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ON THE CONCAVE SOLUTIONS OF THE BLASIUS EQUATION

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Abstract. The Blasius equation is an autonomous, third order, nonlinear differential equation, which results from an appropriate substitution in boundary layer equations. We study in details the concave solutions of initial value problems involving this equation, and apply our results to solve a related boundary value problem

1. Introduction

In the fluid mechanics theory, the Blasius equation

$$f''' + \frac{1}{2} f f'' = 0 \tag{1.1}$$

appears in some boundary layer problems by looking for solutions having a "similarity" form.

By studying the motion of an incompressible viscous fluid near a semi-infinite flat plate, we can drop some terms in the Navier-Stokes equations and derive the Prandtl boundary layer equations. If we assume, moreover, that the tangential velocity at the outer limit of the boundary layer is constant, similarity solutions can be obtained by solving the equation (1.1) on $[0, \infty)$, with the boundary conditions

$$f(0) = f'(0) = 0 \quad \text{and} \quad f'(\infty) = 1. \tag{1.2}$$

H. Blasius **[3]** was the first to show that this problem provided a special solution to the Prandtl boundary layer equations. In fact, the Blasius equation is a particular case of that of Falkner-Skan **[7]**

$$f''' + \frac{m+1}{2} f f'' + m(1 - f'^2) = 0$$

describing the same phenomena, when the velocity at infinity has some dependance on m. See **[10]**, **[11]** and **[14]**.

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The problem of steady free convection about a vertical flat plate embedded in a saturated porous medium leads to a similar boundary value problem. If we assume that convection takes place in a thin layer around the wall, a method similar to those proposed by Prandtl can be used to simplify the equations of Darcy describing the convective flow, and similarity solutions can be obtained. If moreover the temperature on the wall is constant, the problem to be solved is the equation (1.1) with the boundary conditions

$$f(0)=0,\ f'(0)=1 \ \text{ and } \ f'(\infty)=0. \tag{1.3}$$

If the prescribed wall temperature is some power function, the same approach leads to the equation

$$f''' + \frac{\alpha+1}{2}ff'' - \alpha f'^2 = 0$$

and (1.1) corresponds to the case $\alpha = 0$. See **[1]**, **[2]**, **[4]** and **[6]**.

The boundary value problem (1.1)–(1.2) dates back about one century and has been abundantly studied. One of the most important paper on this subject is the one of H. Weyl **[15]**, where existence and uniqueness are proved using integral operators. Elementary proofs using differential inequalities are given by W. A. Coppel **[5]**, and P. Hartmann **[8]**. See also K. K. Tam **[13]**. This problem was first solved numerically (and undoubtedly by hand) by H. Blasius **[3]**.

The problem (1.1)–(1.3), considered more recently by physicians, has been essentially investigated from numerical point of view (see **[4]**) and, to our knowledge, the only papers about the question of existence are the ones of G. V. Ščerbina **[12]** and W. A. Coppel **[5]**.

From the boundary conditions, it follows that the solution has to be convex in the first problem and concave in the second one. This difference is essential as we will see in the following.

Let us now consider the following initial value problem

$$(\mathcal{P}_{a,b,c}) \qquad \begin{cases} f''' + \frac{1}{2}ff'' = 0 \text{ on } [0,T), \\ f(0) = a, \\ f'(0) = b, \\ f''(0) = c, \end{cases}$$

where $a, b, c \in \mathbb{R}$ and $[0,T)$ is the right maximal interval of existence of the solution. If f is a solution of $(\mathcal{P}_{a,b,c})$ then either $f''(t) \equiv 0$ or $f''(t) > 0$ or $f''(t) < 0$ for all $t \in [0,T)$. Therefore, if $c = 0$, we have $T = \infty$ and $f(t) = a + bt$, if $c > 0$ then f is strictly convex, and if $c < 0$ then f is strictly concave. The situation is quite different for the concave solutions and the convex solutions. Indeed, we have

$$\forall t \in [0,T),\ f''(t) = c\mathrm{e}^{-\frac{1}{2}\int_0^t f(s)\,ds}, \tag{1.4}$$

and if $c > 0$, we deduce from the convexity of f that we have $f(t) \geq a + bt$ and thus we can write

$$\forall t \in [0, T),\ 0 < f''(t) \leq c\mathrm{e}^{-\frac{1}{2}at - \frac{1}{4}bt^2}.$$

It follows that $T = \infty$. For $c < 0$, the relation (1.4) does not allow to obtain anything about T. In fact, for $c < 0$, we can have $T < \infty$; for example, for $\tau > 0$, the function

$$g_\tau : t \longmapsto \frac{6}{t - \tau} \tag{1.5}$$

is the solution of the problem $(\mathcal{P}_{a,b,c})$ with $a = -\frac{6}{\tau}$, $b = -\frac{6}{\tau^2}$, $c = -\frac{12}{\tau^3}$ and its right maximal interval of existence is $[0, \tau)$.

Let us now introduce the following general boundary value problem involving the Blasius equation

$$\begin{cases} f''' + \frac{1}{2}ff'' = 0 \ \text{ on } \ [0, \infty), \\ f(0) = a,\ f'(0) = b, \\ f'(\infty) = \lambda. \end{cases} \tag{1.6}$$

This problem with $b \in [0, 1)$ and $\lambda = 1$ is considered by P. Hartmann **[8]**; in this case the solution is convex. For $b \geq 0$ and $\lambda < b$, the solution has to be concave.

In what follows, we will study in details the initial value problem $(\mathcal{P}_{a,b,c})$ with $a, b \in \mathbb{R}$ fixed and c describing $(-\infty, 0]$.

Obviously, if f is a solution of the Blasius equation, then $t \longmapsto \kappa f(\kappa t)$ is also a solution, for any positive constant κ. Therefore, the problem $(\mathcal{P}_{a,b,c})$ can be reduced to a two-parameter problem in the cases $a = -1$, $a = 0$ and $a = 1$. We volontary choose to not use this scaling, because we look at $(\mathcal{P}_{a,b,c})$ as a one-parameter problem (say c) and essentially, our results do not depend on a and b.

We will show that there exists a $c_* \leq 0$ such that for $c \in [c_*, 0]$ the solution f_c exists over the whole interval $[0, \infty)$ and for $c < c_*$ the solution tends to $-\infty$ as t tends to some $T_c < \infty$. Moreover we will study the asymptotic behaviour as $t \to \infty$ of the solutions, and prove that the correspondance $c \longmapsto f_c'(\infty)$ is an increasing one-to-one mapping of $[c_*, 0]$ onto $[0, b]$. The method we use is based on a Comparison-Principle, only valid in the concave case, and on elementary estimates, which allow to see how the solutions of the initial value problems $(\mathcal{P}_{a,b,c})$ move as c goes from $-\infty$ to 0. In this manner we obtain, in particular, that the boundary value problem (1.6) with $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $\lambda \in (-\infty, b]$ has one and only one solution if $b \geq 0$ and $\lambda \in [0, b]$, and no solution if $\lambda < 0$.

In **[5]**, W. A. Coppel gives qualitative properties of the concave solutions of the Blasius equation. Some of our results are in **[5]**, but the finalities are different, and in particular the boundary value problem (1.6) is not solved in the general case.

In the part devoted to the Blasius equation, the author proves that for any given constants $\nu > 0$ and γ, the equation (1.1) has one and only one solution defined on $[0,\infty)$ such that

$$f(t) \to \nu, \; f''(t) \sim \gamma \mathrm{e}^{-\nu t} \;\text{ as }\; t \to \infty.$$

The problem (1.6) with $a = 0$, $b \geq 0$ and $\lambda \geq 0$ is considered by G. V. Ščerbina [**12**, Theorem 3]. But the proof given in this paper, for the concave case (i.e. $\lambda \in [0, b)$) is quite hard to follows, and not complete, since it is assumed, **a priori**, that the solution of the initial value problem is always nonnegative, say exists on the whole interval $[0,\infty)$. This is not true for every c, as we will see in the following. The fact that there exist some $c < 0$ such that the solution is global has to be proved. On the other hand, the question of uniqueness is not investigated, and the author even seems to consider that nonuniqueness could arise.

2. Comparison Principle and Blow-Up Results

Among the concave solutions of the Blasius equation, we will distinguish two kind of behaviour; if f is a solution on $[0,T)$ of the problem $(\mathcal{P}_{a,b,c})$ with $c \leq 0$, we will say

$$\begin{aligned} f \;\text{ is of type (I)} &\iff f \;\text{ is non decreasing} \\ f \;\text{ is of type (II)} &\iff f \;\text{ is decreasing from some }\; t_1 \in [0,T). \end{aligned}$$

For a type (I) solution, we have $f \geq a$ and (1.4) implies that

$$\forall t \in [0,T), \; |f''(t)| \leq |c|\mathrm{e}^{-\frac{1}{2}at}$$

and $T = \infty$. For any type (II) solution, except the affine one, we are going to prove that $T < \infty$. To this end, let us show that we can compare two concave solutions of the Blasius equation.

Proposition 2.1. (Comparison Principle) *Let $t_0 \in \mathbb{R}$ and for $i = 1,2$ let f_i be the solution on $[t_0, T_i)$ of the initial value problem*

$$\begin{cases} f_i''' + \frac{1}{2} f_i f_i'' = 0 \quad \text{on} \quad [t_0, T_i), \\ f_i(t_0) = a_i, \\ f_i'(t_0) = b_i, \\ f_i''(t_0) = c_i, \end{cases}.$$

where $a_i, b_i \in \mathbb{R}$ and $c_i \leq 0$. If $a_1 \geq a_2$, $b_1 \geq b_2$, $c_1 \geq c_2$, then $T_1 \geq T_2$ and if one of these inequalities is strict, we have

$$f_1 > f_2, \; f_1' > f_2' \;\text{ and }\; f_1'' > f_2'' \;\text{ on }\; (0, T_2)$$

except for $c_1 = c_2 = 0$, where only the two first inequalities hold.

Proof. If $c_1 = c_2 = 0$, then $f_i(t) = a_i + b_i(t - t_0)$ and we have $f_1 > f_2$ and $f_1' > f_2'$ on (t_0, ∞).

Let us now assume that $(c_1, c_2) \neq (0, 0)$. Let us set $h = f_1 - f_2$ and prove that there exists $\eta > 0$ such that

$$h'' > 0 \quad \text{on} \quad (t_0, t_0 + \eta). \tag{2.1}$$

If $c_1 > c_2$, then $h''(t_0) > 0$ and it is clear. If $c_1 = c_2$ and $a_1 > a_2$ then we have $h''(t_0) = 0$ and

$$h'''(t_0) = -\frac{1}{2} f_1(t_0) f_1''(t_0) + \frac{1}{2} f_2(t_0) f_2''(t_0) = -\frac{1}{2} c_1(a_1 - a_2) > 0,$$

therefore (2.1) holds. If $c_1 = c_2$, $a_1 = a_2$ and $b_1 > b_2$ then $h''(t_0) = h'''(t_0) = 0$ and

$$\begin{aligned} h^{(4)}(t_0) &= -\frac{1}{2} f_1'(t_0) f_1''(t_0) + \frac{1}{2} f_2'(t_0) f_2''(t_0) - \frac{1}{2} f_1(t_0) f_1'''(t_0) + \frac{1}{2} f_2(t_0) f_2'''(t_0) \\ &= -\frac{1}{2} c_1(b_1 - b_2) > 0, \end{aligned}$$

and thus we get (2.1) in this case again.

Suppose now h'' vanishes on $(t_0, T_1) \cap (t_0, T_2)$ and let $t_1 > t_0$ such that $h''(t_1) = 0$ and $h'' > 0$ on (t_0, t_1). Necessarily, we have

$$h'''(t_1) \leq 0. \tag{2.2}$$

On the other hand,

$$h'''(t_1) = -\frac{1}{2} f_1(t_1) f_1''(t_1) + \frac{1}{2} f_2(t_1) f_2''(t_1) = -\frac{1}{2} f_1''(t_1) h(t_1),$$

and, since $\forall t \in (t_0, t_1)$,

$$h'(t) = b_1 - b_2 + \int_{t_0}^{t} h''(s)\, ds > 0 \text{ and } h(t_1) = a_1 - a_2 + \int_{t_0}^{t_1} h'(s)\, ds > 0,$$

we get $h'''(t_1) > 0$ and a contradiction to (2.2). Finally, $h'' > 0$ on $(t_0, T_1) \cap (t_0, T_2)$ and also $h' > 0$, $h > 0$. The inequality $T_1 \geq T_2$ follows from the concavity of f_1 and the fact that $f_1(t) \geq f_2(t)$ for all t for which $f_1(t)$ and $f_2(t)$ exist. □

Remark 2.1. The previous Comparison Principle could be deduced, at least partially, from a theorem due to E. Kamke [**9**], but the simplicity of the proof in the particular case of the Blasius equation incited us to give it. See also [**5**], and quasi-monotonicity concept widely developed by W. Walter [**14**].

Proposition 2.2. *Let f be a solution on $[0,T)$ of the problem $(\mathcal{P}_{a,b,c})$ with $c<0$. If f is of type (II), then $T<\infty$.*

Proof. Since f is concave and of type (II), there exists $s\in[0,T)$ such that

$$\alpha_1 = f(s) < 0,\ \ \beta_1 = f'(s) < 0 \ \text{ and } \ \gamma_1 = f''(s) < 0.$$

Let us choose τ such that

$$\tau > s + \max\left\{\frac{6}{-\alpha_1}, \sqrt{\frac{6}{-\beta_1}}, \sqrt[3]{\frac{12}{-\gamma_1}}\right\}$$

and consider the function g_τ defined by (1.5). By setting

$$\alpha_0 = g_\tau(s),\ \ \beta_0 = g_\tau'(s) \ \text{ and } \ \gamma_0 = g_\tau''(s),$$

we have $\alpha_0 > \alpha_1$, $\beta_0 > \beta_1$ and $\gamma_0 > \gamma_1$, and applying Proposition 2.1, we get $T \le \tau$. □

Remark 2.2. In **[14]**, the problem $(\mathcal{P}_{0,0,-2})$ is considered and it is shown, by introducing appropriate super- and subfunctions, how to get bounds for the point T where the solution becomes infinite. The method consists of writing the solution as

$$f(t) = \sum_{k=0}^{\infty} \frac{a_k}{3^k}\, t^{3k+2} \tag{2.3}$$

with

$$a_0 = -1,\ a_1 = -\frac{1}{20},\ a_2 = -\frac{11}{2240},\ a_3 = -\frac{5}{9856},\dots$$

and constructing super- and subfunctions from finite segments of this power series expansion for small t and functions as those defined in (1.5). By this way, the following estimate is obtained:

$$3.098 < T < 3.151.$$

Note that, since the expansion (2.3) has only negative coefficients, T is equal to the radius of convergence of this series.

3. Structure of the Set of the Concave Solutions of the Blasius Equation

In this part, for a, b fixed, we will study the set

$$\mathcal{S}_{a,b} = \big\{f_c : [0,T_c) \longrightarrow \mathbb{R};\ c\in(-\infty,0]\big\},$$

where we denote by f_c the solution of $(\mathcal{P}_{a,b,c})$ and by $[0,T_c)$ its right maximal interval of existence.

First, let us give the following very useful continuity result:

Proposition 3.1. *The function $\psi : (t,c) \longmapsto (f_c(t), f_c'(t), f_c''(t))$ defined on the set*

$$U = \{(t,c) \in [0,\infty) \times (-\infty, 0];\ t < T_c\},$$

is continuous, and the function $c \longmapsto T_c$, taking its values in $(0,\infty]$, is lower semicontinuous on $(-\infty, 0]$.

Proof. Of course, this can be obtained from general continuity results (see, for example, [**8**], Ch. V, p. 94, Theorem 2.1), but we would like to give an elementary proof using the Comparison Principle and Gronwall's inequality.

Let $(t,c) \in U$. There exist $\gamma_1 < \gamma_2 \le 0$ and $\eta > 0$ such that

$$(t,c) \in [0, t+\eta] \times [\gamma_1, \gamma_2] \subset U.$$

Let us now consider a sequence (c_n) in $[\gamma_1, \gamma_2]$. It follows from Proposition 2.1 that

$$f_{\gamma_1} \le f_{c_n} \le f_{\gamma_2}. \tag{3.1}$$

By setting $h_n = |f_{c_n} - f_c|$ and using again Proposition 2.1, we see that h_n is equal either to $f_{c_n} - f_c$ or to $f_c - f_{c_n}$. Moreover, h_n, h_n' and h_n'' are positive on $(0, t+\eta]$. Therefore, for all $s \in [0, t+\eta]$, we have

$$\begin{aligned} h_n'''(s) &= -\frac{1}{2} f_{c_n}(s) h_n''(s) - \frac{1}{2} f_c''(s) h_n(s) \\ &\le \frac{1}{2}\left(\sup_{\xi\in[0,t+\eta]} |f_{c_n}(\xi)|\right) h_n''(s) + \frac{1}{2}\left(\sup_{\xi\in[0,t+\eta]} |f_c''(\xi)|\right) h_n(s) \\ &\le C_1 h_n''(s) + C_2 h_n'(s) \end{aligned}$$

by using (3.1) and the convexity of h_n together with $h_n(0) = 0$, to get $h_n(s) \le s h_n'(s)$, and where

$$C_1 = \frac{1}{2}\left(\sup_{\xi\in[0,t+\eta]} (|f_{\gamma_1}(\xi)| + |f_{\gamma_2}(\xi)|)\right) \quad \text{and} \quad C_2 = \frac{1}{2}(t+\eta)\left(\sup_{\xi\in[0,t+\eta]} |f_c''(\xi)|\right).$$

By integrating, we get, for all $s \in [0, t+\eta]$,

$$\begin{aligned} 0 \le h_n''(s) &\le |c_n - c| + C_1 h_n'(s) + C_2 h_n(s) \\ &\le |c_n - c| + C h_n'(s) \end{aligned}$$

with $C = C_1 + (t+\eta)C_2$. After using Gronwall's inequality and twice integrating, we obtain

$$\forall s \in [0, t+\eta],\ 0 \le h_n'(s) \le \frac{1}{C}|c_n - c| \mathrm{e}^{C(t+\eta)} \quad \text{and} \quad 0 \le h_n(s) \le \frac{1}{C^2}|c_n - c| \mathrm{e}^{C(t+\eta)}.$$

The continuity of $\psi = (\psi_0, \psi_1, \psi_2)$ then follows immediately, since for $i = 0$, 1, 2 and for every sequence (t_n, c_n) in $[0, t+\eta] \times [\gamma_1, \gamma_2]$ which converges to (t, c), we have as $n \to \infty$,

$$|\psi_i(t_n, c_n) - \psi_i(t, c)| = |f_{c_n}^{(i)}(t_n) - f_c^{(i)}(t)| \leq h_n^{(i)}(t_n) + |f_c^{(i)}(t_n) - f_c^{(i)}(t)| \longrightarrow 0.$$

We shall prove the second assertion by contradiction. Let $c \leq 0$ and assume that there exists $T \in (0, T_c)$ and a sequence (c_n) which converges to c, such that $T_{c_n} < T$. In accordance with Proposition 2.1, we must have $c_n < c$. Let us set $h_n = f_c - f_{c_n}$. The functions h_n, h_n' and h_n'' are positive on $(0, T_{c_n})$ and following the same way as above, we get

$$\forall t \in [0, T_{c_n}),\ 0 \leq h_n'(t) \leq \frac{1}{C}(c - c_n)\mathrm{e}^{CT}. \tag{3.2}$$

Since $f_{c_n}'(t)$ tends to $-\infty$ as $t \to T_{c_n}$, there exists a point $t_n \in (0, T_{c_n})$ satisfying $f_{c_n}'(t_n) = f_c'(T) - 1$. Then,

$$h_n'(t_n) = f_c'(t_n) - f_{c_n}'(t_n) = f_c'(t_n) - f_c'(T) + 1 \geq 1,$$

and we get a contradiction with (3.2). □

As we saw in the introduction, f_0 is affine and defined on $[0, \infty)$ by $f_0(t) = a + bt$, and for $c < 0$, the function f_c is strictly concave on $[0, T_c)$.

If $b \leq 0$, then for any $c < 0$, the solution f_c is decreasing on $[0, T_c)$ and $T_c < \infty$.

For $b > 0$, it is **a priori** not clear to see if $\mathcal{S}_{a,b} \setminus \{f_0\}$ contains either only type (I) solutions, or only type (II) solutions, or both of them. The answer is given in the following theorem :

Theorem 3.1. *Let $a \in \mathbb{R}$ and $b > 0$. Then there exist $c_* < 0$ such that for $c \in [c_*, 0]$, the solution f_c is of type (I), and for $c \in (-\infty, c_*)$, the solution f_c is of type (II).*

Proof. Taking into account the Comparison Principle, we can set

$$c_* = \inf\{c \leq 0\ ;\ f_c \text{ is of type (I)}\}.$$

and the proof will result from the following lemmas.

Lemma 3.1. *The infimum c_* is finite.*

Proof. Let $c < 0$ and assume that $f = f_c$ is of type (I). Then we have $0 < f' \leq b$. Integrating the equation (1.1) on $[0, t]$, we obtain

$$f''(t) - c + \frac{1}{2}f(t)f'(t) - \frac{1}{2}ab = \frac{1}{2}\int_0^t f'(s)^2\, ds, \tag{3.3}$$

in this way we get

$$\forall t > 0,\ f''(t) + \frac{1}{2}f(t)f'(t) < c + \frac{1}{2}ab + \frac{1}{2}b^2 t,$$

and integrating again,

$$\forall t > 0,\ 0 < f'(t) + \frac{1}{4}f(t)^2 < \frac{1}{4}a^2 + b + \left(c + \frac{1}{2}ab\right)t + \frac{1}{4}b^2t^2.$$

The inequality is fulfilled either if $c \geq -\frac{1}{2}ab$, or if $c < -\frac{1}{2}ab$ and

$$\left(c + \frac{1}{2}ab\right)^2 - b^2\left(\frac{1}{4}a^2 + b\right) < 0,$$

from which, we easily get

$$c > -\frac{1}{2}ab - \frac{1}{2}b\sqrt{a^2 + 4b}.$$

In conclusion,

$$c \leq -\frac{1}{2}ab - \frac{1}{2}b\sqrt{a^2 + 4b} \implies f \quad \text{is of type (II)}, \tag{3.4}$$

and thus c_* is finite. □

Lemma 3.2. *Let f_* be the solution of $(\mathcal{P}_{a,b,c_*})$. Then f_* is of type (I).*

Proof. Let us assume that f_* is of type (II) and denote by t_0 the point in $(0, T_*)$ such that

$$f_*'(t_0) = -1. \tag{3.5}$$

Let us consider a decreasing sequence (c_n) which converges to c_* and denote by f_n the solution of $(\mathcal{P}_{a,b,c_n})$. Since $c_n > c_*$, the function f_n is of type (I) and

$$\forall t \in [0, \infty),\ f_n'(t) > 0. \tag{3.6}$$

On the other hand, it follows from Proposition 3.1 that

$$\lim_{n\to\infty} f_n'(t_0) = f_*'(t_0),$$

which contradicts (3.5) and (3.6). □

Lemma 3.3. *Let $f \in \mathcal{S}_{a,b}$. If f is of type (II), then*

$$\forall t \in [0,T), \;\; f(t) < \sqrt{a^2+4b}.$$

Proof. Let $f \in \mathcal{S}_{a,b}$. Multiplying the Blasius equation by t, and integrating by parts, we get the following identity

$$tf''(t) - f'(t) + b + \frac{1}{2}tf(t)f'(t) = \frac{1}{4}f(t)^2 - \frac{1}{4}a^2 + \frac{1}{2}\int_0^t sf'(s)^2\,ds. \tag{3.7}$$

Now, if f is of type (II) and if we denote by t_1 the point in $(0,T)$ such that $f'(t_1)=0$, we deduce from (3.7) that

$$\frac{1}{4}f(t_1)^2 < b + \frac{1}{4}a^2 + t_1 f''(t_1) < b + \frac{1}{4}a^2,$$

from which

$$|f(t_1)| < \sqrt{a^2+4b}.$$

This completes the proof, since f achieves its maximum at the point t_1. □

Lemma 3.4. *Let (c_n) be an increasing sequence which converges to c_*. Denote by f_n the solution of $(\mathcal{P}_{a,b,c_n})$ and by $[0,T_n)$ its right maximal interval of existence. Then*

$$\lim_{n\to\infty} T_n = \infty.$$

Proof. It follows directly from the lower semicontinuity of the mapping $c \longmapsto T_c$. □

Lemma 3.5. *The infimum c_* is negative.*

Proof. Let us assume that $c_* = 0$. Then $f_*(t) = f_0(t) = a+bt$, for all $t \in [0,\infty)$. Let (c_n) be an increasing sequence which converges to 0 and denote by f_n the solution of $(\mathcal{P}_{a,b,c_n})$ defined on $[0,T_n)$. Choose T such that

$$a + bT - \sqrt{a^2+4b} > 1.$$

Thanks to Lemma 3.4, we see that there exists an integer N such that $T_n > T$ for $n \geq N$. Then, from Lemma 3.3 we get

$$\forall t \in [0,T], \;\; f_n(t) \leq \sqrt{a^2+4b}.$$

Therefore, it follows from the choice of T that

$$f_0(T) - f_n(T) \geq a + bT - \sqrt{a^2+4b} > 1. \tag{3.8}$$

But, we deduce from Proposition 3.1 that

$$\lim_{n\to\infty} f_n(T) = f_0(T),$$

which is contradicting (3.8). □

Remark 3.1. Since f_* is of type (I), then (3.4) gives a strict lower bound for c_*. To improve this bound, we follow an idea given in **[5]** and use the fact that $f'_*(\infty) = f''_*(\infty) = 0$ (see Lemma 4.2 and Remark 4.1 in the next section). First, let us note that the function $\varphi = f'^2_* - 2f_*f''_*$ has as its derivative $f^2_*f''_*$ and consequently is decreasing. So, if we set $g = f'_*$, we have

$$\forall t > 0,\ 4g''(t) = \varphi(t) - g(t)^2 < \varphi(0) - g(t)^2.$$

Multiplying by $g'(t) < 0$ and integrating between 0 and ∞, we get

$$c_*^2 + abc_* - \frac{b^3}{3} < 0$$

and thus, since $c_* < 0$, this give the following inequality:

$$c_* > -\frac{b}{2}\left(a + \sqrt{a^2 + \frac{4b}{3}}\right).$$

To get an upper bound seems to be not very clear in the general case. In fact, the proof of the previous lemma does not give any estimate for c_*. However, for $a \geq 0$, it is possible to obtain an upper bound. To this end, we remark that, if f is a type (II) solution of $(\mathcal{P}_{a,b,c})$, and if we denote by t_0 the point where f vanishes, then f' is convex on $[0, t_0]$. Therefore, considering $t_1 \in (0, t_0)$ such that $f'(t_1) = 0$, we deduce from (3.3) that

$$-c > \frac{ab}{2} + \frac{1}{2}\int_0^{t_1} f'(s)^2\, ds > \frac{ab}{2} + \frac{1}{2}\int_0^{-\frac{b}{c}} (cs+b)^2\, ds = \frac{ab}{2} - \frac{b^3}{6c}$$

from which, by studying the sign of the polynomial $6X^2 + 3abX - b^3$, we get

$$c < -\frac{b}{4}\left(a + \sqrt{a^2 + \frac{8b}{3}}\right).$$

So, we have

$$c \geq -\frac{b}{4}\left(a + \sqrt{a^2 + \frac{8b}{3}}\right) \implies f \text{ is of type (I).}$$

Finally, when $a \geq 0$,

$$-\frac{b}{2}\left(a+\sqrt{a^2+\frac{4b}{3}}\right) < c_* \leq -\frac{b}{4}\left(a+\sqrt{a^2+\frac{8b}{3}}\right).$$

This estimate, only valid for $a \geq 0$, would be worth improving. In the case $a = 0$ and $b = 1$, this becomes $c_* \in \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{6}}\right]$ and is in agreement with the value choosen for c_* by P. Cheng and W. J. Minkowycz [**4**], to compute numerically the function f_*. In fact, to solve numerically the boundary problem (1.6) with $\lambda = 0$, the authors use the Runge-Kutta method for the Cauchy problem $(\mathcal{P}_{0,1,c_*})$ with $c_* = -0.4440$.

4. Asymptotic Behaviour of the Type (I) Solutions

In this part, the asymptotic behaviour, as $t \to \infty$, of type (I) solutions of $(\mathcal{P}_{a,b,c})$ will be discussed and as an application, we will get existence and uniqueness results for the boundary problem (1.6).

Let $a \in \mathbb{R}$ and $b > 0$. According to the previous section, there exists a negative real number $c_* = c_*(a,b)$ such that, for $c \in [c_*, 0]$, the function f_c, solution of $(\mathcal{P}_{a,b,c})$, is of type (I). Moreover, $0 < f_c' \leq b$ and since $f_c'' < 0$, there exists

$$\lambda = \lim_{t\to\infty} f_c'(t)$$

and $\lambda \in [0,b]$. We then can define the following function

$$\varphi_{a,b}\colon [c_*, 0] \longrightarrow [0,b] \text{ such that } \varphi_{a,b}(c) = f_c'(\infty).$$

From the Comparison Principle, we deduce that $\varphi_{a,b}$ is increasing. Indeed, if $c_1 > c_2$, then

$$\varphi_{a,b}(c_1) - \varphi_{a,b}(c_2) = f_{c_1}'(\infty) - f_{c_2}'(\infty) = \int_0^\infty (f_{c_1}''(s) - f_{c_2}''(s))\, ds > 0.$$

In what follows, we are going to show that $\varphi_{a,b}$ is an one-to-one mapping of $[c_*, 0]$ onto $[0,b]$. First, let us prove some lemmas.

Lemma 4.1. *Let f be a type (I) solution of $(\mathcal{P}_{a,b,c})$ with $c < 0$. If we set $\lambda = f'(\infty)$, then there exists $\mu \in (a, \sqrt{a^2+4b-4\lambda})$ such that*

$$\lim_{t\to\infty}(f(t) - \lambda t) = \mu. \tag{4.1}$$

Moreover, for all $t > 0$ we have

$$\lambda t + a < f(t) < \lambda t + \sqrt{a^2+4b-4\lambda}. \tag{4.2}$$

Proof. Since f' is decreasing, we have $f' - \lambda > 0$ and thus, the function

$$t \longmapsto f(t) - \lambda t - a$$

is increasing and positive on $(0, \infty)$. Therefore, there exists $\mu \in (a, \infty]$ such that

$$\lim_{t\to\infty} (f(t) - \lambda t) = \mu.$$

Assume that $\mu = \infty$. Then

$$\forall t \geq 0, \ \frac{f'''(t)}{-f''(t)} = \frac{1}{2} f(t) \geq \frac{1}{2}(f(t) - \lambda t)$$

implying that

$$\lim_{t\to\infty} \frac{f'''(t)}{-f''(t)} = \infty.$$

Therefore, there exists $t_0 > 0$ such that, for all $t \geq t_0$, one has $f'''(t) \geq -f''(t)$, which gives, by integrating between $s \geq t_0$ and ∞,

$$\forall s \geq t_0, \ -f''(s) \geq -\lambda + f'(s).$$

Next, integrating between t_0 and $t > t_0$, we get

$$\forall t \geq t_0, \ -f'(t) + f'(t_0) \geq f(t) - \lambda t - f(t_0) + \lambda t_0$$

which is a contradiction, since the left side is bounded, whereas the right side tends to infinity. Therefore, $\mu < \infty$ and we have

$$\forall t > 0, \ \lambda t + a < f(t) < \lambda t + \mu.$$

To get (4.2), let us introduce the auxiliary nonnegative function

$$u(t) = f'(t) + \frac{1}{4}(f(t) - \lambda t)^2.$$

From (4.1), we see that u is bounded. Moreover, we have

$$u''(t) = -\frac{1}{2}\lambda t f''(t) + \frac{1}{2}(f'(t) - \lambda)^2 > 0$$

and u is convex. Therefore u is decreasing and thus

$$\frac{a^2}{4} + b = u(0) > u(\infty) = \lambda + \frac{1}{4}\mu^2.$$

This completes the proof. □

Lemma 4.2. *If $\lambda_* = f_*'(\infty)$, then $\lambda_* = 0$.*

Proof. Let us assume that $\lambda_* > 0$ and consider an increasing sequence (c_n) which converges to c_*. Denote by f_n the solution of $(\mathcal{P}_{a,b,c_n})$ and by $[0, T_n)$ its right maximal interval of existence. We know, thanks to Lemma 3.4, that

$$\lim_{n\to\infty} T_n = \infty.$$

Choose now T such that

$$a + \lambda_* T - \sqrt{a^2 + 4b} > 1. \tag{4.3}$$

Then, there exists an integer N such that for $n \geq N$, we have $T_n > T$ and

$$\lim_{n\to\infty} f_n(T) = f_*(T) \tag{4.4}$$

in accordance with Proposition 3.1. But, it follows from (4.2), Lemma 3.3 and (4.3) that

$$f_*(T) - f_n(T) > \lambda_* T + a - \sqrt{a^2 + 4b} > 1,$$

which contradicts (4.4). Consequently $\lambda_* = 0$. □

Remark 4.1. If we set $\mu_* = f_*(\infty)$, then $\mu_* > 0$. Otherwise, if $\mu_* \leq 0$, both functions f_* and f_*''' would be negative. Then, f_*' should be concave, decreasing and positive, which is impossible. Moreover, one can show that

$$f_*''(t) = \mathcal{O}(\mathrm{e}^{-\mu_* t}),\ f_*'(t) = \mathcal{O}(\mathrm{e}^{-\mu_* t}) \ \text{ and } \ f_*(t) - \mu_* = \mathcal{O}(\mathrm{e}^{-\mu_* t})$$

for $t \to \infty$. See **[5]**.

The function $c \longmapsto \mu$ with μ defined by (4.1), seems to be decreasing from $[c_*, 0]$ onto $[a, \mu_*]$ and look at that could be an interesting task.

We are now able to give the main result of this section.

Theorem 4.1. *The function $\varphi_{a,b}$ defined by $\varphi_{a,b}(c) = f_c'(\infty)$ is an increasing one-to-one mapping of $[c_*, 0]$ onto $[0, b]$.*

Proof. Taking into account Lemma 4.2, the fact that $\varphi_{a,b}(0) = b$ and the monotony of $\varphi_{a,b}$, we just have to show the continuity. So, let us consider $c \in [c_*, 0]$ and let (c_n) be a sequence in $[c_*, 0]$, which converges to c. Set $\lambda = \varphi_{a,b}(c)$ and $\lambda_n = \varphi_{a,b}(c_n)$. To complete the proof, it remains to show that any convergent subsequence $(\lambda_{\psi(n)})$ of (λ_n) converges to λ. Consider such a subsequence, set

$$\tilde{\lambda} = \lim_{n\to\infty} \lambda_{\psi(n)},$$

and assume that $\lambda > \tilde{\lambda}$. Let $N \in \mathbb{N}$ be such that,

$$n \geq N \implies \lambda_{\psi(n)} < \frac{\lambda + \tilde{\lambda}}{2} < \lambda. \tag{4.5}$$

Choose now T such that

$$\frac{\lambda - \tilde{\lambda}}{2} T + a - \sqrt{a^2 + 4b} > 1. \tag{4.6}$$

If we denote by f, f_n the solutions of $(\mathcal{P}_{a,b,c})$ and $(\mathcal{P}_{a,b,c_n})$, and if we set $h_n = f - f_{\psi(n)}$, then

$$\lim_{n\to\infty} h_n(T) = 0. \tag{4.7}$$

But, it follows from (4.2), (4.5) and (4.6) that

$$\begin{aligned} h_n(T) &= f(T) - f_{\psi(n)}(T) \\ &> \lambda T + a - \lambda_{\psi(n)} T - \sqrt{a^2 + 4b - 4\lambda_{\psi(n)}} \\ &> \lambda T + a - \frac{\lambda + \tilde{\lambda}}{2} T - \sqrt{a^2 + 4b} \\ &= \frac{\lambda - \tilde{\lambda}}{2} T + a - \sqrt{a^2 + 4b} > 1, \end{aligned}$$

which contradicts (4.7). By assuming $\lambda < \tilde{\lambda}$, the same way leads to a contradiction too. Consequently, $\lambda = \tilde{\lambda}$. □

Corollary 4.1. *Let $a \in \mathbb{R}$, $b \in \mathbb{R}$ and $\lambda \in (-\infty, b]$. The boundary value problem*

$$\begin{cases} f''' + \frac{1}{2} f f'' = 0 \quad \text{on} \quad [0, \infty), \\ f(0) = a \qquad f'(0) = b, \\ f'(\infty) = \lambda, \end{cases} \tag{4.8}$$

has one and only one solution when $b \geq 0$ and $\lambda \in [0, b]$, and no solution if $\lambda < 0$.

Proof. The first assertion follows from Theorem 4.1, and the second one from Proposition 2.2. □

Remark 4.2. As we saw in the introduction, the boundary value problem (4.8), with $\lambda > b$, involves necessarily convex solutions. See **[8]**, for the case $b \in [0, 1)$ and $\lambda = 1$.

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