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ON SECOND ORDER DIFFERENTIAL INCLUSIONS WITH PERIODIC BOUNDARY CONDITIONS

M. BENCHOHRA AND S. K. NTOUYAS

ABSTRACT. In this paper a fixed point theorem for condensing maps combined with upper and lower solutions are used to investigate the existence of solutions for second order differential inclusions with periodic boundary conditions.

1. Introduction

This note is concerned with the existence of solutions for the periodic multivalued problem:

$$y''(t) \in F(t, y(t)), \quad \text{for a.e. } \ t \in J = [0, T] \tag{1.1}$$

$$y(0) = y(T), \ y'(0) = y'(T) \tag{1.2}$$

where $F\colon J \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is a compact and convex multivalued map.

The method of upper and lower solutions has been successfully applied to study the existence of multiple solutions for initial and boundary value problems of the first and second order.

This method has been used only in the context of single-valued differential equations. We refer to the books of Bernfeld-Lakshmikantham [**BeLa**], Heikkila-Lakshmikantham [**HeLa**], Ladde-Lakshmikantham-Vatsala [**LaLaVa**], to the thesis of De Coster [**Dec**], to the papers of Carl-Heikkila-Kumpulainen [**CaHeKu**], Cabada [**Cab**], Frigon [**Fri**], Frigon-O'Regan [**FrOr**], Heikkila-Cabada [**HeCa**], Lakshmikantham-Leela [**LaLe**], Nieto [**Nie**], [**Nie1**], Nkashama [**Nka**] and the references therein.

First order differential inclusions with periodic boundary conditions was studied by the authors in [**BeNt**]. Here we extend the results of [**BeNt**] to second order differential inclusions with periodic boundary conditions. Our approach is based on the existence of upper and lower solutions and on a fixed point theorem for condensing maps due to Martelli [**Mar**].

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2. Preliminaries

We will briefly recall some basic definitions and facts from multivalued analysis that we will use in the sequel.

$AC(J, \mathbb{R})$ is the space of all absolutely continuous functions $y\colon J \longrightarrow \mathbb{R}$.

$W^{2,1}(J, \mathbb{R})$ denotes the Banach space of functions $y\colon J \longrightarrow \mathbb{R}$ whose first derivative are absolutely continuous and whose second derivative y'' (which exists almost everywhere) is an element of $L^1(J, \mathbb{R})$ with the norm

$$\|y\|_{W^{2,1}} = \|y\|_{L^1} + \|y'\|_{L^1} + \|y''\|_{L^1} \quad \text{for all } y \in W^{2,1}(J, \mathbb{R}).$$

Condition

$$y \leq \bar{y} \quad \text{if and only if} \quad y(t) \leq \bar{y}(t) \quad \text{for all} \quad t \in J$$

defines a partial ordering in $W^{2,1}(J, \mathbb{R})$. If α, $\beta \in W^{2,1}(J, \mathbb{R})$ and $\alpha \leq \beta$, we denote

$$[\alpha, \beta] = \{y \in W^{2,1}(J, \mathbb{R}) : \alpha \leq y \leq \beta\}.$$

Let $(X, \|.\|)$ be a normed space. A multivalued map $G\colon X \longrightarrow 2^X$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. G is bounded on bounded sets if $G(B) = \cup_{x \in B} G(x)$ is bounded in X for all bounded subsets B of X (i.e. $\sup\limits_{x \in B}\{\sup\{\|y\| : y \in G(x)\}\} < \infty$). G is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set N of X containing $G(x_0)$, there exists an open neighbourhood M of x_0 such that $G(M) \subseteq N$.

G is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B \subset X$.

If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in Gx_n$ imply $y_* \in Gx_*$).

G has a fixed point if there is $x \in X$ such that $x \in Gx$.

In the following $CC(X)$ denotes the set of all nonempty compact and convex subsets of X.

An upper semi-continuous map $G\colon X \longrightarrow 2^X$ is said to be condensing [**Mar**] if for any bounded subset $B \subseteq X$, we have $\alpha(G(B)) < \alpha(B)$, where α denotes the Kuratowski measure of noncompacteness [**BaGo**]. We remark that a compact map is the easiest example of a condensing map. For more details on multivalued maps see the books of Deimling [**Dei**] and Hu and Papageorgiou [**HuPa**].

The multivalued $F\colon J \longrightarrow CC(\mathbb{R})$ is said to be measurable, if for every $y \in \mathbb{R}$, the function $t \longmapsto d(y, F(t)) = \inf\{\|y - z\| : z \in F(t)\}$ is measurable.

Definition 1. A multivalued map $F\colon J \times \mathbb{R} \longrightarrow 2^{\mathbb{R}}$ is said to be an L^1-Carathéodory if

(i) $t \longmapsto F(t,y)$ is measurable for each $y \in \mathbb{R}$;
(ii) $y \longmapsto F(t,y)$ is upper semicontinuous for almost all $t \in J$;
(iii) For each $k > 0$, there exists $h_k \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t,y)\| = \sup\{|v| : v \in F(t,y)\} \le h_k(t)$$

for all $|y| \le k$ and for almost all $t \in J$.

So let us start by defining what we mean by a solution of problem (1.1)–(1.2).

Definition 2. A function $y \in W^{2,1}(J,\mathbb{R})$ is said to be a solution of (1.1)–(1.2) if there exists a function $v \in L^1(J,\mathbb{R})$ such that $v(t) \in F(t,y(t))$ a.e. on J, $y''(t) = v(t)$ a.e. on J, $y(0) = y(T)$ and $y'(0) = y'(T)$.

The following concept of lower and upper solutions for (1.1)–(1.2) has been introduced by Halidias and Papageorgiou in **[HaPa]** for second order multivalued boundary value problems. It will be the basic tools in the approach that follows.

Definition 3. A function $\alpha \in W^{2,1}(J,\mathbb{R})$ is said to be a lower solution of (1.1)–(1.2) if there exists $v_1 \in L^1(J,\mathbb{R})$ such that $v_1(t) \in F(t,\alpha(t))$ a.e. on J, $\alpha''(t) \ge v_1(t)$ a.e. on J, $\alpha(0) = \alpha(T)$ and $\alpha'(0) \ge \alpha'(T)$.

Similarly a function $\beta \in W^{2,1}(J,\mathbb{R})$ is said to be an upper solution of (1.1)–(1.2) if there exists $v_2 \in L^1(J,\mathbb{R})$ such that $v_2(t) \in F(t,\beta(t))$ a.e. on J, $\beta''(t) \le v_2(t)$ a.e. on J, $\beta(0) = \beta(T)$ and $\beta'(0) \le \beta'(T)$.

For the multivalued map F and for each $y \in C(J,\mathbb{R})$ we define $S^1_{F,y}$ by

$$S^1_{F,y} = \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,y(t)) \text{ for a.e. } t \in J\}.$$

Our main result is based on the following:

Lemma 1 (**[LaOp]**). *Let I be a compact real interval and X be a Banach space. Let $F\colon I \times X \longrightarrow CC(X)$; $(t,y) \to F(t,y)$ measurable with respect to t for any $y \in X$ and u.s.c. with respect to y for almost each $t \in I$ and $S^1_{F,y} \neq \emptyset$ for any $y \in C(I,X)$ and let Γ be a linear continuous mapping from $L^1(I,X)$ to $C(I,X)$, then the operator*

$$\Gamma \circ S^1_F \colon C(I,X) \longrightarrow CC(C(I,X)),\ y \longmapsto (\Gamma \circ S^1_F)(y) := \Gamma(S^1_{F,y})$$

is a closed graph operator in $C(I,X) \times C(I,X)$.

Lemma 2 (**[Mar]**). *Let $G\colon X \longrightarrow CC(X)$ be a condensing map. If the set*

$$M := \{v \in X : \lambda v \in G(v) \text{ for some } \lambda > 1\}$$

is bounded, then G has a fixed point.

3. Main Result

We are now in a position to state and prove our existence result for the problem (1.1)–(1.2).

Theorem 1. *Suppose $F\colon J\times\mathbb{R}\longrightarrow CC(\mathbb{R})$ is an L^1-Carathéodory multivalued map. In addition assume that the following condition is satisfied*

(H) *there exist α and β in $W^{2,1}(J,\mathbb{R})$ lower and upper solutions respectively for the problem (1.1)–(1.2) such that $\alpha\le\beta$.*

Then the problem (1.1)–(1.2) has at least one solution $y\in W^{2,1}(J,\mathbb{R})$ such that

$$\alpha(t)\le y(t)\le\beta(t)\quad \textit{for all}\quad t\in J.$$

Proof. Transform the problem into a fixed point problem. Consider the following modified problem

$$y''(t)-y(t)\in F_1(t,y(t)),\quad \text{a.e. } t\in J, \tag{3.1}$$

$$y(0)=y(T),\ y'(0)=y'(T) \tag{3.2}$$

where $F_1(t,y)=F(t,(\tau y)(t))-(\tau y)(t)$ and $\tau\colon C(J,\mathbb{R})\longrightarrow C(J,\mathbb{R})$ is the truncation operator defined by

$$(\tau y)(t)=\begin{cases}\alpha(t), & \text{if } y(t)<\alpha(t);\\ y(t), & \text{if } \alpha(t)\le y\le\beta(t);\\ \beta(t)), & \text{if } \beta(t)<y(t).\end{cases}$$

Remark 1. Notice that F_1 is an L^1-Carathéodory multivalued map with compact convex values and there exists $\phi\in L^1(J,\mathbb{R}_+)$ such that

$$\|F_1(t,y(t))\|\le\phi(t)+\max(\sup_{t\in J}|\alpha(t)|,\sup_{t\in J}|\beta(t)|)$$

for a.e. $t\in J$ and all $y\in C(J,\mathbb{R})$.

Clearly, a solution to (3.1)–(3.2) is a fixed point of the operator $N\colon C(J,\mathbb{R})\longrightarrow 2^{C(J,\mathbb{R})}$ defined by

$$Ny:=\Big\{h\in C(J,\mathbb{R}):h(t)=\int_0^T G(t,s)[v(s)-(\tau y)(s)]\,ds:\ v\in\tilde{S}^1_{F,y}\Big\}$$

where $G(t,s)$ is the Green function corresponding to the problem

$$\begin{aligned} y''-y&=g(t),\ t\in J\\ y(0)&=y(T),\ y'(0)=y'(T)\end{aligned}$$

$$\begin{aligned}
\tilde{S}^1_{F,y} &= \{v \in S^1_{F,\tau y} : v(t) \le v_1(t) \text{ a.e. on } A_1 \text{ and } v(t) \ge v_2(t) \text{ a.e. on } A_2\},\\
S^1_{F,\tau y} &= \{v \in L^1(J,\mathbb{R}) : v(t) \in F(t,(\tau y)(t)) \text{ for a.e. } t \in J\},\\
A_1 &= \{t \in J : y(t) < \alpha(t) \le \beta(t)\},\\
A_2 &= \{t \in J : \alpha(t) \le \beta(t) < y(t)\}.
\end{aligned}$$

Remark 2. (i) For each $y \in C(J,\mathbb{R})$ the set $S^1_{F,y}$ is nonempty (see Lasota and Opial [**LaOp**]).

(ii) For each $y \in C(J,\mathbb{R})$ the set $\tilde{S}^1_{F,y}$ is nonempty. Indeed, by (i) there exists $v \in S^1_{F,\tau y}$. Set

$$w = v_1\chi_{A_1} + v_2\chi_{A_2} + v\chi_{A_3},$$

where

$$A_3 = \{t \in J : \alpha(t) \le y(t) \le \beta(t)\}.$$

Then by decomposability $w \in \tilde{S}^1_{F,y}$.

We shall show that N has a fixed point. The proof will be given in several steps. We first shall show that N is a completely continuous multivalued map, u.s.c. with convex closed values.

Step 1: Ny **is convex for each** $y \in C(J,\mathbb{R})$**.**

Indeed, if h, $\overline{h}$ belong to Ny, then there exist $v \in \tilde{S}^1_{F,y}$ and $\overline{v} \in \tilde{S}^1_{F,y}$ such that

$$h(t) = \int_0^T G(t,s)[v(s) - (\tau y)(s)]\,ds, \quad t \in J$$

and

$$\overline{h}(t) = \int_0^T G(t,s)[\overline{v}(s) - (\tau y)(s)]\,ds, \quad t \in J.$$

Let $0 \le k \le 1$. Then for each $t \in J$ we have

$$[kh + (1-k)\overline{h}](t) = \int_0^T G(t,s)[kv(s) + (1-k)\overline{v}(s) - (\tau y)(s)]\,ds.$$

Since $\tilde{S}^1_{F,y}$ is convex (because F has convex values) then

$$kh + (1-k)\overline{h} \in Ny.$$

Step 2: N **is completely continuous.**

Let $B_r := \{y \in C(J,\mathbb{R}) : \|y\|_\infty = \sup\{|y(t)| : t \in J\} \le r\}$ be a bounded set in $C(J,\mathbb{R})$ and $y \in B_r$, then for each $h \in Ny$ there exists $v \in \tilde{S}^1_{F,y}$ such that

$$h(t) = \int_0^T G(t,s)[v(s) - (\tau y)(s)]\,ds, \quad t \in J.$$

Thus for each $t \in J$ we get

$$\begin{aligned}|h(t)| &\leq \int_0^T |G(t,s)|[|v(s)| + |(\tau y)(s)|]\, ds\\ &\leq \max_{(t,s)\in J\times J} |G(t,s)|[\|\phi_R\|_{L^1} + T\max(r, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)] = K.\end{aligned}$$

Furthermore

$$\begin{aligned}|h'(t)| &\leq \int_0^T |G'_t(t,s)|[|v(s)| + |(\tau y)(s)|]\, ds\\ &\leq \max_{(t,s)\in J\times J} |G'_t(t,s)|[\|\phi_R\|_{L^1} + T\max(r, \sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)] = K_1.\end{aligned}$$

We note that $G(t,s)$ and $G'_t(t,s)$ are continuous on $J\times J$. Thus N maps bounded set of $C(J,\mathbb{R})$ into uniformly bounded and equicontinuous set of $C(J,\mathbb{R})$.

Step 3: N has a closed graph.

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$, and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. $h_n \in N(y_n)$ means that there exists $v_n \in \tilde{S}_{F,y_n}$ such that

$$h_n(t) = \int_0^T G(t,s)[v_n(s) - (\tau y_n)(s)]\, ds, \quad t \in J.$$

We must prove that there exists $v_* \in \tilde{S}_{F,y_*}$ such that

$$h_*(t) = \int_0^T G(t,s)[v_*(s) - (\tau y_*)(s)]\, ds, \quad t \in J.$$

Since $y_n \to y_*,\ h_n \to h_*$ and τ is a continuous function we have that

$$\left\|\Big(h_n + \int_0^T G(t,s)(\tau y_n)(s)\, ds\Big) - \Big(h_* + \int_0^T G(t,s)(\tau y_*)(s)\, ds\Big)\right\|_\infty \longrightarrow 0,$$

as $n \longrightarrow \infty$.

Now, we consider the linear continuous operator

$$\Gamma\colon L^1(J,\mathbb{R}) \longrightarrow C(J,\mathbb{R})$$

$$v \longmapsto \Gamma(v)(t) = \int_0^T G(t,s)v(s)\, ds.$$

From Lemma 1, it follows that $\Gamma \circ \tilde{S}_F$ is a closed graph operator.

Moreover, we have that

$$\Big(h_n(t) + \int_0^T G(t,s)(\tau y_n)(s)\, ds\Big) \in \Gamma(\tilde{S}_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows from Lemma 1 that

$$h_*(t) + \int_0^T G(t,s)(\tau y_*)(s)\,ds = \int_0^t G(t,s)v_*(s)\,ds$$

for some $v_* \in \tilde{S}_{F,y_*}$.

Therefore N is a completely continuous multivalued map, u.s.c. with convex closed values.

Step 4: Now, we are going to show that the set

$$M := \{y \in C(J,\mathbb{R}) : \lambda y \in N(y) \text{ for some } \lambda > 1\}$$

is bounded.

Let $\lambda y \in N(y)$, $\lambda > 1$. Then there exists $v \in \tilde{S}^1_{F,y}$ such that

$$y(t) = \lambda^{-1} \int_0^T G(t,s)[v(s) - (\tau y)(s)]\,ds, \quad t \in J.$$

Thus

$$|y(t)| \leq |G(t,s)| \int_0^T [|v(s)| + |(\tau y)(s)|]\,ds, \quad t \in J.$$

Thus we obtain

$$\|y\|_\infty \leq |G(t,s)|[\|\phi\|_{L^1} + T\max(\sup_{t\in J}|\alpha(t)|, \sup_{t\in J}|\beta(t)|)].$$

This shows that M is bounded. Hence, Lemma 2 applies and N has a fixed point which is a solution to problem (3.1)–(3.2).

Step 5: We shall show that the solution y of (3.1)–(3.2) satisfies

$$\alpha(t) \leq y(t) \leq \beta(t) \quad \text{for all} \quad t \in J.$$

Let y be a solution to (3.1)–(3.2). We prove that

$$\alpha(t) \leq y(t) \quad \text{for all} \quad t \in J.$$

Assume that $y - \alpha$ attains a negative minimum on J at t_0, that is

$$(y-\alpha)(t_0) = \min\{y(t) - \alpha(t) : t \in J\} < 0.$$

We shall distinguish the following cases.

Case 1. $t_0 \in (0,T)$.

Then there exists $t_0^* \in (t_0, T)$ such that

$$y(t) - \alpha(t) < 0 \quad \text{for all} \quad t \in (t_0, t_0^*).$$

By the definition of τ there exists $v \in L^1(J, \mathbb{R})$ with $v(t) \leq v_1(t)$ a.e. on (t_0, t_0^*) and $v(t) \in F(t, \alpha(t))$ a.e. on (t_0, t_0^*).

Since $y'(t_0) - \alpha'(t_0) = 0$ and using the fact that α is a lower solution to (1.1)–(1.2) we obtain that for $t \in (t_0, t_0^*)$

$$\begin{aligned} y'(t) - \alpha'(t) &= \int_{t_0}^{t} (y''(s) - \alpha''(s))\, ds \\ &\leq \int_{t_0}^{t} [v(s) + y(s) - \alpha(s) - v_1(s)]\, ds \\ &< 0. \end{aligned}$$

This proves that $y(t_0) - \alpha(t_0)$ is not a minimum of $y - \alpha$ which is a contradiction.

Case 2. $\min\{y(t) - \alpha(t) : t \in J\} = y(0) - \alpha(0) = y(T) - \alpha(T) < 0$.
Then we obtain

$$y'(0) - \alpha'(0) \geq 0 \geq y'(T) - \alpha'(T)$$

and from the definition of a lower solution

$$y'(0) - \alpha'(0) \leq y'(T) - \alpha'(T).$$

Hence, $y'(0) - \alpha'(0) = 0$ and for $t > 0$ small

$$\begin{aligned} y'(t) - \alpha'(t) &= \int_{0}^{t} (y''(s) - \alpha''(s))\, ds \\ &\leq \int_{0}^{t} [v(s) + y(s) - \alpha(s) - v_1(s)]\, ds \\ &< 0 \end{aligned}$$

which is a contradiction.

Analogously, we can prove that

$$y(t) \leq \beta(t) \quad \text{for all} \quad t \in J.$$

This shows that the problem (3.1)–(3.2) has a solution in the interval $[\alpha, \beta]$. Since

$$\tau(y) = y \quad \text{for all} \quad y \in [\alpha, \beta],$$

then y is a solution to (1.1)–(1.2). $\square$

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M. Benchohra, Département de Mathématiques, Université de Sidi Bel Abbès, BP 89, 22000 Sidi Bel Abbès, Algérie; *e-mail*: benchohra@yahoo.com

S. K. Ntouyas, Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece; *e-mail*: sntouyas@cc.uoi.gr