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ON THE COMPUTATION OF SYMBOLIC POWERS OF SOME CURVES IN $\boldsymbol{A}^4$

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Abstract. In the paper the powers of the prime ideal P of monomial Gorenstein curves in affine 4-space are investigated. The equality of the second ordinary and symbolic powers is shown and 6 elements generating the symbolic cube of P over the ordinary one are found. An example of symbolic Rees algebra of finite type is presented.

1. Introduction

Monomial curves $C(n_1, n_2, n_3)$ in affine space $\boldsymbol{A}^3$ with the generic zero $(t^{n_1}, t^{n_2}, t^{n_3})$ appear frequently in mathematic considerations. It is known that minimal number $\mu(P)$ of the associated prime P of this curve is either 2 or 3 and $\mu(P) = 2$ (i.e. P is a complete intersection) iff the second ordinary and symbolic powers coincide, $P^2 = P^{(2)}$. This is also equivalent to the fact that the numeric semigroup $S = \langle n_1, n_2, n_3\rangle$ is symmetric. In case $\mu(P) = 3$ the curve C is a set-theoretic complete intersection and $P^2 \neq P^{(2)}$; $P^{(2)} = \left(P^2, \Delta\right)$, see **[7]**. Schenzel and Vasconcelos also showed that in some cases the symbolic Rees algebra $S(P) = \oplus_{n\geq 0} P^{(n)}$ is an A-algebra of finite type.

W. V. Vasconcelos noted in **[8]** there were very few general descriptions of the equations of the symbolic cube algebra $R\left[It, I^{(2)}t^2, I^{(3)}t^3\right]$ of an ideal I of a regular local ring R. Schenzel found them for a certain classe of non-complete intersection prime ideals of monomial curves in $\boldsymbol{A}^3$ (see **[7]**) and showed that the module $P^{(3)}/P.P^{(2)}$ is generated by at most 3 elements (see **[5]**). More complete picture on symbolic powers and blowup algebras can be found in Vasconcelos nice book **[8]**. The above mentioned results are on pp. 201, 203, 221.

Our aim is to extend some techniques and results of **[7]** in order to get information on ideals of some monomial curves $C(n_1, n_2, n_3, n_4)$ in $\boldsymbol{A}^4$. Here the symmetry of $S = \langle n_1, n_2, n_3, n_4\rangle$ does not imply C is complete intersection; the ideal P of the curve C can be generated by 5 elements given by the Pfaffians of 4×4 minors obtained from a 5×5 skew-symmetric matrix $\boldsymbol{I}_5(P)$ by deleting the

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i-th row and column (see **[3]** and **[1]**). It is not difficult to show that this matrix is (in notation of **[1]**, see Proposition 2.1):

$$\boldsymbol{I}_5(P) = \begin{pmatrix} 0 & 0 & x_3^{\alpha_{13}} & x_1^{\alpha_{21}} & x_2^{\alpha_{32}} \\ 0 & 0 & x_1^{\alpha_{31}} & x_4^{\alpha_{14}} & x_3^{\alpha_{43}} \\ -x_3^{\alpha_{13}} & -x_1^{\alpha_{31}} & 0 & x_2^{\alpha_{42}} & x_4^{\alpha_{24}} \\ -x_1^{\alpha_{21}} & -x_4^{\alpha_{14}} & -x_2^{\alpha_{42}} & 0 & 0 \\ -x_2^{\alpha_{32}} & -x_3^{\alpha_{43}} & -x_4^{\alpha_{24}} & 0 & 0 \end{pmatrix}$$

From **[3]** one can see that P^2 is generated by all 4×4 minors of this matrix. This can be shown also by direct calculation. It is also known that P is in the linkage class of a complete intersection (i.e. P is a licci ideal, see **[9]** or **[3]**). From **[4]**, Corollary 2.9 it follows then that the second ordinary and symbolic powers coincide but the third ones do not, $P^3 \neq P^{(3)}$.

We will prove this equality resp. non-equality again by computing some lengths (Corollaries 3.6 and 3.8) and show that symbolic cube $P^{(3)} = \left(P^3, d_1, \ldots, d_6\right)$ for some explicit given elements d_i (Theorem 3.9). Though the latter is made just for several cases one can see how it works in general. In the last section we compute the symbolic Rees algebra of the prime ideal P of the curve $\left(t^5, t^6, t^7, t^8\right)$ and show it is an A-algebra of finite type. By virtue of **[9]** the ideal P is then set-theoretic complete intersection as known also by Bresinsky **[2]**.

The computation of test examples (some of them are included in this paper, see Example 3.1 and the Example in the 4th section) has been made using Computer algebra system Macaulay created by D. Bayer and M. Stillman.

2. Monomial Curves

Let n_i, $i = 1, 2, 3, 4$ be positive integers with g.c.d. $(n_1, n_2, n_3, n_4) = 1$ and $C\,(n_1, n_2, n_3, n_4)$ a curve in $\boldsymbol{A}_k^4$, k an arbitrary field, given parametricaly by $x_i = T^{n_i}$ for $i = 1, 2, 3, 4$. Let P be the corresponding prime ideal in $A = k\,[x_1, x_2, x_3, x_4]_{(x_1, x_2, x_3, x_4)}$. Putting $\deg x_i = n_i$ A becomes a graded k-algebra and P a homogeneous prime ideal with $\dim(p) = 1$ and height $ht\,(p) = 3$.

Denote $S = \langle n_1, n_2, n_3, n_4\rangle$ the additive semigroup generated by n_1, n_2, n_3, n_4, $S = \{z \in \boldsymbol{Z}; z = \sum z_i n_i, z_i \in \boldsymbol{N} \cup \{0\}\}$. Assume no proper subset of $\{n_1, n_2, n_3, n_4\}$ generates S. The semigroup S is said to be symmetric if there is an integer $m \in \boldsymbol{Z}$ such that for all $z \in \boldsymbol{Z}$: $z \in S \Leftrightarrow m - z \notin S$ (see e.g. **[6]** or **[1]**). H. Bresinsky has shown that if S is symmetric then the prime ideal P has either 3 generators (i.e. P is a complete intersection) or P is generated minimally by 5 exactly (up to isomorphism) given elements (see **[1]**, Theorems 3 and 5):

Proposition 2.1. *The semigroup $S = \langle n_1, n_2, n_3, n_4\rangle$ is symmetric and P not a complete intersection if and only if $P = (f_1, f_2, f_3, f_4, g)$, where the polynomials $f_1 = x_1^{\alpha_1} - x_3^{\alpha_{13}} x_4^{\alpha_{14}}$, $f_2 = x_2^{\alpha_2} - x_1^{\alpha_{21}} x_4^{\alpha_{24}}$, $f_3 = x_3^{\alpha_3} - x_1^{\alpha_{31}} x_2^{\alpha_{32}}$, $f_4 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}$ are unique up to isomorphism and $g = x_3^{\alpha_{43}} x_1^{\alpha_{21}} - x_2^{\alpha_{32}} x_4^{\alpha_{14}}$.*

In this case then

$$\begin{aligned} n_1 &= \alpha_2\alpha_3\alpha_{14} + \alpha_{32}\alpha_{13}\alpha_{24}, \\ n_2 &= \alpha_3\alpha_4\alpha_{21} + \alpha_{31}\alpha_{43}\alpha_{24}, \\ n_3 &= \alpha_1\alpha_4\alpha_{32} + \alpha_{14}\alpha_{42}\alpha_{31}, \\ n_4 &= \alpha_1\alpha_2\alpha_{43} + \alpha_{42}\alpha_{21}\alpha_{13}, \end{aligned}$$

with $\alpha_i > 0$, $0 < \alpha_{ji} < \alpha_i$, $1 \le i, j \le 4$ *and* $\alpha_1 = \alpha_{31} + \alpha_{21}$, $\alpha_2 = \alpha_{32} + \alpha_{42}$, $\alpha_3 = \alpha_{13} + \alpha_{43}$, $\alpha_4 = \alpha_{24} + \alpha_{14}$.

3. Symbolic Powers

Let (A, M) be local noetherian ring with its maximal ideal M. For a prime ideal P denote $P^{(n)} = P^n A_P \cap A$ the n-th symbolic power of P.

For our calculations we need some lemmas.

Lemma 3.1. *Let Q be an M-primary ideal of a local ring (A, M) and x an element of A. Then*

$$L\left(A/\left(Q, x\right)\right) = L\left(A/Q\right) - L\left(A/Q : x\right).$$

Lemma 3.2. *Let P be a prime ideal of a regular local ring (A, M) with* $\dim A = 4$ *and* $\dim P = \dim A/P = 1$. *Let I be an ideal and $x \notin P$ an element such that*

1. $P^n \subseteq I \subseteq P^{(n)}$
2. $L\left(A/\left(I, x\right)\right) = \binom{n+2}{3} \cdot e\left(x, A/P\right).$

Then $I = P^{(n)}$.

The lemma and its proof is a modification of the Lemma 2.3 of **[7]**. Note that x is a parameter in A/P and $L\left(A_P/P^n.A_P\right) = \binom{n+2}{3}$ since $P^n.A_P$ is a power of the maximal ideal in the 3-dimensional regular local ring A_P.

Putting I $= P^n$ we can find whether $P^n = P^{(n)}$ or not. Then the converse is also true:

Lemma 3.3. *Let P, x, A be as above. Then $P^n = P^{(n)}$ if and only if* $L(A/(P^n, x)) = L(A_P/P^n.A_P).e\left(x, A/P\right)$.

Proof. If $P^n = P^{(n)}$ then A/P^n is Cohen-Macaulay and $L\left(A/\left(P^n, x\right)\right) = e\left(x, A/P^n\right) = L\left(A_P/P^n.A_P\right).e\left(x, A/P\right)$. If the equation holds then $L(A/(P^n, x)) = e\left(x, A/P^n\right)$, the ring A/P^n is Cohen-Macaulay and $P^n = P^{(n)}$. □

In the following let $P = (f_1, f_2, f_3, f_4, g)$ be the non-complete intersection prime ideal of local ring $A = k\left[x_1, x_2, x_3, x_4\right]_{(x_1, x_2, x_3, x_4)}$ corresponding to the curve $C\left(n_1, n_2, n_3, n_4\right)$ with $S = \langle n_1, n_2, n_3, n_4\rangle$ symmetric as in Proposition 2.1. Then it holds

Corollary 3.1. *$P^2 = P^{(2)}$ iff $L\left(A/\left(P^2, x_i\right)\right) = 4 \cdot n_i$ and $P^3 = P^{(3)}$ iff $L\left(A/\left(P^3, x_i\right)\right) = 10 \cdot n_i$.*

Proof. Putting $n = 2$ (or 3) we get $L\left(A_P/P^n.A_P\right) = 4$ (or 10). We use the previous lemma for $x = x_i$. Though it is known that $e\left(x_i, A/P\right) = n_i$, we prove it for some cases (used in the next Proposition) by our techniques. Consider four conditions on the numbers α_{ij} as follows :

a) $\alpha_{32} \leq \alpha_{42}$ and $\alpha_{13} \leq \alpha_{43}$
b) $\alpha_{32} \geq \alpha_{42}$ and $\alpha_{13} \leq \alpha_{43}$
c) $\alpha_{32} \leq \alpha_{42}$ and $\alpha_{13} \geq \alpha_{43}$
d) $\alpha_{32} \geq \alpha_{42}$ and $\alpha_{13} \geq \alpha_{43}$.

Then in the first 3 cases we show that $e\left(x_1, A/P\right) = n_1$ and in the last one that $e\left(x_4, A/P\right) = n_4$. In all cases the element x_1 (x_4) is a parameter in one-dimensional Cohen-Macaulay ring A/P and $e\left(x_i, A/P\right) = L\left(A/\left(P, x_i\right)\right)$. Thus it is enough to calculate the length of A/Q for $Q = \left(x_i, P\right)$, $i = 1, 4$.

In case a) we have $x_4^{\alpha_4+\alpha_{14}} \in Q$. Putting $Q_0 = \left(x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{\alpha_4+\alpha_{14}}\right)$ and $a_1 = x_2^{\alpha_{32}} x_4^{\alpha_{14}}$, $a_2 = x_3^{\alpha_{13}} x_4^{\alpha_{14}}$, $a_3 = x_4^{\alpha_4} - x_2^{\alpha_{42}} x_3^{\alpha_{43}}$, we get that $Q = \left(x_1, Q_0, a_1, a_2, a_3\right)$. Since

$$\begin{aligned} Q : a_1 &= (x_2^{\alpha_{42}}, x_3^{\alpha_3}, x_4^{\alpha_4}) =: Q_1 \\ (Q, a_1) : a_2 &= (x_2^{\alpha_{32}}, x_3^{\alpha_{43}}, x_4^{\alpha_4}) =: Q_2 \\ (Q, a_1, a_2) : a_3 &= (x_2^{\alpha_{32}}, x_3^{\alpha_{13}}, x_4^{\alpha_{14}}) =: Q_3 \end{aligned}$$

it holds

$$\begin{aligned} L\left(A/Q\right) = L\left(A/Q_o\right) - \sum L\left(A/Q_i\right) &= \alpha_2\alpha_3\left(\alpha_4 + \alpha_{14}\right) \\ &- \alpha_{42}\alpha_3\alpha_4 - \alpha_{32}\alpha_{43}\alpha_4 - \alpha_{32}\alpha_{13}\alpha_{14} = \alpha_2\alpha_3\alpha_{14} + \alpha_{32}\alpha_{13}\alpha_{24} = n_1. \end{aligned}$$

The calculations for the cases b) and c) are the same as the first one. In the last case $\alpha_{32} \geq \alpha_{42}$ and $\alpha_{13} \geq \alpha_{43}$ we show that $e\left(x_4, A/P\right) = n_4$. Here $Q = \left(x_4, P\right)$ and $x_3^{\alpha_3+\alpha_{43}} \in Q$. The calculations are then similar and we see that

$$\square \qquad e\left(x_4, A/P\right) = L\left(A/\left(P, x_4\right)\right) = \alpha_1\alpha_2\alpha_{43} + \alpha_{42}\alpha_{21}\alpha_{13} = n_4.$$

$\square$

Proposition 3.1. *Let $P = \left(f_1, f_2, f_3, f_4, g\right)$ be as before. Assume one of the conditions a),b),c),d) from the proof of the the Corollary 3.4 holds. Then in the cases a), b) and c) it follows that*

$$L\left(A/\left(P^2, x_1\right)\right) = 4 \cdot n_1$$

and in the case d) it holds

$$L\left(A/\left(P^2, x_4\right)\right) = 4 \cdot n_4.$$

Corollary 3.2. *The second symbolic and ordinary powers of the above prime ideal P coincide, i.e. $P^2 = P^{(2)}$.*

Remark 3.1. It follows then that A/P^2 is a Cohen-Macaulay ring. Though the ring A/P is Gorenstein (see [**2**] and [**3**]), the ring A/P^2 need not to be Gorenstein. Moreover in all (for me) know examples the type of Cohen-Macaulay ring A/P^2 is 10, so in these cases A/P^2 is not Gorenstein which is of type 1 (see [**6**, p. 195]). One can see this also from the resolution of A/P^2 (see [**3**]). It seems to me that it holds in general for all non-complete intersection Gorenstein prime ideals of monomial curves in $\boldsymbol{A}^4$.

Proof. The case a). It is easy to see that

$$\left(P^2, x_1\right) = \left(x_1, x_2^{2\alpha_2}, x_3^{2\alpha_3}, x_4^{2\alpha_4}, a_1, a_2, \ldots, a_{12}\right)$$

with

$$\begin{aligned}
a_1 &= x_2^{\alpha_2+\alpha_{32}} x_4^{\alpha_{14}}, & a_2 &= x_2^{2\alpha_{32}} x_4^{2\alpha_{14}},\\
a_3 &= x_2^{\alpha_{32}} x_4^{\alpha_4+\alpha_{14}}, & a_4 &= x_3^{\alpha_3+\alpha_{13}} x_4^{\alpha_{14}},\\
a_5 &= x_3^{2\alpha_{13}} x_4^{2\alpha_{14}}, & a_6 &= x_3^{\alpha_{13}} x_4^{\alpha_4+\alpha_{14}},\\
a_7 &= x_2^{\alpha_2} x_3^{\alpha_3}, & a_8 &= x_2^{\alpha_{32}} x_3^{\alpha_{13}} x_4^{2\alpha_{14}},\\
a_9 &= x_2^{\alpha_2} x_3^{\alpha_{13}} x_4^{\alpha_{14}}, & a_{10} &= x_2^{\alpha_{32}} x_3^{\alpha_3} x_4^{\alpha_{14}},\\
a_{11} &= x_2^{\alpha_2+\alpha_{42}} x_3^{\alpha_{43}}, & a_{12} &= x_2^{\alpha_{42}} x_3^{\alpha_3+\alpha_{43}}.
\end{aligned}$$

Put $I_0 = \left(x_2^{2\alpha_2}, x_3^{2\alpha_3}, x_4^{2\alpha_4}\right)$, $I_1 = (I_0, a_1)$, $\ldots$ and $I_k = (I_{k-1}, a_k)$ for all $k = 1, 2, \ldots, 12$.

Let's denote $I_{k-1} : a_k = J_k$ and $L\left(A/J_k\right) = A_k$. Then we get

$$\begin{aligned}
J_1 &= \left(x_2^{\alpha_{42}}, x_3^{2\alpha_3}, x_4^{\alpha_4+\alpha_{24}}\right), \ J_2 = \left(x_2^{\alpha_{42}}, x_3^{2\alpha_3}, x_4^{2\alpha_{24}}\right),\\
J_3 &= \left(x_2^{\alpha_{32}}, x_3^{2\alpha_3}, x_4^{\alpha_{24}}\right),\\
J_4 &= \left(x_2^{\alpha_2+\alpha_{32}}, x_3^{\alpha_{43}}, x_4^{\alpha_4+\alpha_{24}}, x_2^{2\alpha_{32}} x_4^{\alpha_{14}}, x_2^{\alpha_{32}} x_4^{\alpha_4}\right),\\
J_5 &= \left(x_2^{2\alpha_{32}}, x_3^{\alpha_{43}}, x_4^{2\alpha_{24}}, x_2^{\alpha_{32}} x_4^{\alpha_{24}}\right), \ J_6 = J_8 = \left(x_2^{\alpha_{32}}, x_3^{\alpha_{13}}, x_4^{\alpha_{24}}\right),\\
J_7 &= \left(x_2^{\alpha_2}, x_3^{\alpha_3}, x_4^{2\alpha_{14}}, x_2^{\alpha_{32}} x_4^{\alpha_{14}}, x_3^{\alpha_{13}} x_4^{\alpha_{14}}\right), \ J_9 = \left(x_2^{\alpha_{32}}, x_3^{\alpha_{43}}, x_4^{\alpha_{14}}\right),\\
J_{10} &= \left(x_2^{\alpha_{42}}, x_3^{\alpha_{13}}, x_4^{\alpha_{14}}\right), \ J_{11} = J_{12} = \left(x_2^{\alpha_{32}}, x_3^{\alpha_{13}}, x_4^{\alpha_{14}}\right),\\
A_1 &= 2\alpha_{42}\alpha_3\left(\alpha_4 + \alpha_{24}\right), \ A_2 = 4\alpha_{42}\alpha_3\alpha_{24},\\
A_3 &= 2\alpha_{32}\alpha_3\alpha_{24}, \ A_4 = 2\alpha_{32}\alpha_{43}\alpha_{14} + 3\alpha_{32}\alpha_{43}\alpha_{24} + \alpha_{42}\alpha_{43}\alpha_{14},\\
A_5 &= 3\alpha_{32}\alpha_{43}\alpha_{24}, \ A_6 = \alpha_{32}\alpha_{13}\alpha_{24},\\
A_7 &= 2\alpha_{32}\alpha_{13}\alpha_{14} + \alpha_{42}\alpha_{13}\alpha_{14} + \alpha_{32}\alpha_{43}\alpha_{14} + \alpha_{42}\alpha_{43}\alpha_{14}\\
A_8 &= \alpha_{32}\alpha_{13}\alpha_{24}, \ A_9 = \alpha_{32}\alpha_{43}\alpha_{14},\\
A_{10} &= \alpha_{42}\alpha_{13}\alpha_{14}, \ A_{11} = A_{12} = \alpha_{32}\alpha_{13}\alpha_{14}.
\end{aligned}$$

Using Lemma 3.1 and relations from Proposition 2.1 we get then $L\left(A/\left(P^2,x_1\right)\right) = 2\left(8\alpha_2\alpha_3\alpha_4+\alpha_3+\alpha_4\right)-\sum A_i = 4n_1$.

The proof of case d). The assumptions here imply that $x_3^{2(\alpha_3+\alpha_{43})}$ is in P^2 and $\left(P^2,x_4\right)=\Big(x_4,x_1^{2\alpha_1},x_2^{2\alpha_2},x_3^{2(\alpha_3+\alpha_{43})},d_1,d_2,\dots,d_{14}\Big)$ with

$$
\begin{aligned}
d_1 &= x_2^{\alpha_2+\alpha_{42}}x_3^{\alpha_{43}}, \quad d_2 = x_2^{2\alpha_{42}}x_3^{2\alpha_{43}}, \quad d_3 = x_2^{\alpha_{42}}x_3^{\alpha_3+2\alpha_{43}},\\
d_4 &= x_1^{\alpha_1+\alpha_{21}}x_3^{\alpha_{43}}, \quad d_5 = x_1^{2\alpha_{21}}x_3^{2\alpha_{43}}, \quad d_6 = x_1^{\alpha_{21}}x_3^{\alpha_3+\alpha_{43}},\\
d_7 &= x_1^{\alpha_1}x_2^{\alpha_2}, \quad d_8 = x_1^{\alpha_{31}}x_2^{\alpha_2+\alpha_{32}}, \quad d_9 = x_1^{\alpha_{21}}x_2^{\alpha_2}x_3^{\alpha_{43}},\\
d_{10} &= x_1^{\alpha_{21}}x_2^{\alpha_{42}}x_3^{2\alpha_{43}}, \quad d_{11} = x_1^{\alpha_1}x_2^{\alpha_{42}}x_3^{\alpha_{43}},\\
d_{12} &= x_1^{\alpha_1}x_3^{\alpha_3}-x_1^{\alpha_1+\alpha_{31}}x_2^{\alpha_{32}}, \quad d_{13} = x_2^{\alpha_{42}}x_3^{\alpha_3+\alpha_{43}}-x_1^{\alpha_{31}}x_2^{\alpha_2}x_3^{\alpha_{43}},\\
d_{14} &= x_3^{2\alpha_3}-2x_1^{\alpha_{31}}x_2^{\alpha_{32}}x_3^{\alpha_3}+x_1^{2\alpha_{31}}x_2^{2\alpha_{32}}.
\end{aligned}
$$

Then we can calculate as before that

$$L\left(A/\left(P^2,x_4\right)\right) = 8\cdot\alpha_1\alpha_4\left(\alpha_3+\alpha_{43}\right)-\sum D_i = 4n_4,$$

having $D_i = L\left(A/\left(I_{k-1}:d_i\right)\right)$ and $I_k = \left(I_{k-1},d_k\right)$. □

Proposition 3.2. *Let $P=(f_1,f_2,f_3,f_4,g)$ be the non-complete intersection prime ideal corresponding to the curve $C(n_1,n_2,n_3,n_4)$ with $S=\langle n_1,n_2,n_3,n_4\rangle$ symmetric; assume one of the following four conditions on the numbers α_{ij} is satisfied*

a-1) $\alpha_{32}\le\alpha_{42},\alpha_{13}\le\alpha_{43}$ *and* $\alpha_{14}\le\alpha_{24}$ *or*
a-2) $\alpha_{32}\le\alpha_{42},\alpha_{13}\le\alpha_{43}$ *and* $\alpha_{24}\le\alpha_{14}$ *or*
b-1) $\alpha_{42}\le\alpha_{32},\alpha_{13}\le\alpha_{43}$ *and* $\alpha_{14}\le\alpha_{24}$ *or*
b-2) $\alpha_{42}\le\alpha_{32},\alpha_{13}\le\alpha_{43}$ *and* $\alpha_{24}\le\alpha_{14}$.

Then $L\left(A/\left(P^3,x_1\right)\right) = 10\cdot n_1+6\cdot t$, *where* $t=\alpha_{32}\alpha_{13}\alpha_{14}$ *in case* a-1), $t=\alpha_{32}\alpha_{13}\alpha_{24}$ *in case* a-2), $t=\alpha_{42}\alpha_{13}\alpha_{14}$ *in case* b-1), $t=\alpha_{42}\alpha_{13}\alpha_{24}$ *in case* b-2), *i.e. t is a product of* 3 *numbers:* $\min\{\alpha_{32},\alpha_{42}\}$, $\min\{\alpha_{13},\alpha_{43}\}$, $\min\{\alpha_{14},\alpha_{24}\}$.

The proof of this proposition is made in the same way as the proof of the fact $L\left(A/\left(P^2,x_1\right)\right)=4\cdot n_1$ (see Proposition 3.1). It is just more complicated and the calculations are rather long and unpleasant. In the simplest case a-1) the ideal $\left(P^3,x_1\right)$ is generated by monomials and contains $x_4^{3\alpha_4}$; in the other cases this ideal is no more monomial and the lowest power of x_4 in it is $x_4^{2\alpha_4+2\alpha_{14}}$ in case a-2) and $x_4^{3\alpha_4+3\alpha_{14}}$ in the last 2 cases.

Corollary 3.3. *Under the assumptions of the previous Proposition* 3.2 *is* $P^3\neq P^{(3)}$.

Remark 3.2. From the Corollaries 3.2 and 3.3 we get that S is symmetric and $P^3=P^{(3)}$ iff P is a complete intersection.

Let's clear the further general computations by an example.

Example 3.1. The numerical semigroup $\mathbf{S}_k = \langle 5, 5k+1, 5k+2, 5k+3\rangle$ is symmetric for all $k \geq 1$. The corresponding prime ideal of the curve $C(5k+2, 5; 5k+1; 5k+3)$ is $p = (x_1^2 - x_3x_4, x_2^{2k+1} - x_1x_4, x_3^2 - x_1x_2^k, x_4 - x_2^{k+1}x_3, x_1x_3 - x_2^kx_4)$. Take $k=1$ and $S_1 = \langle 5,6,7,8\rangle$. We use in the following x, y, z, w instead of x_1, x_2, x_3, x_4. Then the prime ideal for $C\,(5,6,7,8)$ given by computer program Macaulay is $P = (F_1, \ldots, F_5)$ where $F_1 = y^2 - xz$, $F_2 = yz - xw$, $F_3 = z^2 - yw$, $F_4 = x^3 - zw$, $F_5 = x^2y - w^2$. There are relations

$$\begin{aligned}
R_1&: \quad zF_1 - yF_2 + xF_3 = 0\\
R_2&: \quad wF_1 - zF_2 + yF_3 = 0\\
R_3&: \quad -wF_2 - yF_4 + xF_5 = 0\\
R_4&: \quad -x^2F_1 - wF_3 - zF_4 + yF_5 = 0\\
R_5&: \quad -x^2F_2 - wF_4 + zF_5 = 0.
\end{aligned}$$

From the relations R_1, R_2, R_3 we can derive a new one

$$y\left(F_2^2F_5 - F_1F_3F_5 - F_2F_3F_4\right) = w\left(F_1^2F_5 + F_2^2F_3\right) \;.$$

Since $y \notin P$ there is an element d_1 such that $yd_1 = F_1^2F_5 + F_2^2F_3$ and $d_1 \in P^{(3)}$. In this way we can find further relations and elements $d_2 \ldots, d_6$:

$$\begin{aligned}
&wd_2 = F_5^3 - xF_1F_4F_5 - xF_2F_4^2,\; zd_3 = xF_1F_2^2 - F_3^2F_4,\\
&wd_4 = F_3F_4F_5 - F_2F_5^2 + xF_1F_2F_4,\; zd_5 = F_3^3 - xF_1^2F_2 + F_1F_3F_5,\\
&zd_6 = F_3F_4F_5 - xF_1F_2F_4 + xF_2^2F_3.
\end{aligned}$$

Then

$$\begin{aligned}
&d_1 \equiv yz^4 - y^3w^2 - y^2z^2w\,(\bmod x)\,, d_2 \equiv w^5\,(\bmod x)\,,\\
&d_3 \equiv z^4w - 2yz^2w^2 + y^2w^3\,(\bmod x)\,, d_4 \equiv z^3w^2 - 2\mathrm{yzw}^3\,(\bmod x)\,,\\
&d_5 \equiv z^5 - 3yz^3w + 2y^2zw^2\,(\bmod x)\,, d_6 \equiv z^2w^3 - yw^4\,(\bmod x)\,.
\end{aligned}$$

It is easy to see that all the elements d_i are different.

We show that $P^{(3)} = (P^3, d_1, \ldots, d_6)$. First we see that $(P^3, x) = (x, (y,z,w)^6)$ and $L(A/(P^3, x)) = 56 = 10n_1 + 6$.

Put $Q_0 = \big(P^3, x\big)$, $Q_i = (Q_{i-1}, d_i)$ for $i = 1, 2, \ldots, 6$, then

$$Q_{i-1} : d_i = (y,z,w)\,(\bmod x)\,.$$

Then $L(A/(P^3, x, d_1, \ldots, d_6)) = 56 - 6 = 50 = 10 \cdot n_1$ and $P^{(3)} = (P^3, d_1, d_2, \ldots, d_6)$ by virtue of Lemma 3.2.

We can now generalize these computations:

Theorem 3.1. *Let P, f_1, f_2, f_3, f_4, g be as in the Proposition* 3.2. *There are* 6 *elements $d_1, \dots, d_6 \in P^{(3)}$ such that $\left(P^3, d_1, \dots, d_6\right) = P^{(3)}$ and the following relations hold:*

(putting $\varepsilon_1 = \max\{0, \alpha_{31} - \alpha_{21}\}$, $\delta_1 = \max\{0, \alpha_{21} - \alpha_{31}\}$)

1. *if $\alpha_{32} \le \alpha_{42}$, $\alpha_{13} \le \alpha_{43}$ and $\alpha_{14} \le \alpha_{24}$ then*

$$\begin{aligned}
x_2^{\alpha_{32}} d_1 &= x_4^{\alpha_{24}-\alpha_{14}} f_4 g^2 - x_3^{\alpha_{43}-\alpha_{13}} f_2^2 f_3 \\
x_4^{\alpha_{14}} d_2 &= x_1^{\delta_1} f_1 f_4^2 - x_1^{\varepsilon_1} x_2^{\alpha_{42}-\alpha_{32}} f_2 g^2 \\
x_3^{\alpha_{13}} d_3 &= x_1^{\delta_1} x_2^{\alpha_{42}-\alpha_{32}} f_3^3 - x_1^{\varepsilon_1}\left(x_4^{\alpha_{24}-\alpha_{14}} f_1^2 g + f_1 f_2 f_3\right) \\
x_4^{\alpha_{14}} d_4 &= x_1^{\delta_1} x_3^{\alpha_{43}-\alpha_{13}} f_1 f_3 f_4 - x_1^{\varepsilon_1}\left(f_4 g^2 + x_3^{\alpha_{43}-\alpha_{13}} f_1 f_2 g\right) \\
x_4^{\alpha_{14}} d_5 &= x_1^{\delta_1} f_4^3 - x_2^{\alpha_{42}-\alpha_{32}} x_3^{\alpha_{43}-\alpha_{13}}\left(x_1^{\delta_1} f_2 f_3 f_4 + x_1^{\varepsilon_1} f_2^2 g\right) \\
x_2^{\alpha_{32}} d_6 &= x_1^{\delta_1}\left(x_3^{\alpha_{43}-\alpha_{13}} f_1 f_2 f_3 - f_3 f_4 g\right) - x_1^{\varepsilon_1} f_2 g^2
\end{aligned}$$

2. *if $\alpha_{32} \le \alpha_{42}$, $\alpha_{13} \le \alpha_{43}$ and $\alpha_{24} \le \alpha_{14}$*

$$\begin{aligned}
x_2^{\alpha_{32}} d_1 &= f_4 g^2 - x_3^{\alpha_{43}-\alpha_{13}} x_4^{\alpha_{14}-\alpha_{24}} f_2^2 f_3 \\
x_4^{\alpha_{24}} d_2 &= x_1^{\delta_1} f_1 f_4^2 - x_1^{\varepsilon_1} x_2^{\alpha_{42}-\alpha_{32}} f_2 g^2 \\
x_3^{\alpha_{13}} d_3 &= x_1^{\delta_1} x_2^{\alpha_{42}-\alpha_{32}} x_4^{\alpha_{14}-\alpha_{24}} f_3^3 - x_1^{\varepsilon_1}\left(f_1^2 g + x_4^{\alpha_{14}-\alpha_{24}} f_1 f_2 f_3\right) \\
x_4^{\alpha_{24}} d_4 &= x_1^{\delta_1} x_3^{\alpha_{43}-\alpha_{13}} f_1 f_3 f_4 - x_1^{\varepsilon_1}\left(f_4 g^2 + x_3^{\alpha_{43}-\alpha_{13}} f_1 f_2 g\right) \\
x_4^{\alpha_{24}} d_5 &= x_1^{\delta_1} f_4^3 - x_2^{\alpha_{42}-\alpha_{32}} x_3^{\alpha_{43}-\alpha_{13}}\left(x_1^{\delta_1} f_2 f_3 f_4 + x_1^{\varepsilon_1} f_2^2 g\right) \\
x_2^{\alpha_{32}} d_6 &= x_1^{\delta_1}\left(x_3^{\alpha_{43}-\alpha_{13}} f_1 f_2 f_3 - f_3 f_4 g\right) - x_1^{\varepsilon_1} f_2 g^2.
\end{aligned}$$

Remark 3.3. Comparing a-1) and a-2) one can see how the relations change if some inequality does. So if in a-1) changes $\alpha_{13} \le \alpha_{43}$ to $\alpha_{43} \le \alpha_{13}$ (we get b-1) case from the Proposition 3.2) the relation for d_1 changes from $x_2^{\alpha_{32}} d_1 = x_4^{\alpha_{24}-\alpha_{14}} f_4 g^2 - x_3^{\alpha_{43}-\alpha_{13}} f_2^2 f_3$ to $x_2^{\alpha_{32}} d_1 = x_3^{\alpha_{13}-\alpha_{43}} x_4^{\alpha_{24}-\alpha_{14}} f_4 g^2 - f_2^2 f_3$ etc. Relations not involving x_3 in positive power stay unchanged (e.g. $x_4^{\alpha_{14}} d_2 = x_1^{\delta_1} f_1 f_4^2 - x_1^{\varepsilon_1} x_2^{\alpha_{42}-\alpha_{32}} f_2 g^2$) but relations with $x_3^{\alpha_{13}}$ on the left side change the power to $x_3^{\alpha_{43}}$:

$$x_3^{\alpha_{43}} d_3 = x_1^{\delta_1} x_2^{\alpha_{42}-\alpha_{32}} f_3^3 - x_1^{\varepsilon_1}\left(x_4^{\alpha_{24}-\alpha_{14}} f_1^2 g + f_1 f_2 f_3\right).$$

Proof of the Theorem 3.2. From the Proposition 3.2 we know that $L(A/(P^3, x_1)) = 10 \cdot n_1 + 6 \cdot t$, where $t = \alpha\beta\gamma$ is a product of 3 numbers: $\alpha = \min\{\alpha_{32}, \alpha_{42}\}$, $\beta = \min\{\alpha_{13}, \alpha_{43}\}$, $\gamma = \min\{\alpha_{14}, \alpha_{24}\}$.

Using the notation of Bresinsky (see Proposition 2.1) we have 5 relations:

$$\begin{aligned}
R_1\colon&\quad x_3^{\alpha_{43}} f_1 + x_4^{\alpha_{14}} f_3 - x_1^{\alpha_{31}} g = 0 \\
R_2\colon&\quad x_2^{\alpha_{32}} f_1 + x_1^{\alpha_{21}} f_3 - x_3^{\alpha_{13}} g = 0 \\
R_3\colon&\quad x_3^{\alpha_{43}} f_2 + x_2^{\alpha_{32}} f_4 + x_4^{\alpha_{24}} g = 0 \\
R_4\colon&\quad x_4^{\alpha_{14}} f_2 + x_1^{\alpha_{21}} f_4 + x_2^{\alpha_{42}} g = 0 \\
R_5\colon&\quad x_4^{\alpha_{24}} f_1 + x_1^{\alpha_{31}} f_2 + x_2^{\alpha_{42}} f_3 + x_3^{\alpha_{13}} f_4 = 0
\end{aligned}$$

(they can be obtained from the matrix equation $\mathbf{I}_5(P).\mathbf{J}^T = \mathbf{0}$ where $\mathbf{J} = (-f_4, -f_2, -g, f_3, f_1)$ and symbol $\mathbf{I}_5(P)$ is defined in Section 1.1).

As in the above Example 3.2 we can derive from these relations new ones and to find elements $d_1, \ldots, d_6$ which have in different situations little modified form.

We prove our claim for a-1) ; the other cases are in principle the same. In the following we need in some cases to differ between the numbers α_{21} and α_{31}. Denote the wanted elements d_i, d_i', d_i'' in case $\alpha_{31} > \alpha_{21}$, $\alpha_{31} = \alpha_{21}$, resp. $\alpha_{31} < \alpha_{21}$.

Then in case a-1)

$$\begin{aligned}
d_1 &\equiv d_1' \equiv d_1'' \equiv x_2^{\alpha_{32}} x_4^{2\alpha_4} - x_2^{\alpha_2} x_3^{\alpha_{43}} x_4^{\alpha_4} - x_2^{\alpha_2+\alpha_{42}} x_3^{2\alpha_{43}} \;(\bmod x_1)\,,\\
d_2 &\equiv x_3^{\alpha_{13}} x_4^{2\alpha_4} - 2x_2^{\alpha_{42}} x_3^{\alpha_3} x_4^{\alpha_4} + x_2^{2\alpha_{42}} x_3^{\alpha_3+\alpha_{43}} \;(\bmod x_1)\,,\\
d_2' &\equiv d_2 + x_2^{2\alpha_2} x_4^{\alpha_{14}} \;(\bmod x_1)\,,\\
d_2'' &\equiv d_2' - d_2 \equiv x_2^{2\alpha_2} x_4^{\alpha_{14}} \;(\bmod x_1)\,,\\
d_3 &\equiv x_2^{\alpha_{42}-\alpha_{32}} x_3^{2\alpha_3+\alpha_{43}} \;(\bmod x_1)\,,\\
d_3' &\equiv d_3 + x_2^{\alpha_{32}} x_3^{\alpha_{13}} x_4^{\alpha_4+\alpha_{14}} + x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_{14}} \;(\bmod x_1)\,,\\
d_3'' &\equiv d_3' - d_3 \equiv x_2^{\alpha_{32}} x_3^{\alpha_{13}} x_4^{\alpha_4+\alpha_{14}} + x_2^{\alpha_2} x_3^{\alpha_3} x_4^{\alpha_{14}} \;(\bmod x_1)\,,\\
d_4 &\equiv x_2^{\alpha_{42}} x_3^{\alpha_3+2\alpha_{43}} - x_3^{\alpha_3+\alpha_{43}} x_4^{\alpha_4} \;(\bmod x_1)\,,\\
d_4' &\equiv d_4 + x_2^{2\alpha_{32}} x_4^{\alpha_4+2\alpha_{14}} \;(\bmod x_1)\,,\\
d_4'' &\equiv d_4' - d_4 \equiv x_2^{2\alpha_{32}} x_4^{\alpha_4+2\alpha_{14}} \;(\bmod x_1)\,,\\
d_5 &\equiv x_4^{2\alpha_4+\alpha_{24}} - 3x_2^{\alpha_{42}} x_3^{\alpha_{43}} x_4^{\alpha_4+\alpha_{24}} + 2x_2^{2\alpha_{42}} x_3^{2\alpha_{43}} x_4^{\alpha_{24}} \;(\bmod x_1)\,,\\
d_5' &\equiv d_5 - x_2^{2\alpha_2+\alpha_{42}} x_3^{\alpha_{43}-\alpha_{13}} \;(\bmod x_1)\,,\\
d_5'' &\equiv d_5 - d_5' \equiv x_2^{2\alpha_2+\alpha_{42}} x_3^{\alpha_{43}-\alpha_{13}} \;(\bmod x_1)\,,\\
d_6 &\equiv 2x_2^{\alpha_{42}} x_3^{\alpha_3+\alpha_{43}} x_4^{\alpha_{14}} - x_3^{\alpha_3} x_4^{\alpha_4+\alpha_{14}} \;(\bmod x_1)\,,\\
d_6' &\equiv d_6 - x_2^{\alpha_2+\alpha_{32}} x_4^{2\alpha_{14}} \;(\bmod x_1)\,,\\
d_6'' &\equiv d_6 - d_6' \equiv x_2^{\alpha_2+\alpha_{32}} x_4^{2\alpha_{14}} \;(\bmod x_1)\,.
\end{aligned}$$

Following the idea from the above example put $Q_0 = \left(P^3, x_1\right)$, $Q_i = (Q_{i-1}, d_i)$ for $i = 1, 2, \ldots, 6$, then

$$Q_{i-1} : d_i = (x_2^{\alpha_{32}}, x_3^{\alpha_{13}}, x_4^{\alpha_{14}}) \;(\bmod x_1) \text{ for all } i.$$

Then we have $L\left(A/\left(P^3, x_1, d_1, \ldots, d_6\right)\right) = (10.n_1 + 6 \cdot t) - 6 \cdot t$ with $t = \alpha_{32}\alpha_{13}\alpha_{14}$. It follows then that $\left(P^3, d_1, \ldots, d_6\right) = P^{(3)}$.

4. Finite Symbolic Algebras

Let us turn to the curve $C(5,6,7,8)$ from the previous example. Put $D = (d_1, \ldots, d_6)$ and $R = \mathrm{A}\left[Pt, Dt^3\right]$.

Let $\varphi\colon A\,[T_1, T_2, \dots, T_{11}] \to R$ be the natural epimorphism sending

$$\begin{aligned} T_i &\mapsto f_i t && \text{for } i = 1,2,3,4 \\ T_5 &\mapsto gt && \text{and} \\ T_i &\mapsto d_{i-5}t^3 && \text{for } i = 6,\dots,11. \end{aligned}$$

Then φ induce isomorphisms

$$\begin{aligned} \varphi'\colon A\,[T_1,\dots,T_{11}]\,/\operatorname{Ker}\varphi &\to R \qquad \text{and} \\ \varphi^*\colon k\,[T_1,\dots,T_{11}]\,/\,(\operatorname{Ker}\varphi, M) &\to R/MR. \end{aligned}$$

We need to compute the analytic spread $a(P^{(3)})$ of $P^{(3)}$ in order to use the following lemma for deciding on finiteness of $S\,(P)$ (see [**7**, Theorem 2.1 and Corollary 2.2]).

Lemma 4.4. *For a P-primary ideal Q of an unmixed local ring (A, M) the following conditions are equivalent:*

1. *$S\,(Q)$ is an A-algebra of finite type.*
2. *There is an integer k such that $a(Q^{(k)}A_p) < \dim A_p$ for all prime ideals $p \supset P$.*

Using computer program Macaulay we get that

$$\operatorname{Ker}\varphi^* = I = (g_1, \dots, g_{40}) = \left(T_2^3 - T_1T_2T_3 + T_1^2T_4, \dots, T_6T_7 + T_{11}^2\right)$$

and $\dim R/MR = 3$. Because $P^{(3)} = \left(P^3, d_1, \dots, d_6\right)$,

$$a(P^{(3)}) = \dim R\left(P^{(3)}\right)/MR\left(P^{(3)}\right) = \dim R/MR = 3 < 4 = \dim A_M,$$

thus the symbolic Rees algebra $S\,(P)$ is an A-algebra of finite type.

Note that the computations were on the edge of our computer possibilities; e.g. we could not correctly compute the standard basis of $\operatorname{Ker}\varphi^*$ for the curve $C\,(5,7,8,9)$ though it is not too much “complicated” curve.

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