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ON UNIFORM EXPONENTIAL STABILITY OF PERIODIC EVOLUTION OPERATORS IN BANACH SPACES

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Abstract. The aim of this paper is to obtain some discrete-time characterizations for the uniform exponential stability of periodic evolution operators in Banach spaces. We shall also obtain a discrete-time variant for Neerven's theorem using Banach sequence spaces and a new proof for Neerven's theorem.

1. Introduction

Let X be a real or a complex Banach space. The norm on X and on the space $B(X)$ of all bounded linear operators on X will be denoted by $||\cdot||$.

Definition 1.1. A familly $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ of bounded linear operators is called an **evolution operator** if the following properties are satisfied:

e_1) $\Phi(t,t) = I$, the identity operator on X;
e_2) $\Phi(t,s)\Phi(s,t_0) = \Phi(t,t_0)$, for all $t \geq s \geq t_0 \geq 0$;
e_3) for all $x \in X$ the function $\Phi(t,\cdot)x$ is continuous on $[0,t]$ and the function $\Phi(\cdot,t_0)x$ is continuous on $[t_0,\infty)$;
e_4) there exist $M \geq 1$, $\omega > 0$ such that

$$||\Phi(t,s)|| \leq Me^{\omega(t-s)}, \quad \forall t \geq s \geq 0.$$

Definition 1.2. An evolution operator $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ is said to be

i) **uniformly exponentially stable** (and we denote by **u.e.s.**) if there are $N \geq 1$ and $\nu > 0$ such that

$$||\Phi(t,s)|| \leq Ne^{-\nu(t-s)}, \quad \forall t \geq s \geq 0;$$

ii) **periodic** if there exists $\tau > 0$ such that

$$\Phi(t+\tau, s+\tau) = \Phi(t,s), \quad \forall t \geq s \geq 0.$$

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Remark 1.1. If $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is a C_0-semigroup on the Banach space X then $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ defined by

$$\Phi(t,s) = T(t-s), \quad \forall t \geq s \geq 0$$

is easily checked to be a τ-periodic evolution operator, for all $\tau > 0$.

In stability theory of C_0-semigroups in Banach spaces a notable result is given by:

Theorem 1.1 (Neerven). *Let X be a complex Banach space and let $\mathbf{T}$ be a C_0-semigroup on X. If E is a Banach function space over $\mathbf{R}_+$ with $\lim\limits_{t\to\infty} \Psi_E(t) = \infty$ and with the property that for every $x \in X$ the map $t \to ||T(t)x||$ belongs to E then $\mathbf{T}$ is u.e.s.*

Nerveen's proof (see [**5**]) is valid only for complex Banach spaces. In his proof Neerven used a lemma which is not valid in real Banach spaces and in [**3**] he gives an example which shows this fact.

In this paper we shall give the discrete-time variant of the Neerven's theorem but for periodic evolution operators. We shall also obtain neccesary and sufficient conditions for uniform exponential stability of periodic evolution operators. All our results are valid in real or complex Banach spaces.

2. Banach Function Spaces

Let (Ω, Σ, μ) be a positive σ-finite measure space. By $M(\mu)$ we denote the linear space of μ-measurable functions $f\colon \Omega \to \mathbf{C}$, identifying the functions which are equal μ-a.e.

Definition 2.1. A **Banach function norm** is a function $N\colon M(\mu) \to [0,\infty]$ with the following properties:

n_1) $N(f) = 0$ if and only if $f = 0$ μ-a.e.;
n_2) if $|f| \leq |g|$ μ-a.e. then $N(f) \leq N(g)$;
n_3) $N(af) = |a|N(f)$, for all scalars $a \in \mathbf{C}$ and all f with $N(f) < \infty$;
n_4) $N(f+g) \leq N(f) + N(g)$, for all $f, g \in M(\mu)$.

Let $E = E_N$ be the set defined by:

$$E := \{f \in M(\mu) : ||f||_E := N(f) < \infty\}.$$

It is easily seen that $(E, ||\cdot||_E)$ is a normed linear space. If E is complete then E is called *Banach function space* over Ω.

Remark 2.1. E is an ideal in $M(\mu)$, i.e. if $|f| \leq |g|$ μ-a.e. and $g \in E$ then also $f \in E$ and $||f||_E \leq ||g||_E$.

Remark 2.2. If $f_n \to f$ in norm in E, then there exists a subsequence (f_{k_n}) converging to f pointwise (see [**1**]).

Let $(\Omega, \Sigma, \mu) = (\mathbf{R}_+, \mathcal{L}, m)$ where $\mathcal{L}$ is the σ-algebra of all Lebesgue measurable sets $A \subset \mathbf{R}_+$ and m the Lebesgue measure. For a Banach function space over $\mathbf{R}_+$ we define

$$\Psi_E \colon \mathbf{R}_+ \to \bar{\mathbf{R}}_+, \quad \Psi_E(t) := \begin{cases} ||\chi_{[0,t)}||_E, & \text{if } \chi_{[0t)} \in E \\ \infty, & \text{if } \chi_{[0,t)} \notin E \end{cases}$$

where $\chi_{[0,t)}$ denotes the characteristic function of $[0, t)$. The function Ψ_E is called **the fundamental function** of the Banach space E.

In what follows we shall denote by $\mathcal{B}(\mathbf{R}_+)$ the set of all Banach function spaces with the property that $\lim\limits_{t\to\infty} \Psi_E(t) = \infty$.

Let $(\Omega, \Sigma, \mu) = (\mathbf{N}, \mathcal{P}(\mathbf{N}), \mu_c)$ where μ_c is the countable measure and E a Banach function space over $\mathbf{N}$ (in this case E is called **Banach sequence space**). We define

$$\Psi_E \colon \mathbf{N}^* \to \bar{\mathbf{R}}_+, \quad \Psi_E(n) := \begin{cases} ||\chi_{\{0,\ldots,n-1\}}||_E, & \text{if } \chi_{\{0,\ldots,n-1\}} \in E \\ \infty, & \text{if } \chi_{\{0,\ldots,n-1\}} \notin E \end{cases}$$

called **the fundamental function of** E.

In what follows we denote by $\mathcal{B}(\mathbf{N})$ the set of all Banach sequence spaces E with $\lim\limits_{n\to\infty} \Psi_E(n) = \infty$.

Remark 2.3. If E is a Banach function space over $\mathbf{R}_+$ and $(t_n)_n$ an increasing sequence of real positive numbers with $\lim\limits_{n\to\infty} t_n = \infty$ we define

$$S_E := \left\{ (\alpha_n)_n : \sum_{n=0}^{\infty} \alpha_n \chi_{[t_n, t_{n+1})} \in E \right\}.$$

It is easily checked that S_E is a Banach sequence space with respect to the norm

$$||(\alpha_n)_n||_{S_E} := \left\| \sum_{n=0}^{\infty} \alpha_n \chi_{[t_n, t_{n+1})} \right\|_E .$$

Moreover, we have that $E \in \mathcal{B}(\mathbf{R}_+)$ if and only if $S_E \in \mathcal{B}(\mathbf{N})$.

In what follows we shall give some examples of Banach sequence spaces.

Example 2.1. If $p \in [1, \infty)$ then $E = l^p$ with

$$||s||_p = \left(\sum_{n=0}^{\infty} |s(n)|^p \right)^{\frac{1}{p}}$$

has the property that $E \in \mathcal{B}(\mathbf{N})$.

Example 2.2. If $p \in [1, \infty)$ and $\alpha = (\alpha_n)$ is a sequence of strict positive real numbers with

$$\sum_{n=0}^{\infty} \alpha_n = \infty$$

then the space $E = l^p_\alpha$ of all sequences $s\colon \mathbf{N} \to \mathbf{C}$ with the property

$$\sum_{n=0}^{\infty} \alpha_n |s(n)|^p < \infty$$

is a Banach sequence space with respect to the norm:

$$||s||_{l^p_\alpha} = \left(\sum_{n=0}^{\infty} \alpha_n |s(n)|^p \right)^{\frac{1}{p}}.$$

Because

$$\Psi_{l^p_\alpha}(n) = \left(\sum_{j=0}^{n-1} \alpha_j \right)^{\frac{1}{p}}$$

it follows that $l^p_\alpha \in \mathcal{B}(\mathbf{N})$.

Example 2.3. If $p \in [1, \infty)$ and $k = (k_n)$ is a sequence of natural numbers with the following properties:

(i) $k_n \geq n$, for all $n \in \mathbf{N}$;

(ii) $\overline{\lim\limits_{n\to\infty}} (k_n - n) = \infty$

then the space E^p_k of all sequences $s\colon \mathbf{N} \to \mathbf{C}$ with the property

$$||s||_{E^p_k} = \sup_{n \in \mathbf{N}} \left(\sum_{j=n}^{k_n} |s(j)|^p \right)^{\frac{1}{p}}$$

is a Banach sequence space with $E^p_k \in \mathcal{B}(\mathbf{N})$.

3. Preliminary Results

First we will need the following technical lemmas.

Lemma 3.1. *Let A be a bounded linear operator on a Banach space X whose spectral radius $r(A) \geq 1$. Then for all $\varepsilon \in (0,1)$ and $n \in \mathbf{N}$ there is $x \in X$ with $||x|| = 1$ and*

$$||A^j x|| \geq \varepsilon, \quad \textit{for all } j \in \{0, \dots, n\}.$$

Proof. Let $\lambda \in \sigma(A)$ with $|\lambda| = r(A)$. Then there is $(x_n)_n \subset X$ with $||x_n|| = 1$ and $Ax_n - \lambda x_n \to 0$.

Since $A^j x_n - \lambda^j x_n \to 0$, for every $j \in \mathbf{N}$, it follows that for all $\varepsilon \in (0,1)$ and $n \in \mathbf{N}$ there exists $n_0 \in \mathbf{N}$ such that

$$||A^j x_{n_0} - \lambda^j x_{n_0}|| < 1 - \varepsilon, \quad \text{for all } j \in \{0, \dots, n\}.$$

Hence

$$|\lambda|^j = ||\lambda^j x_{n_0}|| \leq ||\lambda^j x_{n_0} - A^j x_{n_0}|| + ||A^j x_{n_0}|| < 1 - \varepsilon + ||A^j x_{n_0}||$$

which implies

$$||A^j x_{n_0}|| > |\lambda|^j - 1 + \varepsilon \geq \varepsilon. \qquad \square$$

Lemma 3.2. *Let E be a Banach sequence space and let (A_n) be a sequence of bounded linear operators on X with the property that for every $x \in X$ the sequence*

$$s_x \colon \mathbf{N} \to \mathbf{R}_+, \quad s_x(n) = ||A_n x||$$

belongs to E. Then there exists $M > 0$ such that

$$||s_x||_E \leq M||x||, \quad \textit{for all } x \in X.$$

Proof. Let $E(X)$ be the set of all sequences $s \colon \mathbf{N} \to X$ with $||s|| \in E$. $E(X)$ is a Banach space with respect to the norm

$$|s|_{E(X)} := ||\,||s||\,||_E.$$

We consider the map $S \colon X \to E(X)$ defined by

$$S(x)(n) = A_n x.$$

Using the closed graph theorem it is sufficient to show that the linear map S is closed.

Indeed, if $x_n \to x$ in X and $S(x_n) \to s$ in $E(X)$ then from Remark 2.2 it follows that there exists a subsequence (x_{k_n}) such that $S(x_{k_n}) \to s$ pointwise.

Since for every $j \in \mathbf{N}$ we have

$$S(x_{k_n})(j) = A_j x_{k_n} \longrightarrow A_j x = S(x)(j)$$

it follows that $s = S(x)$, which proves the closedness of S. $\square$

Lemma 3.3. *Let A be a bounded operator on X and E a Banach sequence space with $E \in \mathcal{B}(\mathbf{N})$. If there exists a sequence $(k_n) \subset \mathbf{N}$ such that for every $x \in X$ the sequence*

$$s_x : \mathbf{N} \to \mathbf{R}_+, \quad s_x(n) = ||A^{k_n} x||$$

belongs to E then the spectral radius $r(A) < 1$.

Proof. Suppose the contrary, i.e. $r(A) \geq 1$. Let $n \in \mathbf{N}^*$ and

$$p_n = \max\{k_0, k_1, \ldots, k_n\}.$$

From Lemma 3.1. it follows that there exists $x_n \in X$ with $||x_n|| = 1$ and

$$||A^j x_n|| > \frac{1}{2}, \qquad \text{for all } j \in \{0, \ldots, p_n\}.$$

Hence we have

$$\chi_{\{0,\ldots,n-1\}} < 2 s_{x_n}$$

which implies

$$\Psi_E(n) = ||\chi_{\{0,\ldots,n-1\}}||_E \leq 2||s_{x_n}||_E \leq 2M$$

where M is given by Lemma 3.2.

This fact contradicts the assumption $E \in \mathcal{B}(\mathbf{N})$. □

Lemma 3.4. *Let $\Phi = \{\Phi(t,s)\}_{t \geq s \geq 0}$ be a τ-periodic evolution operator and $V = \Phi(\tau, 0)$. Then Φ is u.e.s. if and only if $r(V) < 1$.*

Proof. For all $n \in \mathbf{N}^*, V^n = \Phi(n\tau, 0)$. Hence, it follows that if Φ is u.e.s. then $r(V) < 1$.

Conversely, if $r(V) < 1$ there exists $\nu > 0$ such that $r(V) < e^{-\nu\tau}$ and exists $n_0 \in \mathbf{N}^*$ with

$$||V^n|| \leq e^{-\nu n \tau}, \qquad \forall n \geq n_0.$$

For the beginning let us prove that there exists $K > 0$ with

$$||\Phi(t,0)|| \leq K e^{-\nu t}, \qquad \forall t \geq 0.$$

We denote by $M_1 = \sup\{||\Phi(t,s)|| : t, s \in [0,\tau], t \geq s\}$. If $t = n\tau + r$ with $n \in \mathbf{N}$ and $r \in [0, \tau)$ then

$$||\Phi(t,0)|| \leq ||\Phi(t, n\tau)||\,||\Phi(n\tau, 0)|| \leq K e^{-\nu t}, \quad \forall t \geq 0,$$

where $K = \max\{M_1^{n_0} e^{n_0 \tau \nu}, M_1 e^{\tau\nu}\}$.

Let now $t \geq s \geq 0$, $t = n\tau + r$, $s = k\tau + u$, with $n \geq k$ and $r, u \in [0, \tau)$. If $n = k$ then

$$||\Phi(t,s)|| = ||\Phi(r,u)|| \leq M_1 \leq M_1 e^{\nu\tau} e^{-\nu(t-s)},$$

and if $n \geq k+1$ then

$$\begin{aligned} ||\Phi(t,s)|| &\leq ||\Phi(t,(k+1)\tau)||\,||\Phi((k+1)\tau, s)|| \\ &\leq ||\Phi(t-(k+1)\tau, 0)||\,||\Phi(\tau, u)|| \leq M_1 K e^{\nu\tau} e^{-\nu(t-s)}. \end{aligned}$$

It follows that Φ is u.e.s. □

4. The Main Results

In this section we shall give necessary and sufficient conditions for uniform exponential stability of periodic evolution operators in Banach spaces.

Our main result is:

Theorem 4.1. *Let $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be a periodic evolution operator on a Banach space X. Then the following assertions are equivalent:*

i) *Φ is u.e.s.;*

ii) *there are a Banach sequence space $E \in \mathcal{B}(\mathbf{N})$ and a sequence $(t_n) \subset \mathbf{R}_+$ such that for all $x \in X$ the map*

$$s_x \colon \mathbf{N} \to \mathbf{R}_+, \quad s_x(n) = ||\Phi(t_n, 0)x||$$

belongs to E.

Proof. **Necessity.** It is sufficient to consider

$$E = l^1 \in \mathcal{B}, t_n = n.$$

Sufficiency. Since Φ is periodic there is $\tau > 0$ such that

$$\Phi(t+\tau, s+\tau) = \Phi(t,s), \qquad \forall t \geq s \geq 0.$$

We denote by $V = \Phi(\tau, 0)$. Let $k_n = [\frac{t_n}{\tau}] + 1$, for all $n \in \mathbf{N}$.

Since Φ has exponential growth there exist $M \geq 1, \omega > 0$ such that

$$||\Phi(t,s)|| \leq Me^{\omega(t-s)}, \qquad \forall t \geq s \geq 0.$$

Then

$$||V^{k_n}x|| = ||\Phi(\tau k_n, 0)x|| \leq ||\Phi(\tau k_n, t_n)||\,||\Phi(t_n,0)x|| \leq Me^{\omega\tau}||\Phi(t_n,0)x||,$$

for all $x \in X$ and $n \in \mathbf{N}$.

Since for all $x \in X, s_x$ belongs to E and E is an ideal we obtain that for every $x \in X$ the map

$$u_x \colon \mathbf{N} \to \mathbf{R}_+, \quad u_x(n) = ||V^{k_n}x||$$

belongs to E. Using Lemma 3.3. we obtain that $r(V) < 1$ and from Lemma 3.4. Φ is u.e.s. ∎

In what follows we are going to apply Theorem 4.1. to certain Banach sequence spaces.

Corollary 4.1. *Let $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be a periodic evolution operator on a Banach space X. The following assertions are equivalent:*

i) *Φ is u.e.s.;*

ii) *there are two sequences of positive real numbers (α_n) and (t_n) such that:*

ii') $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$*;*

ii") $\sum\limits_{n=0}^{\infty} \alpha_n ||\Phi(t_n,0)x||^p < \infty,$ *for all $x \in X$ and $p \in [1,\infty)$;*

iii) *there are two sequences of positive real numbers (α_n) and (t_n) such that:*

iii') $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$*;*

iii") *there exists $p \in [1,\infty)$ such that*

$$\sum_{n=0}^{\infty} \alpha_n ||\Phi(t_n,0)x||^p < \infty, \ \textit{for all } x \in X.$$

Proof. (i) $\Rightarrow$ (ii): It is sufficient to choose $\alpha_n = 1$ and $t_n = n$.
(ii) $\Rightarrow$ (iii): It is trivial.
(iii) $\Rightarrow$ (i): Without lost of generality we may assume that $\alpha_n > 0$ for all $n \in \mathbf{N}$.

Let l^p_α be the Banach sequence space considered in Example 2.2. Using the hypothesis it follows that for every $x \in X$ the sequence

$$s_x\colon \mathbf{N} \to \mathbf{R}_+, \quad s_x(n) = ||\Phi(t_n,0)x||$$

defines an element of l^p_α. Using the Theorem 4.1. it follows that Φ is u.e.s. $\square$

Corollary 4.2. *Let $(k_n) \subset \mathbf{N}$ be a sequence such that $k_n \geq n$ for all $n \in \mathbf{N}$ and*

$$\overline{\lim_{n\to\infty}} (k_n - n) = \infty.$$

Let $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be a periodic evolution operator on the Banach space X. Then Φ is u.e.s. if and only if there are $p \in [1,\infty)$ and a sequence $(t_n) \subset \mathbf{R}_+$ such that

$$\sup_{n\in\mathbf{N}} \sum_{j=n}^{k_n} ||\Phi(t_j,0)x||^p < \infty, \quad \textit{for all } x \in X.$$

Proof. **Necessity** is trivial (for $p = 1$ and $t_n = n$).
Sufficiency. It follows by applying Theorem 4.1. for the space E^p_k defined in Example 2.3. $\square$

As a consequence of Theorem 4.1. we obtain a generalization for Neerven's theorem

Theorem 4.2. *Let $\Phi = \{\Phi(t,s)\}_{t\geq s\geq 0}$ be a periodic evolution operator on the Banach space X and E a Banach function space with $E \in \mathcal{B}(\mathbf{R}_+)$. If for every $x \in X$ the map*

$$f_x \colon \mathbf{R}_+ \to \mathbf{R}_+, \qquad f_x(t) = ||\Phi(t,0)x||$$

belongs to E then Φ is u.e.s.

Proof. Let τ be the period of Φ and

$$S_E = \left\{ (\alpha_n) : \sum_{n=0}^{\infty} \alpha_n \chi_{[n\tau,(n+1)\tau)} \in E \right\}.$$

From Remark 2.3. using the hypothesis that $E \in \mathcal{B}(\mathbf{R}_+)$ it follows that $S_E \in \mathcal{B}(\mathbf{N})$. Let

$$s_x \colon \mathbf{N} \to \mathbf{R}_+, \quad s_x(n) = ||\Phi((n+1)\tau,0)x||.$$

We define

$$g_x \colon \mathbf{R}_+ \to \mathbf{R}_+, \quad g_x(t) = \sum_{n=0}^{\infty} s_x(n)\chi_{[n\tau,(n+1)\tau)}(t).$$

For every $x \in X$ we have that

$$g_x(t) = \sum_{n=0}^{\infty} ||\Phi((n+1)\tau,0)x||\chi_{[n\tau,(n+1)\tau)}(t), \quad \forall t \geq 0.$$

Let $t \in \mathbf{R}_+$. There is $n \in \mathbf{N}$ such that $t \in [n\tau,(n+1)\tau)$. It follows that

$$\begin{aligned} g_x(t) &= ||\Phi((n+1)\tau,0)x|| \leq ||\Phi((n+1)\tau,t)||\,||\Phi(t,0)x|| \\ &= ||\Phi(\tau,t-n\tau)||\,||\Phi(t,0)x|| \leq M_2||\Phi(t,0)x||, \end{aligned}$$

where $M_2 = \sup\{||\Phi(\tau,s)|| : s \in [0,\tau]\}$.

Hence

$$g_x(t) \leq M_2\, f_x(t), \qquad \forall t \geq 0.$$

Since $f_x \in E$ and E is an ideal we obtain that $g_x \in E$. Using the definition of S_E we have that $s_x \in S_E$.

By applying Theorem 4.1 it follows that Φ is u.e.s.

References

1. Meyer-Nieberg P., *Banach Lattices*, Springer Verlag, Berlin – Heidelberg – New York, 1991.

2. Nagel R. (Ed.), *One-parameter semigroups of positive operators*, Lecture Notes in Math., Vol. 1184, Springer-Verlag, New York, 1986.

3. van Neerven J. M. A. M., *On the orbits of an operator with spectral radius one*, Czech. Math. J. **45**(120) (1995), 495–502.

4. ______, *Exponential Stability of Operators and Operator Semigroups*, J. Func. Anal. **130** (1995), 293–309.

5. ______, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Theory Advances and Applications, vol. 88, Birkhauser, 1996.

6. Pazy A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, Berlin – Heidelberg – New York, 1983.

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