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SUBALTERNATIVE ALGEBRAS

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Abstract. An algebra is called subalternative if the associator of any three linearly dependent elements is their linear combination. We prove that in characteristic $\neq 2, 3$ any such algebra is Maltsev-admissible and can be identified with a hyperplan in certain unital alternative algebra.

1. Introduction

In [**2**] we discussed **subassociative algebras** in which any associator is a linear combination of its arguments. A subassociative algebra is always Lie-admissible. In any associative unital algebra G, one can make a hyperplan H, which does not contain the unit, a subassociative algebra by projecting the multiplication from G into H; any subassociative algebra in characteristic not 2, 3 can be made in this way.

The following article is a continuation of [**2**]. We shall generalize the previous results to **subalternative algebras**, in which any associator of linearly dependent elements is their linear combination. It is not surprising that any such algebra is Maltsev-admissible and can be constructed on a hyperplan of some alternative algebra (except of course pathological cases in characteristic 2 or 3).

In [**4**] there are classified anticommutative algebras (over an algebraically closed field of characteristic $\neq 2$) in which there exist bilinear forms $(x, y) \mapsto N(x, y)$ satisfying the identities: $N(x, y) = N(y, x)$ (symmetry), $N(xy, z) = N(x, yz)$ (invariancy), $(xy)y = N(x, y)y - N(y, y)x$ (which is a special form of subalternativity, since $(xy)y = [x, y, y] = -[y, y, x]$). We will prove that the existence and the properties of such a form are consequences of subalternativity, even if the base field is not algebraically closed, which gives a still wider significance to the above mentioned classification.

Throughout the article we will suppose the conventions and definitions from [**2**].

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2. Preliminary Facts

Definition 1. Let H be an algebra with a multiplication $(x,y) \mapsto xy$, over a field $\mathbb{F}$. H is **subalternative algebra** if $\forall (x,y) \in H^2\ \exists (\alpha,\beta,\gamma,\delta) \in \mathbb{F}^4$:

$$[x,x,y] = \alpha x + \beta y,\ \ [x,y,y] = \gamma x + \delta y.$$

Definition 2. A subalternative algebra H from Definition 1 is **proper** if there exists such a bilinear form $A\colon H^2 \to \mathbb{F}$ that the following holds:

$$[x,x,y] = A(x,y)x - A(x,x)y\,, \tag{1}$$

$$[x,y,y] = A(y,y)x - A(x,y)y\,, \tag{2}$$

$$A(x^2,y) = A(x,xy)\,, \tag{3}$$

$$A(xy,y) = A(x,y^2)\,. \tag{4}$$

Because of the identity

$$[x,y,x] = [x,x+y,x+y] - [x,x,x] - [x,x,y] - [x,y,y] \tag{5}$$

the associator $[x,y,x]$ in a subalternative algebra is also a linear combination of its arguments. This implies the following proposition.

Proposition 3. *An algebra H over $\mathbb{F}$ is subalternative if and only if for any linerly dependent triple $\{x,y,z\} \subset H$, the associator $[x,y,z]$ is a linear combination of its arguments.*

Of course, any subassociative algebra is subalternative, and any proper subassociative algebra is proper subalternative.

The next proposition is obvious.

Proposition 4. *Any subalternative algebra of dimension ≤ 3 is subassociative.*

Propositions 5 and 6 from [**2**] are (mutatis mutandis) correct also for alternative and subalternative algebras.

Proposition 5. *Let G be an alternative algebra with multiplication $(a,b) \mapsto a*b$ and with a unit e. Further let $P\colon G \to \mathbb{F}$ be a linear functional and $P(e) = 1$. Define in $H := \operatorname{Ker} P$ a new multiplication*

$$(x,y) \mapsto xy := x*y - A(x,y)e\,,$$

where $A(x,y) := P(x*y)$. Then H is a proper subalternative algebra and A is the bilinear form from Definition 2.

Proposition 6. *Let H be a proper subalternative algebra from Definition 2, and $G := \mathbb{F}e \oplus H$, where $e \notin H$. Introduce in G a new multiplication*

$$(\alpha e + x, \beta e + y) \mapsto (\alpha e + x) * (\beta e + y) := \\ = (\alpha\beta + A(x,y))e + \alpha y + \beta x + xy\,.$$

G with this multiplication is an alternative algebra with unit e.

Corollary 7. *Let H be a proper subalternative algebra from Definition 2. Then for any linearly dependent triple $\{x, y, z\} \subset H$ there holds:*

$$[x, y, z] = A(y, z)x - A(x, y)z\,, \tag{6}$$

$$A(xy, z) = A(x, yz)\,. \tag{7}$$

Proposition 8. *Improper subassociative algebra is also improper subalternative.*

Proof. For the twodimensional strange subassociative algebras from [**2**, Tables 7 and 8], it is enough to look over the associators $[p, p, q]$ and $[q, p, p]$.

In the case of threedimensional strange subassociative algebras [**2**, Table 9], the best way to check the proposition is to use a computer.

Now, suppose that the observed algebra is improper non–strange (with a dimension > 2 and $\operatorname{chr} \mathbb{F} = 2$). Then:

$$[x, y, z] = A'(y, z)x + B'(x, z)y + C'(x, y)z$$

identically, for certain bilinear forms A', B', C'.

$$\begin{aligned}[x, y, y] &= A'(y, y)x + B'(x, y)y + C'(x, y)y \\ &= A(y, y)x + A(x, y)y\end{aligned}$$

Then: $A(x, y) = B'(x, y) + C'(x, y) = A'(x, y)$ by [**2**, (15)] for any x, y linearly independent, and then also $A(y, y) = A'(y, y)$.

$$\begin{aligned}[x, x, y] &= A'(x, y)x + B'(x, y)x + C'(x, x)y \\ &= A(x, y)x + A(x, x)y\end{aligned}$$

This gives $B'(x, y) = 0$ for any x, y, which is impossible. □

Remark. In fact, we proved a little more: if a subassociative algebra has a bilinear form A with the identities (1) and (2), it is proper subassociative.

From Propositions 4, 5, 8 and Corollary 7 we find the following interesting consequences:

a) Alternative algebra of dimension ≤ 3 is associative.
b) Unital alternative algebra of dimension ≤ 4 is associative.

For $U \subset H$, let $\mathrm{alg}_H U$ be the subalgebra of H generated by U, and for $V \subset G$ let $\mathrm{alg}_G V$ be the subalgebra of G generated by V; here H and G are the algebras from Propositions 5 and 6. Any element from $\mathrm{alg}_H\{u, v\}$ is also from $\mathrm{alg}_G\{u, v, e\}$. If $x \in \mathrm{alg}_H\{u, v\}$ then there exists such $\alpha \in \mathbb{F}$ that $\alpha e + x \in \mathrm{alg}_G\{u, v\}$. According to Artin's theorem, $\mathrm{alg}_G\{u, v\}$ is associative subalgebra of G. If x, y, z are from $\mathrm{alg}_H\{u, v\}$ and hence for certain α, β, γ also $\alpha e + x$, $\beta e + y$, $\gamma e + z$ from $\mathrm{alg}_G\{u, v\}$, then

$$\begin{aligned} 0 &= [\alpha e + x, \beta e + y, \gamma e + z]_G = [x, y, z]_G \\ &= [x, y, z]_H + A(x, y)z - A(y, z)x + (A(xy, z) - A(x, yz))e \,. \end{aligned}$$

So we have

Proposition 9. *Let H be a proper subalternative algebra. Then for any $(u, v) \in H^2$, $\mathrm{alg}_H\{u, v\}$ is a proper subassociative algebra.*

3. Proper Subalternative Algebras

Theorem 10. *Let H be a subalternative algebra over $\mathbb{F}$, $\mathrm{chr}\,\mathbb{F} \neq 2$, excluding also the case $\dim H = 2$, $\mathbb{F} = \mathbb{Z}_3$. Then there exists such a bilinear form A that* (1) *and* (2) *hold.*

Proof. If $\dim H \leq 3$, H is subassociative and the theorem holds. Hence we shall suppose also that $\dim H > 3$.

$$[x, z, y] + [z, x, y] = [x + z, x + z, y] - [x, x, y] - [z, z, y] \in \mathrm{lin}\{x, y, z\} \,.$$

By [**2**, Lemma 7] there exist three bilinear forms A_1, A_2, A_3, such that:

$$[x, z, y] + [z, x, y] = A_1(y, z)x + A_2(x, z)y + A_3(x, y)z.$$

For $z = x$ we get:

$$[x, x, y] = \frac{1}{2}(A_1(y, x) + A_3(x, y))x + \frac{1}{2}A_2(x, x)y \,.$$

Analogous conclusion holds for $[y, x, x]$. Hence, there exist four bilinear forms A, B, C, D, such that the following identities hold:

$$[x, x, y] = A(x, y)x - B(x, x)y \,, \tag{8}$$

$$[y, x, x] = C(x, x)y - D(y, x)x \,. \tag{9}$$

(7) and (8) give for $x = y$ a new identity:

$$A(x,x) + D(x,x) = B(x,x) + C(x,x)\,. \tag{10}$$

From (5) if follows:

$$\begin{aligned}[x,y,x] = (A(y,x) + D(y,x) - B(y,x) - B(x,y))x +\\ + (A(x,x) - C(x,x))y\,.\end{aligned} \tag{11}$$

If we have for some $x : x^2 = \lambda x$, the associator $[x,x,x]$ gives us

$$B(x,x) = A(x,x)\,. \tag{12}$$

Next, suppose that $x^2 \neq \lambda x$. If we use (8), (9) and (11) in Teichmüller equation $E(x,x,x,x)$, we get:

$$0 = (3A(x,x) - B(x,x) - 2C(x,x))x^2 + (\ldots)x\,.$$

Therefore,

$$B(x,x) = 3A(x,x) - 2C(x,x)\,. \tag{13}$$

Suppose additionally that $B(x,x) \neq A(x,x)$. Then $A(x,x) \neq C(x,x)$ and from $E(x,x,x^2,x)$ it follows: $xx^2 = \alpha_1 x + \beta_1 x^2$. Then: $x^2x = [x,x,x] + xx^2 = \alpha_2 x + \beta_1 x^2$. Further: $A(x,x^2)x - B(x,x)x^2 = [x,x,x^2] = x^2x^2 - x(\alpha_1 x + \beta_1 x^2) = x^2x^2 - (\alpha_1 + \beta_1^2)x^2 - \alpha_1\beta_1 x$, and $x^2x^2 = \alpha_3 x + \beta_2 x^2$. alg$\{x\}$ is therefore a twodimensional algebra and hence subassociative. If it is proper subassociative, it possesses two bilinear forms A', B' for which the following identities hold: $B'(u,u) = 0$ and

$$[u,v,w] = A'(v,w)u + B'(u,w)v - (A'(u,v) + 2B'(u,v))w\,.$$

From $[x,x,x^2]$ and $[x^2,x,x]$ we get $B(x,x) = A'(x,x)$ and $C(x,x) = A'(x,x)$. But then we find the contradiction in (13). Hence, alg$\{x\}$ is an algebra from [**2**, Table 8] and $x = \pm p$. From this table and (8) and (9), the associators $[p,p,p]$ and $[p,p,q]$ determine $A(p,p) = 1 - \alpha$ and the associator $[q,p,p]$ also $C(p,p) = 1 - \alpha$ and the contradiction is final.

So, (12) holds in any case. (10) gives also $D(x,x) = C(x,x)$ and (8), (9) and (11) can be formulated with only two bilinear forms:

$$[x,x,y] = A(x,y)x - A(x,x)y\,, \tag{14}$$

$$[y,x,x] = D(x,x)y - D(y,x)x\,, \tag{15}$$

$$[x,y,x] = (D(y,x) - A(x,y))x + (A(x,x) - D(x,x))y\,. \tag{16}$$

We have seen that if $x^2 \neq \lambda x$ then (13) holds and consequently $A(x,x) = D(x,x)$. We claim that this is always true. So, suppose that $x^2 = \lambda x$ and $A(x,x) \neq D(x,x)$. Choose y linearly independent from x. From $E(x,y,x,x)$ it follows: $yx = \alpha x + \frac{\lambda}{2} y$ (for some α). From $E(x,x,y,x)$ it follows: $xy = \beta x + \frac{\lambda}{2} y$ (for some β). But then $[x,y,x] = \frac{\beta-\alpha}{2}\lambda x$, which is in a contradiction with (16).

The linearized form of $A(x,x) = D(x,x)$ is

$$A(x,y) + A(y,x) = D(x,y) + D(y,x)\,, \tag{17}$$

and the linearized form of (15) is

$$[y,x,z] + [y,z,x] = -D(y,z)x + (D(x,z) + D(z,x))y - D(y,x)z\,. \tag{18}$$

From $E(y,x,x,x)$ it follows:

$$[y,x,x^2] - [y,x^2,x] = D(yx,x)x - D(y,x)x^2\,. \tag{19}$$

We add the identity (18), with $z = x^2$, to (19):

$$[y,x,x^2] = (\dots)x + (\dots)y - D(y,x)x^2\,. \tag{20}$$

From $E(x,y,x,x)$ it follows:

$$[x,y,x^2] = (\dots)x - A(x,y)x^2\,. \tag{21}$$

In the identity

$$[x+y, x+y, x^2] - [x,x,x^2] - [y,y,x^2] = [x,y,x^2] + [y,x,x^2]$$

we use (14) on the left side and (20) and (21) on the right side:

$$0 = \alpha x + (A(y,x) - D(y,x))x^2 + \beta y$$

If we choose $y \notin \mathrm{lin}\{x, x^2\}$, we find $\beta = 0$, and since β depends only to x, it then follows identically:

$$(A(y,x) - D(y,x))x^2 + \alpha x = 0\,. \tag{22}$$

Suppose that x and y are such elements that $A(y,x) \neq D(y,x)$. Then $x^2 = \lambda x$ and because of (17) also $y^2 = \mu y$. Since $A(y, \gamma x + \delta y) - D(y, \gamma x + \delta y) = \gamma(A(y,x) - D(y,x))$, we have $w^2 = \nu w$ for each $w = \gamma x + \delta y$. If $\dim \mathrm{alg}\{x,y\} = 2$, this is an algebra from [**2**, Table 5] (without condition $pq \neq qp$). But in this algebra $A = D$; hence, $\dim \mathrm{alg}\{x,y\} > 2$. Since $(x+y)^2 = \tau(x+y)$ and then $xy + yx = (\tau - \lambda)x + (\tau - \mu)y$, it must be: $z := xy \notin \mathrm{lin}\{x,y\}$. From the associators $[x,y,y]$, $[y,x,y]$, $[x,x,y]$ and $[x,y,x]$ we get that zy, yz, xz and zx are linear combinations of x, y and z. If z and z^2 were linearly independent, it would be $A(y,z) = D(y,z)$ and consequently $A(y, x+z) \neq D(y, x+z)$ and $(x+z)^2 = \xi(x+z)$, which means that z^2 is a linear combination of x, y and z. Therefore, $\dim \mathrm{alg}\{x,y\} = 3$ and according to [**2**, Theorem 12] we conclude that $\mathrm{alg}\{x,y\}$ is proper subassociative and again $A = D$. □

Theorem 11. *Let H be a subalternative algebra with a bilinear form A for which* (1) *and* (2) *hold.*

(i) *If* $\operatorname{chr}\mathbb{F} \neq 3$ *or if H is a subassociative algebra, then H is a proper subalternative algebra.*
(ii) *If* $\operatorname{chr}\mathbb{F} = 3$ *and H is not subassociative then the following weaker identities hold:*

$$A(xy,x) = A(x,yx)\,, \tag{23}$$
$$A(x^2,y) + A(yx,x) = A(x,xy) + A(y,x^2)\,. \tag{24}$$

Proof. As it was pointed out in Remark after Proposition 8, the subassociativity is sufficient for properness. Therefore we may suppose that $\dim H > 3$.

Denote:

$$\begin{aligned} R(x,y) &:= A(x^2,y) - A(x,xy)\,,\\ M(x,y) &:= A(xy,x) - A(x,yx)\,,\\ L(x,y) &:= A(yx,x) - A(y,x^2)\,. \end{aligned}$$

$$\begin{aligned} [x^2,x,y] &= [x^2,x+y,x+y] + [x,x,yx] - [x^2,x,x] - [x^2,y,y]\\ &\quad - x[x,y,x] - [x,x,y]x - [x,xy,x]\\ &= A(x,y)x^2 - A(x^2,x)y - (R(x,y) + M(x,y))x\,, \end{aligned} \tag{25}$$

considering (5) and consequently

$$[x,y,x] = (A(y,x) - A(x,y))x\,.$$

$$\begin{aligned} [x,x^2,y] &= [x,x^2+y,x^2+y] + [xy,x,x] - [x,x^2,x^2] - [x,y,y]\\ &\quad - x[y,x,x] - [x,y,x]x - [x,yx,x]\\ &= A(x^2,y)x - A(x,x^2)y - (M(x,y) + L(x,y))x \end{aligned} \tag{26}$$

If we put (25) and (26) into the identity

$$[x+x^2,x+x^2,y] = [x,x,y] + [x^2,x^2,y] + [x^2,x,y] + [x,x^2,y]\,,$$

we get:

$$R(x,y) + 2M(x,y) + L(x,y) = 0\,. \tag{27}$$

Further,

$$
\begin{aligned}
(28) \qquad [y,x^2,x] &= [x^2+y,x^2+y,x]+[x,x,yx]-[x^2,x^2,x]-[y,y,x] \\
&\quad -x[x,y,x]-[x,x,y]x-[x,xy,x] \\
&= A(x^2,x)y-A(y,x^2)x-(R(x,y)+M(x,y))x\,.
\end{aligned}
$$

Considering (27) and (28), we also get:

$$
\begin{aligned}
(29) \qquad [y,x,x^2] &= [y,x+x^2,x+x^2]-[y,x,x]-[y,x^2,x^2]-[y,x^2,x] \\
&= A(x,x^2)y-A(y,x)x^2-(M(x,y)+L(x,y))x\,.
\end{aligned}
$$

Including (25) and (26) into $E(x,x,x,y)$, we find:

$$(A(x^2,x)-A(x,x^2))y=(L(x,y)-2R(x,y))x\,.$$

Choosing x and y linearly independent we get $A(x^2,x)=A(x,x^2)$, which then implies:

$$L(x,y)=2R(x,y)\,. \tag{30}$$

Similarly we find from (28), (29) and $E(y,x,x,x)$:

$$R(x,y)=2L(x,y)\,. \tag{31}$$

If $\operatorname{chr}\mathbb{F}\neq 3$, from (30) and (31) we already find $L=R=0$. From $R(x+y,x)=0$ follows also $M=0$.

In the case $\operatorname{chr}\mathbb{F}=3$, (30) and (31) are equivalent with (24) and further (27) with (23). □

Theorem 12. *Let H be a subalternative algebra over $\mathbb{F}$. Each of the following conditions*

(i) $\dim H\leq 3$*;*
(ii) *H is subassociative;*
(iii) $\operatorname{chr}\mathbb{F}\neq 2,3$*;*
(iv) *H is proper subalternative;*

implies that H is Maltsev-admissible.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ H is Lie-admissible, by [**2**, Theorem 13] $\Rightarrow$ H is Maltsev-admissible.

From Theorems 10 and 11 the implication (iii) $\Rightarrow$ (iv) follows. Further, G from Proposition 6 is alternative algebra, the commutator algebra G^- is Maltsev and $H^-\cong G^-/(\mathbb{F}e)^-$ is also Maltsev. □

Theorem 13. *Let H be an anticommutative subalternative algebra over $\mathbb{F}$ with $\operatorname{chr}\mathbb{F} \neq 2$. Then H is proper subalternative and also Maltsev algebra and for the bilinear form A from* (1) *and* (2) *the following identities hold:*

$$A(x,y) = A(y,x)\,, \tag{32}$$

$$A(xy,z) = A(x,yz)\,, \tag{33}$$

$$(xy)y = A(y,y)x - A(x,y)y\,. \tag{34}$$

Proof. If $\dim H \leq 3$, H is subassociative and, according to [**2**, Theorem 13], the theorem above is correct. For $\dim H > 3$, (1) and (2) hold for an adequate A, by Theorem 10. The anticommutativity implies flexibility $[x,y,x] = 0$ and (5) implies (32). (34) is a consequence of (2) and the anticommutativity. If we linearize (23), which is implied either by $\operatorname{chr}\mathbb{F} = 3$ or by $\operatorname{chr}\mathbb{F} \neq 3$ and (7), according to Theorem 11, we get:

$$A(xy,z) + A(zy,x) = A(x,yz) + A(z,yx)\,.$$

Using (32) and anticommutativity, we transform this identity into

$$2A(xy,z) = 2A(x,yz)\,,$$

which is (33), and the properness of the algebra is proved. Then, by Theorem 12, H is Maltsev-admissible and hence Maltsev. □

The classification of algebras from Theorem 13 for $\mathbb{F}$ algebraically closed is described in [**4**, Theorem 3.3].

A natural question for the end: is it possible and significant to generalize the theory treated in this article? We suggest two ways of thinking. Non-commutative Jordan algebras are a kind of natural generalization of alternative algebras and are defined with associator identities; hence, perhaps sub-(non-commutative Jordan) algebras are suitable research target. Secondly, an important example for motivation for further research is the color algebra ([**3**]).

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