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FORCED SUPERLINEAR OSCILLATIONS VIA PICONE'S IDENTITY

J. JAROŠ, T. KUSANO AND N. YOSHIDA

Abstract. In the paper a nonlinear version of an identity known in the literature as the Picone's formula is derived and then it is used to extend the classical Sturmian comparison theory to forced superlinear equations of the second order.

1. Introduction

In this paper we are concerned with the forced second order super-linear ordinary differential equation

$$(P(t)y')' + Q(t)|y|^{\beta} \operatorname{sgn} y = f(t), \quad t \geq t_0, \tag{A}$$

where $\beta > 1$ and $P, Q, f\colon [t_0, \infty) \to R$ are continuous real-valued functions with $P(t) > 0$ for $t \geq t_0$.

By a solution of (A) on an interval $I \subset [t_0, \infty)$ we understand a function $y\colon I \to R$ which is continuously differentiable on I together with Py' and satisfies (A) at every point of I. Such a solution is called **oscillatory** if it is defined on an interval of the form $[t_x, \infty), t_x \geq t_0$, and has arbitrarily large zeros in this interval.

Recently, there was a renaissance of interest in studying forced second-order equations, both linear and nonlinear (see [**12**] and the references cited therein) motivated by the absence of oscillation tests that would be sensitive enough to cover some important specific examples (including Mathieu's equation with a forcing term $y'' + (a + b\cos 2t)y = ct^{\delta} \sin t$ where $a, b, c, \delta \in R$ and forced Emden-Fowler equations of the form (A) with an oscillatory "potential" $Q(t)$) and would not only deduce oscillation of solutions itself, but would also provide information about the number and distribution of zeros in a given interval.

In the linear oscillation theory, one of such powerful techniques is offered by the well-known Picone's formula ([**8**]; see also [**5**] and [**10**]). There were several attempts to extend this formula to nonlinear equations (see, for instance, [**2**]), but none of these extensions applies to forced superlinear equations of the form (A).

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Thus, the purpose of this paper is to show how Picone's formula can be used, in a rather surprising but simple way, to extend the classical Sturmian theory to forced superlinear equations of the form (A). Our main results improve and extend those in [**7**].

2. Sturmian Theorems for the Forced Superlinear Equation (A)

Consider the nonlinear second-order differential equation

$$L_\beta[y] \equiv (P(t)y')' + Q(t)|y|^\beta \operatorname{sgn} y = f(t), \tag{A}$$

where $\beta > 1$ and P, Q and f are continuous real-valued functions on a given interval $I \subset [t_0, \infty)$ and $P(t) > 0$ for all $t \in I$. Denote by $\mathcal{D}_L(I)$ the domain of the operator L_β, i.e., the set of all continuous real-valued functions y defined on I such that y and Py' are continuously differentiable on I.

We begin with the following preliminary result which is a nonlinear version of an identity used by Leighton in proving his well-konwn improvement of the classical Sturm-Picone theorem (see also Swanson [**9**, Lemma 1.3]). The proof is straightforward and it is omitted.

Lemma 1. *If $y \in \mathcal{D}_L(I_0)$ for some non-degenerate subinterval $I_0 \subset I$ and $y(t) \neq 0$ in I_0, then for any $x \in C^1(I_0)$ the following identity holds:*

$$\frac{d}{dt}\left[\frac{x^2}{y}P(t)y'\right] = P(t)x'^2 - \left[Q(t)|y|^{\beta-1} - \frac{f(t)}{y}\right]x^2 - P(t)\left(x' - \frac{x}{y}y'\right)^2 + \frac{x^2}{y}\{L_\beta[y] - f(t)\}. \tag{1}$$

To obtain our first application of the identity (1), assume that $Q(t) \geq 0$ on some interval $[a, b] \subset I$ and consider the quadratic functional J defined by

$$J[\eta] \equiv \int_a^b \left[P(t)\eta'^2 - \beta(\beta-1)^{\frac{1-\beta}{\beta}}[Q(t)]^{\frac{1}{\beta}}|f(t)|^{\frac{\beta-1}{\beta}}\eta^2\right]dt$$

with the domain $U = \{\eta \in C^1[a, b] : \eta(a) = \eta(b) = 0\}$. Then the following nonlinear extension of Leighton's variational theorem is valid.

Theorem 1. *If there exists an $\eta \in U, \eta \not\equiv 0$, such that*

$$J[\eta] \leq 0, \tag{2}$$

then every solution y of (A) *defined on* $[a,b]$ *and satisfying*

$$y(t)f(t) \le 0 \tag{3}$$

in this interval must have a zero in $[a,b]$.

Proof. Suppose, to the contrary, that there exists a solution $y(t) \neq 0$ in $[a,b]$ of (A) that satisfies (3). Then the identity (1) from Lemma 1 with $x(t) = \eta(t)$ is valid and since y solves $L_\beta[y] = f$, (1) reduces to

$$\left[\frac{\eta^2}{y}P(t)y'\right]' = P(t)\eta'^2 - \left[Q(t)|y|^{\beta-1} - \frac{f(t)}{y}\right]\eta^2 - P(t)\left(\eta' - \frac{\eta}{y}y'\right)^2. \tag{4}$$

Taking the nonnegativity of $Q(t)$ and the inequality (3) into account, and considering the expression in the brackets on the right-hand side of (4) as the function of y, we obtain

$$\min_{y\neq 0}\left[Q|y|^{\beta-1} + \frac{|f|}{|y|}\right] = \beta(\beta-1)^{\frac{1-\beta}{\beta}}Q^{\frac{1}{\beta}}|f|^{\frac{\beta-1}{\beta}}. \tag{5}$$

Thus, (4) becomes

$$\left[\frac{\eta^2}{y}P(t)y'\right]' \le P(t)\eta'^2 - \beta(\beta-1)^{\frac{1-\beta}{\beta}}[Q(t)]^{\frac{1}{\beta}}|f(t)|^{\frac{\beta-1}{\beta}}\eta^2 - P(t)\left(\eta' - \frac{\eta}{y}y'\right)^2, \tag{6}$$

and integrating the inequality (6) from a to b yields

$$0 \le J[\eta] - \int_a^b P(t)\left(\eta' - \frac{\eta}{y}y'\right)^2 dt, \tag{7}$$

which is a contradiction unless $J[\eta] = 0$ and $\eta' - \eta y'/y \equiv 0$ in $[a,b]$. The last relation implies that y must be a constant multiple of η, and so we have, in particular, $y(a) = y(b) = 0$. This is the contradiction with the assumption $y(t) \neq 0$ on $[a,b]$. The proof is complete. □

As an immediate consequence of Theorem 1 we have the following oscillation result.

Corollary 1. *Let there exist two sequences of disjoint intervals* (a_n^-, b_n^-), (a_n^+, b_n^+), $t_0 \le a_n^- < b_n^- \le a_n^+ < b_n^+$, $a_n^- \to \infty$ *as* $n \to \infty$ *such that*

$$Q(t) \ge 0 \quad \text{on } [a_n^-, b_n^-] \cup [a_n^+, b_n^+], \tag{8}$$

$$f(t) \le 0 \quad \text{on } [a_n^-, b_n^-], \tag{9}$$

$$f(t) \ge 0 \quad \text{on } [a_n^+, b_n^+], \tag{10}$$

$n = 1, 2, \dots,$ and two sequences of nontrivial continuously differentiable functions $\eta_n^-(t)$ and $\eta_n^+(t)$ defined on $[a_n^-, b_n^-]$ and $[a_n^+, b_n^+]$, respectively, such that

$$\eta_n^-(a_n^-) = \eta_n^-(b_n^-) = \eta_n^+(a_n^+) = \eta_n^+(b_n^+) = 0,$$

$n = 1, 2, \dots,$ and

$$J[\eta_n^{\pm}] \equiv \int_{a_n^{\pm}}^{b_n^{\pm}} \left[P(t)(\eta_n^{\pm})'^2 - \beta(\beta-1)^{\frac{1-\beta}{\beta}} [Q(t)]^{\frac{1}{\beta}} |f(t)|^{\frac{\beta-1}{\beta}} \eta_n^{\pm 2} \right] dt \le 0 \tag{11}$$

for every $n \in N$. Then all solutions of (A) *are oscillatory.*

Remark 1. Theorem 1 and Corollary 1 improve and extend the corresponding results in [**7**].

Our next results will be obtained by comparing the superlinear equation (1) with the Sturm-Liouville equation

$$l[x] \equiv (p(t)x')' + q(t)x = 0, \tag{B}$$

where $p, q\colon [t_0, \infty) \to R$ are continuous functions and $p(t) > 0$ for $t \ge t_0$. Analogously as in the case of the nonlinear differential operator L_β, by $\mathcal{D}_l(I)$ we denote the set of all real-valued functions which are defined and continuous on an interval $I \subset [t_0, \infty)$ and such that both x and px' are continuously differentiable on I.

In what follows, instead of the weaker form of Picone's identity (1) involving an arbitrary C^1-function x, the following stronger version will be used. It can be readily derived by a straightforward calculation and the verification is left to the reader.

Lemma 2. *If $x \in \mathcal{D}_l(I_0)$ and $y \in \mathcal{D}_L(I_0)$ for some non-degenerate subinterval $I_0 \subset I$ and $y(t) \ne 0$ for $t \in I_0$, then*

$$\begin{aligned} &\frac{d}{dt}\left\{ \frac{x}{y}[yp(t)x' - xP(t)y'] \right\} \\ &\quad = [p(t) - P(t)]x'^2 + \left[Q(t)|y|^{\beta-1} - \frac{f(t)}{y} - q(t) \right] x^2 \\ &\qquad + P(t)\left(x' - \frac{x}{y}y' \right)^2 + xl[x] - \frac{x^2}{y}\{L_\beta[y] - f(t)\}. \end{aligned} \tag{12}$$

Theorem 2 (Leighton-type comparison theorem). *If there exists a nontrivial solution $x \in \mathcal{D}_l([a,b])$ of the linear equation $l[x] = 0$ in $[a,b]$ such that $x(a) = x(b) = 0$ and*

$$\begin{aligned} V[x] \equiv \int_a^b &\left[(p(t) - P(t))x'^2 + (\beta(\beta-1)^{\frac{1-\beta}{\beta}} [Q(t)]^{\frac{1}{\beta}} |f(t)|^{\frac{\beta-1}{\beta}} - q(t))x^2 \right] dt \\ &\ge 0, \end{aligned} \tag{13}$$

then every solution y of the forced superlinear equation (A) *satisfying* $y(t)f(t) \le 0$ *in* (a,b) *has a zero in* $[a,b]$.

Proof. Assume that there exists a solution y of (A) such that $y(t) \neq 0$ and $y(t)f(t) \le 0$ in $[a,b]$. An application of the identity (12) together with (5) leads to the Picone-type inequality

$$\begin{aligned} \frac{d}{dt}\left\{\frac{x}{y}[yp(t)x' - xP(t)y']\right\} &\\ \ge [p(t) - P(t)]x'^2 &+ \left[\beta(\beta-1)^{\frac{1-\beta}{\beta}}[Q(t)]^{\frac{1}{\beta}}|f(t)|^{\frac{\beta-1}{\beta}} - q(t)\right]x^2 \\ &+ P(t)\left(x' - \frac{x}{y}y'\right)^2 + xl[x] - \frac{x^2}{y}\{L_\beta[y] - f(t)\}. \end{aligned} \tag{14}$$

Taking the equations (A) and (B) into account and integrating (14) from a to b gives a contradiction to (13) unless $V[x] = 0$ and $y(t) = cx(t)$ for some constant c, i.e., in particular, $y(a) = y(b) = 0$. But this is again the contradiction with the assumption $y(t) \neq 0$ in $[a,b]$. □

Remark 2. Theorem 2 can be proved also indirectly (without refering to (12)) in the following way.

Since $l[x] = 0$ on $[a,b]$ and $x(a) = x(b) = 0$, integration by parts yields

$$\int_a^b [p(t)x'^2 + q(t)x^2]\,dt = 0. \tag{15}$$

Thus, combining (13) with (15) we obtain $V[x] = -J[x] \ge 0$ and the conclusion follows from Theorem 1.

Corollary 2 (Sturm-Picone type comparison theorem). *Let* $Q(t) \ge 0$ *in* $[a,b]$. *If*

$$p(t) \ge P(t) > 0, \tag{16}$$

$$\beta(\beta-1)^{\frac{1-\beta}{\beta}}[Q(t)]^{\frac{1}{\beta}}|f(t)|^{\frac{\beta-1}{\beta}} \ge q(t), \tag{17}$$

in $[a,b]$ *and there exists a nontrivial solution* $x \in \mathcal{D}_l([a,b])$ *of the linear equation* (B) *such that* $x(a) = x(b) = 0$, *then any solution of* (A) *satisfying* $y(t)f(t) \le 0$ *in* (a,b) *has a zero in* $[a,b]$.

As a consequence of Theorem 2, we have the following comparison result relating oscillation property of the forced superlinear equation (A) to that of "conjugacy" of two sequences of associated "minorant" linear Sturm-Liouville equations (B_n^-) and (B_n^+) below considered on sequences of corresponding disjoint intervals $[a_n^-, b_n^-]$ and $[a_n^+, b_n^+]$, respectively.

Corollary 3. *If there exist two sequences of disjoint intervals* (a_n^-, b_n^-), (a_n^+, b_n^+), $t_0 \le a_n^- < b_n^- \le a_n^+ < b_n^+$, $a_n^- \to \infty$ *as* $n \to \infty$ *such that*

$$Q(t) \ge 0 \quad \text{on } [a_n^-, b_n^-] \cup [a_n^+, b_n^+], \tag{18}$$

$$f(t) \le 0 \quad \text{on } [a_n^-, b_n^-], \tag{19}$$

$$f(t) \ge 0 \quad \text{on } [a_n^+, b_n^+], \tag{20}$$

$n = 1, 2, \dots,$ *and two sequences of Sturm-Liouville equations*

$$l_n^-[x] \equiv (p_n^-(t)x')' + q_n^-(t)x = 0, \tag{B_n^-}$$

$$l_n^+[x] \equiv (p_n^+(t)x')' + q_n^+(t)x = 0, \tag{B_n^+}$$

where $p_n^-, q_n^- : [a_n^-, b_n^-] \to R$ *and* $p_n^+, q_n^+ : [a_n^+, b_n^+] \to R$ *are continuous functions with* $p_n^-(t) > 0$ *and* $p_n^+(t) > 0$ *with respective nontrivial solutions* $x_n^- \in \mathcal{D}_{l_n^-}([a_n^-, b_n^-])$ *and* $x_n^+ \in \mathcal{D}_{l_n^+}([a_n^+, b_n^+])$ *satisfying*

$$x_n^-(a_n^-) = x_n^-(b_n^-) = x_n^+(a_n^+) = x_n^+(b_n^+) = 0, \tag{21}$$

$n = 1, 2, \dots,$ *and*

$$\begin{aligned} V[x_n^\pm] \equiv \int_{a_n^\pm}^{b_n^\pm} \Big\{ & [p_n^\pm(t) - P(t)](x_n^\pm)'^2 \\ & + [\beta(\beta-1)^{\frac{1-\beta}{\beta}} (Q(t))^{\frac{1}{\beta}} |f(t)|^{\frac{\beta-1}{\beta}} - q_n^\pm(t)] x_n^{\pm 2} \Big\}\, dt \\ \ge 0, & \end{aligned} \tag{22}$$

for every $n \in N$, *then all solutions of* (A) *are oscillatory.*

In our next comparison result which is an immediate consequence of Corollary 2 and which relates oscillation of (A) to that of the linear equation (B), by **consecutive sign change points** of the oscillatory forcing function f we understand points $t_1, t_2 \in [t_0, \infty)$, $t_1 < t_2$, such that $f(t) \ge 0$ (resp. $f(t) \le 0$) on $[t_1, t_2]$ and $f(t) < 0$ (resp. $f(t) > 0$) on $(t_1 - \epsilon, t_1) \cup (t_2, t_2 + \epsilon)$ for some $\epsilon > 0$ (see [**2**]).

Corollary 4. *Let* $Q(t) \ge 0$ *on* $[t_0, \infty)$,

$$p(t) \ge P(t), \tag{23}$$

$$\beta(\beta-1)^{\frac{1-\beta}{\beta}} [Q(t)]^{\frac{1}{\beta}} |f(t)|^{\frac{\beta-1}{\beta}} \ge q(t), \tag{24}$$

for $t \ge t_0$ *and either* (23) *or* (24) *do not become an identity on any open interval where* $f(t) \equiv 0$. *Moreover, suppose that the linear equation* (B) *is oscillatory* *and the distance between consecutive zeros of any solution of* (B) *is less than the distance between consecutive sign change points of the forcing function* f. *Then every solution of the superlinear equation* (A) *is oscillatory.*

In the last Corollary, by a **quickly oscillating solution** of (B) we mean an oscillatory solution for which the distance between consecutive zero points t_n and t_{n+1} tends to zero as $n \to \infty$. Also, the oscillatory function $f(t)$ is said to be a **moderately oscillating function** if the distance between any consecutive sign change points of f remains bounded from below by some positive constant c.

Corollary 5. *Suppose that* $Q(t) \geq 0$ *for* $t \geq t_0$. *If* (23) *and* (24) *hold, the function* f *is moderately oscillating function and every solution of* (B) *is quickly oscillatory, then every nontrivial solution of* (A) *is oscillatory, too.*

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J. Jaroš, Department of Mathematical Analysis, Faculty of Mathematics and Physics, Comenius University, 842 48 Bratislava, Slovakia

T. Kusano, Department of Applied Mathematics, Faculty of Science, Fukuoka University, Fukuoka, 814-0180, Japan

N. Yoshida, Department of Mathematics, Faculty of Science, Toyama University, Toyama, 930-8555, Japan