_n$ with generating set $X = \{a, b, ba^{i_1}, ba^{i_2}\}$, where $i_1, i_2 \neq 0, i_1 \not\equiv i_2 \pmod n$, and let*

$n \geq 6$. If $\phi\colon G \to G^{-1}$ is an isomorphism such that $\phi(a^i) = a^{-i}$ then, for some p,

$$\{b^{-1}a^p, b^{-1}a^{-i_1+p}, b^{-1}a^{-i_2+p}\} = \{b^{-1}, b^{-1}a^{-i_1k}, b^{-1}a^{-i_2k}\}. \tag{2}$$

Proof. There are just three arcs emanating from the vertex e and terminating in the cycle of the form $(b, ba, ba^2, \dots, b)$ in the digraph G. Similarly, there are just three arcs emanating from the vertex e and terminating in the cycle of the form $(b^{-1}, b^{-1}a^{-1}, b^{-1}a^{-2}, \dots, b^{-1})$ in the digraph G^{-1}. Because $\phi(e) = e$ it holds that $\phi(\{b\langle a\rangle\}) = \{b^{-1}\langle a\rangle\}$, i.e.: the cycle $(b, ba, ba^2, \dots, b)$ is mapped onto the cycle $(b^{-1}, b^{-1}a^{-1}, b^{-1}a^{-2}, \dots, b^{-1})$ by the isomorphism ϕ. In the cycle $(b, ba, ba^2, \dots, b)$ the distance from the vertex ba^i to the vertex ba^j is $d(ba^i, ba^j) \equiv j - i \pmod{n}$. Applying the isomorphism ϕ, we have $d(\phi(ba^i), \phi(ba^j)) \equiv j - i \pmod{n}$, for all i, j. This is true if and only if $\phi(ba^i) = b^{-1}a^{-i+p}$ for some $p \in [0, n-1]$.

Now, there are three arcs emanating from the vertex e and terminating in the set of vertices $\{b^{-1}, (ba^{i_1})^{-1}, (ba^{i_2})^{-1}\} = \{b^{-1}, b^{-1}a^{-i_1k}, b^{-1}a^{-i_2k}\}$ and one arc terminating in the vertex a^{-1}, in the digraph G^{-1}. Similarly, there are three arcs emanating from the vertex e and terminating in the set of vertices $\{b^{-1}, b^{-1}a^{-i_1+p}, b^{-1}a^{-i_2+p}\}$ and one arc terminating in the vertex a^{-1}, in the digraph $\phi(G)$. Comparing the above two sets we obtain (2). □

Summing up the preceding three Lemmas we have:

Lemma 7. *Let G be a Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_m$ with the generating set $X = \{a, b, ba^{i_1}, ba^{i_2}\}$, where $i_1, i_2 \neq 0, i_1 \not\equiv i_2 \pmod{n}$, and let $n \geq 6$. Let there exist an isomorphism $\phi\colon G \to G^{-1}$ such that $\phi(e) = e$. Then, for some p, we have $\{b^{-1}a^p, b^{-1}a^{-i_1+p}, b^{-1}a^{-i_2+p}\} = \{b^{-1}, b^{-1}a^{-i_1k}, b^{-1}a^{-i_2k}\}$.*

Lemma 8. *Let G be a Cayley digraph for the group $\Gamma = Z_n \rtimes_k Z_m$, $n \geq 6$ and $k^2 \equiv 1 \pmod{n}$, $k \neq 1$, with the generating set $X = \{a, b, ba^{i_1}, ba^{i_2}\}$ such that $i_1, i_2 \neq 0, i_1 \not\equiv i_2 \pmod{n}$. Then there exist i_1, i_2 such that $G \not\cong G^{-1}$.*

Proof. Let ϕ be an isomorphism such that $\phi(e) = e$.

i) If $i_1 = 1, i_2 = 3$, then

$$\{b^{-1}a^p, b^{-1}a^{-1+p}, b^{-1}a^{-3+p}\} = \{b^{-1}, b^{-1}a^{-k}, b^{-1}a^{-3k}\},$$

by Lemma 7. We can see that the above is equivalent with

$$\{a^p, a^{-1+p}, a^{-3+p}\} = \{e, a^{-k}, a^{-3k}\}.$$

I) Let $a^p = e$, then $p = 0$ and we have the following two possibilities:
If $a^{-1+p} = a^{-k}$, the $k \equiv 1 \pmod{n}$ and this implies that $k = 1$; a contradiction.
If $a^{-1+p} = a^{-3k}$ then $3k \equiv 1 \pmod{n}$ and at the same time $a^{-3+p} = a^{-k}$; hence $k \equiv 3 \pmod{n}$ and this implies that $k = 3$. Thus $9 \equiv 1 \pmod{n}$ which implies that $n = 8$.

The remaining case to be investigated is $Z_8 \rtimes_3 Z_m$, $ba = a^3b$.

II) Let $a^{-1+p} = e$, then $p = 1$ and the possibilities to be considered are $a^p = a^{-k}$ and $a^p = a^{-3k}$.

Let $a^p = a^{-k}$. Then $k \equiv -1 \pmod n$ that is, $k = n - 1$. Together with $a^{-3+p} = a^{-3k}$ we obtain $3k \equiv 2 \pmod n$. Hence $3(n-1) = 3n - 3 \equiv 2 \pmod n$ and now $5 \equiv 0 \pmod n$ which implies that $n = 5$. But $n \geq 6$; a contradiction.

If $a^p = a^{-3k}$ then $3k \equiv -1 \pmod n$. So $a^{-3+p} = a^{-k}$ and it follows that $k \equiv 2 \pmod n$, thus $k = 2$. Consequently $6 \equiv -1 \pmod n$ which implies that $n = 7$. But $k^2 = 2^2 = 4 \not\equiv 1 \pmod 7$; a contradiction.

III) Let $a^{-3+p} = e$. Then $p = 3$ and we have the following two cases:
If $a^p = a^{-k}$, then $k \equiv -3 \pmod n$. Therefore $k = n - 3$ and so $a^{-1+p} = a^{-3k} \Rightarrow 3k \equiv -2 \pmod n$. From this it follows that $3(n-3) = 3n - 9 \equiv -2 \pmod n$ and accordingly $7 \equiv 0 \pmod n$; so now $n = 7$. But $k = n - 3 = 4$, $k^2 = 4^2 = 16 \not\equiv 1 \pmod 7$; a contradiction.
If $a^p = a^{-3k}$, then $3k \equiv 3 \pmod n$ and at the same time $a^{-1+p} = a^{-k}$. This implies that $k \equiv -2 \pmod n$ and consequently $k = n - 2$. Thus, $3(n-2) = 3n - 6 \equiv 3 \pmod n$ and hence $9 \equiv 0 \pmod n$. From this it follows that $n = 3$ or $n = 9$. Since $n \geq 6$, the only possibility is $n = 9$. But $k = n - 2 = 7$, $k^2 = 7^2 = 49 \not\equiv 1 \pmod 9$; a contradiction.

ii) Let $i_1 = 1$, $i_2 = 4$. We consider this possibility only for the case $Z_8 \rtimes_3 Z_m$, $ba = a^3b$, $n = 8$, $k = 3$. By part i),

$$\{a^p, a^{-1+p}, a^{-4+p}\} = \{e, a^{-k}, a^{-4k}\},$$

thus

$$\{a^p, a^{7+p}, a^{4+p}\} = \{e, a^5, a^4\}.$$

I) Let $a^p = e$. Then $p = 0$ and it follows that $a^{7+p} = a^7$. But $a^7 \neq e, a^7 \neq a^5$ and $a^7 \neq a^4$.
II) Let $a^{7+p} = e$. Then we have $p = 1$ which implies that $a^p = a$. But $a \neq e$, $a \neq a^5$ and $a \neq a^4$.
III) Let $a^{4+p} = e$. Then $p = 4$ and consequently $a^{7+p} = a^3$. As in the above cases, $a^3 \neq e$, $a^3 \neq a^5$ and $a^3 \neq a^4$. □

5. Groups of Type $Z_n \rtimes_k Z_{2m}$ for $n \leq 5$ and Special m

In order to complete our investigation, the remaining cases to be considered are

$$\Gamma_1 = Z_3 \rtimes_2 Z_{2m}, \; ba = a^2b,$$
$$\Gamma_2 = Z_4 \rtimes_3 Z_{2m}, \; ba = a^3b,$$
$$\Gamma_3 = Z_5 \rtimes_4 Z_{2m}, \; ba = a^4b.$$

Lemma 9. *Let* $\Gamma_1 = Z_3 \rtimes_2 Z_{2m}$, $ba = a^2b$, *and let* $2m = 2^r3^sq$; *where* $r \geq 1$, $s \geq 0$, $q > 1$, $\gcd(q,2) = \gcd(q,3) = 1$, *so* $q > 4$. *Let* $2^r3^s = p$. *Then*

$$\Gamma_1 \cong Z_{3q} \rtimes_2 Z_p.$$

Proof. We take the group generated by elements a and b_1, namely $H_1 = \langle a, b_1\rangle$, where $b_1 = b^p$. Then $b_1a = b^pa = a^{2^p}b^p = ab^p = ab_1$, because p is even. Now since $|\langle b_1\rangle| = q$ and $3 \nmid q$ we have $\langle a, b_1\rangle \cong Z_3 \times Z_q \cong Z_{3q}$. We can see that $(a^ib^j)a^{i_1}{b_1}^{j_1}(a^ib^j)^{-1} = a^{i_2}{b_1}^{j_2}$, and so $\langle a, b_1\rangle \lhd \Gamma_1$. Let $b_2 = b^q$, and $\langle b_2\rangle = H_2$. Then $H_1 \cap H_2 = \{e\}$, and $H_1 \cup H_2 = \Gamma_1$.

Accordingly

$$\Gamma_1 = Z_3 \rtimes_2 Z_{pq} \cong Z_{3q} \rtimes_k Z_p. \qquad \square$$

Remark 1. The group $Z_{3q} \rtimes_k Z_p$ has a presentation $Z_{3q} \rtimes_k Z_p = \langle c, d \mid c^{3q} = b^p = e, dc = c^kd\rangle$ where $k^p \equiv 1 \pmod{3q}$ and at the same time $k \equiv 2 \pmod 3$. The isomorphism $\phi\colon Z_3 \rtimes_2 Z_{pq} \to Z_{3q} \rtimes_k Z_p$ has the form $\phi(a) = c^q$ and $\phi(b) = cd$.

Remark 2. By similar arguments,

$$\Gamma_2 = Z_4 \rtimes_3 Z_{pq} \cong Z_{4q} \rtimes_k Z_p,$$

where $p = 2^r$; $r \geq 1$, $\gcd(q,2) = 1$; $q > 1$. Here the group $Z_{4q} \rtimes_k Z_p$ has a presentation $Z_{4q} \rtimes_k Z_p = \langle c, d \mid c^{4q} = b^p = e, dc = c^kd\rangle$ where $k^p \equiv 1 \pmod{4q}$ and $k \equiv 3 \pmod 4$. The isomorphism $\phi\colon Z_4 \rtimes_3 Z_{pq} \to Z_{4q} \rtimes_k Z_p$ is given by $\phi(a) = c^q$ and $\phi(b) = cd$.

Similarly

$$\Gamma_3 = Z_5 \rtimes_4 Z_{pq} \cong Z_{5q} \rtimes_4 Z_p,$$

and $p = 2^r5^s$; $r \geq 1$, $s \geq 0$, $\gcd(q,2) = 1$, $\gcd(q,5) = 1$, $q > 1$. The group $Z_{5q} \rtimes_k Z_p$ has a presentation $Z_{5q} \rtimes_k Z_p = \langle c, d \mid c^{5q} = b^p = e, dc = c^kd\rangle$ where $k^p \equiv 1 \pmod{5q}$ and $k \equiv 4 \pmod 5$. The isomorphism $\phi\colon Z_5 \rtimes_4 Z_{pq} \to Z_{5q} \rtimes_k Z_p$ has the form $\phi(a) = c^q$ and $\phi(b) = cd$.

Lemma 10. *Let*

$$\Gamma_1 = Z_3 \rtimes_2 Z_{pq},$$

where $p = 2^r3^s$; $r \geq 1$, $s \geq 0$, $\gcd(q,2) = 1$, $\gcd(q,3) = 1$, $q > 1$;

$$\Gamma_2 = Z_4 \rtimes_3 Z_{pq},$$

where $p = 2^r$; $r \geq 1$, $\gcd(q,2) = 1$, $q > 1$;

$$\Gamma_3 = Z_5 \rtimes_4 Z_{pq},$$

where $p = 2^r5^s$; $r \geq 1$, $s \geq 0$, $\gcd(q,2) = 1$, $\gcd(q,5) = 1$, $q > 1$.

Then there are generating sets X_i, $i = 1, 2, 3$, such that for $G_i = C(\Gamma_i, X_i)$ it holds that $G_i \ncong {G_i}^{-1}$.

Proof. By Lemma 9 and Remark 1, the groups Γ_i have the form $Z_n \rtimes_k Z_m$, where $n \geq 6$, $c^n = d^m = e$, $dc = c^k d$, $k^m \equiv 1 \pmod{n}$, $k \neq 1$.

i) If $k^2 \not\equiv 1 \pmod{n}$, the result follows from Lemmas 2 and 3.
ii) If $k^2 \equiv 1 \pmod{n}$, the result follows from Lemma 8.

□

6. Reverses of Cayley Digraphs of Valence 3

Lemma 11. *Let $\Gamma = Z_3 \rtimes_2 Z_{2^r 3^s}$, $ba = a^2 b$, $r \geq 1$, $s \geq 1$, $2^r = r_1$, $3^s = s_1$, $2^r 3^s = m$ and let $X = \{a, b, ab^{s_1}\}$. Let $G = C(\Gamma, X)$. If an isomorphism $\phi\colon G \to G^{-1}$ is such that $\phi(e) = e$ then $\phi(a^i b^j) = a^{-i} b^{-j}$.*

Proof. It is obvious that for each $g \in G$ there exists exactly one cycle through g of the length 3 in $G(G^{-1})$, namely $C_a(g)(C^{-1}_{a^{-1}}(g))$. Similarly $C_{ab^{s_1}}(g)(C^{-1}_{(ab^{s_1})^{-1}}(g))$ is the unique cycle of length r_1, through g, and $C_b(g)(C^{-1}_{b^{-1}}(g))$ is the unique cycle of length m, through g in $G(G^{-1})$, respectively. In particular, from this it follows that the arcs of the form (g, gab^{s_1}) must be mapped by an isomorphism to the arcs of the form $(g', g'(ab^{s_1})^{-1})$. Hence an isomorphism (if any) must be the same as in the case with the generating set $\{a, b\}$. Now by Lemma 1 we have $\phi(a^i b^j) = a^{-i} b^{-j}$. □

Lemma 12. *Let $\Gamma = Z_3 \rtimes_2 Z_{2^r 3^s}$, $ba = a^2 b$, $r \geq 1$, $s \geq 1$, $2^r = r_1$, $3^s = s_1$, $2^r 3^s = m$ and let $X = \{a, b, ab^{s_1}\}$. Then $C(\Gamma, X) \ncong C(\Gamma, X^{-1})$.*

Proof. Let $\phi(e) = e$. By Lemma 11, $\phi(a^i b^j) = a^{-i} b^{-j}$. Hence in the digraph G we have $e \xrightarrow{ab^{s_1}} ab^{s_1}$ and for the digraph $\phi(G)$ it holds that $\phi(e) \xrightarrow{(ab^{s_1})^{-1}} (ab^{s_1})^{-1}$. Therefore $(ab^{s_1})^{-1} = a^{-1} b^{-s_1}$ and this implies that $a^{-2^{-s_1}} b^{-s_1} = a^{-1} b^{-s_1}$. Consequently $-2^{-s_1} \equiv -1 \pmod{3}$ and thus $2^{m-s_1} \equiv 1 \pmod{3}$, where $m - s_1$ is odd. But $2^{2h+1} \equiv 2 \pmod{3}$ for all positive integers; a contradiction. □

Lemma 13. *Let $\Gamma = Z_5 \rtimes_4 Z_{2^r 5^s}$, $ba = a^4 b$, $r \geq 1$, $s \geq 1$, $2^r = r_1$, $5^s = s_1$, $2^r 5^s = m$ and let $X = \{a, b, ab^{s_1}\}$. If an isomorphism $\phi\colon G \to G^{-1}$ is such that $\phi(e) = e$ then $\phi(a^i b^j) = a^{-i} b^{-j}$.*

Proof. The proof is similar to the proof of Lemma 11. □

Lemma 14. *Let $\Gamma = Z_5 \rtimes_4 Z_{2^r 5^s}$, $ba = a^4 b$, $r \geq 1$, $s \geq 1$, $2^r = r_1$, $5^s = s_1$, $2^r 3^s = m$ and let $X = \{a, b, ab^{s_1}\}$. Then $C(\Gamma, X) \ncong C(\Gamma, X^{-1})$.*

Proof. Let $\phi(e) = e$. By Lemma 13, $\phi(a^i b^j) = a^{-i} b^{-j}$. Hence in the digraph G it holds that $e \xrightarrow{ab^{s_1}} ab^{s_1}$ and for digraph $\phi(G)$ we have $\phi(e) \xrightarrow{(ab^{s_1})^{-1}} (ab^{s_1})^{-1}$.

Therefore $(ab^{s_1})^{-1} = a^{-1}b^{-s_1}$ and this implies that $a^{-4^{-s_1}}b^{-s_1} = a^{-1}b^{-s_1}$. Consequently $-4^{-s_1} \equiv -1 \pmod 5$ and thus $4^{m-s_1} \equiv 1 \pmod 5$, where $m - s_1$ is odd. But $4^{2h+1} \equiv 4 \pmod 5$ for all positive integers; a contradiction. □

Lemma 15. *Let $\Gamma = D_i$, $i \in \{3,4,5\}$, be the dihedral group of the form $D_i = Z_i \rtimes_{i-1} Z_2$, $ba = a^{i-1}b$. Then $C(\Gamma, X) \cong C(\Gamma, X^{-1})$ for all $X \subset \Gamma$.*

Proof. It is easy to see that $\phi(a^i b^j) = a^{-i+p_j}b^{-j}$ defines an isomorphism $\phi\colon G \to G^{-1}$ for a suitable choice of p_j. (The numbers p_j depend on the generating set X.) □

7. Summary

In order to facilitate the formulation we set

$$\Gamma_i' = Z_i \rtimes_{i-1} Z_{2^r},\ r \geq 2,\ ba = a^{i-1}b, i \in \{3,4,5\}$$

and

$$D_i = Z_i \rtimes_{i-1} Z_2,\ ba = a^{i-1}b,\ i \in \{3,4,5\}.$$

The above results about comparing Cayley digraphs of metacyclic groups with their reverses (disregarding the vertex valence) may now be summed up as follows.

Theorem 1. *Let Γ be an abelian or metacyclic group.*

If Γ is abelian or D_i then $C(\Gamma, X) \cong C(\Gamma, X^{-1})$ for every generating set X.

If Γ is any metacyclic group such that $\Gamma \neq \Gamma_i'$ and $\Gamma \neq D_i$ then there exist generating sets X for Γ such that $C(\Gamma, X) \ncong C(\Gamma, X^{-1})$.

The comparison of the Cayley digraphs $C(\Gamma, X)$ and $C(\Gamma, X^{-1})$ for the groups $\Gamma = {\Gamma_i}'$ remains open.

Acknowledgement. I would like to thank J. Širáň for helpful discussion on the topic of this paper.

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