Acta Math. Univ. Comenianae
Vol. LXIX, 2(2000), pp. 229–232

THE BITRANSITIVE CONTINUOUS MAPS OF THE INTERVAL ARE CONJUGATE TO MAPS EXTREMELY CHAOTIC A.E.

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Abstract. In the eighties, Misiurewicz, Bruckner and Hu provided examples of functions chaotic almost everywhere. In this paper we show — by using much simpler arguments — that any bitransitive continuous map of the interval is conjugate to a map which is extremely chaotic almost everywhere. Using a result of A. M. Blokh we get as a consequence that for any map f with positive topological entropy there is a k such that f^k is semiconjugate to a continuous map which is extremely chaotic almost everywhere.

Let $I = [0,1]$ be the unit interval. By $C(I,I)$ we denote the set of continuous maps $f\colon I \to I$. Let $f \in C(I,I)$, $x \in I$ and $n \geq 0$ be an integer. By $f^n(x)$ we denote the n-th iterate of x under f. By an interval we always mean a non-degenerate interval.

A map $f \in C(I,I)$ is **extremely chaotic in the sense of Li and Yorke** (briefly, **extremely chaotic**) if there is a set $S \subset I$ such that, for every $x, y \in S$ with $x \neq y$, $\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1$ and $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$. The set S is called **(extremely) scrambled set** of f. We say that a function $f \in C(I,I)$ is **extremely chaotic almost everywhere** if there is a scrambled set S of f with $\lambda(S) = 1$, where λ denotes the Lebesgue measure.

In **[5]** and **[3]** Misiurewicz, Bruckner and Hu gave examples of functions chaotic almost everywhere. In this paper we improve these results in two directions. First, we give not only individual examples of functions chaotic almost everywhere — we show that this property is fulfilled by a whole class of functions. Moreover, we show that these functions are not only chaotic but even extremely chaotic almost everywhere.

More precisely, we prove that each bitransitive map $f \in C(I,I)$ is topologically conjugate to a map $g \in C(I,I)$ which is extremely chaotic almost everywhere.

Received September 1, 2000.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 26A18, 37E05, 37D45; Secondary 54H20, 26A30, 37A25.
Key words and phrases. Bitransitive map, chaotic map, scrambled set.
The research was supported, in part, by the Grant Agency of Czech Republic, grant No. 201/00/0859, and the Czech Ministry of Education, contract No. CEZ:J10/98:192400002.

Recall that a map f is **topologically transitive** if, for any open intervals $U, V \subset I$ there exists a positive integer n such that $f^n(U) \cap V \neq \emptyset$; f is **bitransitive** if f^2 is transitive. A function $f \in C(I,I)$ is **semiconjugate** to $g \in C(I,I)$ if there is a surjective map $h \in C(I,I)$ such that $h \circ f = g \circ h$. If h is bijective, then f and g are **conjugate**.

Lemma 1 ([**3**, Proposition 4.4] or [**1**, Proposition 44])**.** *Let $f \in C(I,I)$ be a bitransitive map, J, K compact intervals, $J, K \subset (0,1)$. Then $f^n(J) \supset K$, for any sufficiently large n.*

Lemma 2. *Let $f \in C(I,I)$ be bitransitive, $J \subset (0,1)$ a compact interval, M an infinite set of positive integers, and let $p_n \in (0,1)$ be such that the accumulation points of $\{p_n\}_{n=1}^{\infty}$ are in $\{0,1\}$. Then there are a non-empty nowhere dense perfect set $P \subset J$, and an increasing sequence $\{k(n)\}_{n=1}^{\infty}$ in M with the following properties:*

$$f^{k(n)}(P) \subset \left[p_n - \frac{1}{n}, p_n + \frac{1}{n}\right] \text{ for any } n, \tag{1}$$

and

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| = 1, \text{ for any } x, y \in P, \; x \neq y. \tag{2}$$

Proof. Put $q_n = 1 - p_n$. We let the set P be in the form $P = \bigcap_{n=1}^{\infty} P_n$ where, for any n, P_n is the union of pairwise disjoint compact intervals U_s, $s \in \{0,1\}^n$, and $P_{n+1} \subset P_n$. The intervals U_s are defined inductively by n.

Stage 1: Let U_0, U_1 be disjoint closed subintervals of J. Put $P_1 = U_0 \cup U_1$, and let $k(1,0) < k(1,1)$ be any numbers in M.

Stage $n+1$: Sets $P_1, \dots, P_n$, and positive integers $k(1,0) < k(1,1) < k(2,0) < k(2,1) < k(2,2) < \cdots < k(n,0) < k(n,1) < \cdots < k(n,n)$ in M are available from stage n such that, for any $s = s_1 \dots s_v \in \{0,1\}^v$, $1 \le v \le n$, $0 \le j \le v$,

$$|U_s| \le \frac{1}{v}, \tag{3}$$

$$f^{k(v,j)}(U_s) \subset \left[p_v - \frac{1}{v}, p_v + \frac{1}{v}\right] \text{ if } j = 0 \text{ or } s_j = 0, \tag{4}$$

$$f^{k(v,j)}(U_s) \subset \left[q_v - \frac{1}{v}, q_v + \frac{1}{v}\right] \text{ if } s_j = 1. \tag{5}$$

By Lemma 1 there is an integer $k(n+1,0) > k(n,n)$ in M such that, for any $s \in \{0,1\}^n$, $p_{n+1} \in f^{k(n+1,0)}(U_s)$. Hence, for any $s \in \{0,1\}^n$, there is a compact interval $V_s \subset U_s$ such that

$$f^{k(n+1,0)}(V_s) \subset \left[p_{n+1} - \frac{1}{n+1}, p_{n+1} + \frac{1}{n+1}\right]. \tag{6}$$

Again by Lemma 1, there is a $k(n+1,1) > k(n+1,0)$ in M such that, for any $s \in \{0,1\}^n$, $\{p_{n+1}, q_{n+1}\} \subset f^{k(n+1,1)}(V_s)$. Hence, for any $s = s_1 s_2 \dots s_n \in \{0,1\}^n$ there is a compact interval $V_s^1 \subset V_s$ such that $x \in f^{k(n+1,1)}(V_s^1)$, where $x = p_{n+1}$ if $s_1 = 0$, and $x = q_{n+1}$ otherwise, and such that $|f^{k(n+1,1)}(V_s^1)| \leq 1/(n+1)$. Applying this process n times we obtain integers $k(n+1,1) < k(n+1,2) < \dots < k(n+1,n)$ in M, and compact intervals $V_s^1 \supset V_s^2 \supset \dots \supset V_s^n$ such that, for any $s = s_1 s_2 \dots s_n \in \{0,1\}^n$, and any $1 \leq j \leq n$,

$$f^{k(n+1,j)}(V_s^j) \subset \left[p_{n+1} - \frac{1}{n+1}, p_{n+1} + \frac{1}{n+1}\right] \text{ if } s_j = 0, \tag{7}$$

and

$$f^{k(n+1,j)}(V_s^j) \subset \left[q_{n+1} - \frac{1}{n+1}, q_{n+1} + \frac{1}{n+1}\right] \text{ if } s_j = 1. \tag{8}$$

Finally, let $k(n+1,n+1) > k(n+1,n)$ be from M and such that, for any $s \in \{0,1\}^n$, $f^{k(n+1,n+1)}(V_s^n) \supset \{p_{n+1}, q_{n+1}\}$. Then there are disjoint compact intervals $U_{s0}, U_{s1} \subset V_s^n$ such that

$$|U_{s0}|, |U_{s1}| \leq \frac{1}{n+1}, \tag{9}$$

$$f^{k(n+1,n+1)}(U_{s0}) \subset \left[p_{n+1} - \frac{1}{n+1}, p_{n+1} + \frac{1}{n+1}\right], \tag{10}$$

and

$$f^{k(n+1,n+1)}(U_{s1}) \subset \left[q_{n+1} - \frac{1}{n+1}, q_{n+1} + \frac{1}{n+1}\right]. \tag{11}$$

Thus, we have sets U_s defined for any $s \in \{0,1\}^{n+1}$. They satisfy (3) by (9), (4) by (6), (7), (10), and (5) by (8) and (11). This completes the induction.

Put $P = \bigcap_{k=1}^{\infty} \bigcup_{s \in \{0,1\}^k} U_s$ and, for any n, $k(n) = k(n,0)$. Then P is a nowhere dense perfect set; this follows by (3). By (4), P satisfies (1). It remains to prove (2).

Let x and y be distinct points in P. Then for any positive integer K there are $s, s' \in \{0,1\}^K$ such that $x \in U_s, y \in U_{s'}$. Take K sufficiently large so that the sets U_s and $U_{s'}$ are disjoint. Thus, $s \neq s'$ and hence, $s_j \neq s'_j$ for some j, where $s = s_1 \dots s_K$, $s' = s'_1 \dots s'_K$. Without loss of generality assume $s_j = 0$ and $s'_j = 1$.

Let $n > K$. Then by (4) and (5), $|f^{k(n,j)}(x) - f^{k(n,j)}(y)| \geq |p_n - q_n| - 2/n \to 1$ for $n \to \infty$, which implies (2). $\square$

Theorem 1. *Each bitransitive map $f \in C(I,I)$ is topologically conjugate to a map $g \in C(I,I)$ which is extremely chaotic almost everywhere.*

Proof. Define by induction a sequence $S_1 \subset S_2 \subset \dots$ of perfect extremely scrambled sets, and a sequence $M_1 \supset M_2 \supset \dots$ of infinite sets of positive integers such that

$$f^{d(n)}(S_m) \subset \left[0, \frac{1}{n}\right] \quad \text{for every } n, \tag{12}$$

where $\{d(n)\}_{n=1}^{\infty}$ is an enumeration of M_m. Apply Lemma 2 to $p_n = 1/(n+1)$, $q_n = 1 - p_n$, $J = [1/3, 2/3]$, and M the set of positive integers to get P and $\{k(n)\}_{n=1}^{\infty}$. Denote $S_1 = P$ and $M_1 = \{k(n)\}_{n=1}^{\infty}$.

Now, assume there are sets S_m and M_m satisfying (12). Let J be the middle third of the interval complementary to S_m of maximal length. Apply Lemma 2 to $M = M_m$, $p_n = 1/(n+1)$ for n odd, $p_n = 1 - 1/(n+1)$ for n even, and $q_n = 1 - p_n$, to get P and $\{k(n)\}_{n=1}^{\infty}$. Then $S_{m+1} = S_m \cup P$ is a perfect extremely scrambled set since for $x \in S_m$ and $y \in P$, by (1) and (12), $\lim_{n\to\infty} |f^{k(n)}(x) - f^{k(n)}(y)| = 1$. To complete the induction put $M_{m+1} = \{k(2n+1)\}_{n=1}^{\infty}$.

Let $S = \bigcup_{m=1}^{\infty} S_m$. Then S is an extremely scrambled set of type F_σ, which is c-dense in I. By **[4]**, any c-dense F_σ set is homeomorphic to a set of full Lebesgue measure. So let $\varphi \in C(I,I)$ be a homeomorphism such that $\lambda(\varphi(S)) = 1$. To finish the proof note that $\varphi(S)$ is an extremely scrambled set for $g = \varphi \circ f \circ \varphi^{-1}$.□

Theorem 2. *Let $f \in C(I,I)$ be a map with positive topological entropy. Then, for some $k \geq 1$, f^k is semiconjugate to a map $g \in C(I,I)$ which is extremely chaotic almost everywhere.*

Proof. There exists a positive integer k such that f^k is semiconjugate to a bitransitive map $g \in C(I,I)$ (see **[2]**). Using Theorem 1 we obtain the assertion.□

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