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VECTOR ERGODIC THEOREM IN $L(X)\log L(X)$

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Abstract. Let X be a reflexive Banach space and Ω be a finite measure space. We prove the almost everywhere convergence of the vector multiparameter averages

$$\frac{1}{n_1\dots n_d}\sum_{0\le k_1,\dots k_d<n_j}\alpha^1_{k_1}\dots\alpha^d_{k_d}T_1^{k_1}\dots T_d^{k_d}f$$

for all $f\in L^p(X)$, $1<p<\infty$, and where $\{(\alpha^j_n)\}$ are bounded Besicovitch sequences $(j=1,\dots,d)$ with $T_1,\dots,T_d$ are linear operators acting on $L^1(X)$ and satisfying certain conditions.

For $d=2$, we obtain more general result. Indeed, in this case, we prove the convergence a.e. for $f\in L(X)\log L(X)$. The general case $(d>2)$ requires integrability of the supremum of the norm of these averages. As applications, we give new proof of Zygmund-Fava's Theorem.

1. Introduction

Let (Ω,β,μ) be a σ-finite measure space, X be a Banach space with norm $\|.\|$. Let $L^1(\Omega,X)=L^1(X)$ be the usual Banach space of all X-valued strongly measurable functions f on Ω. By $T_1,\dots,T_d$ we denote a family of linear operators (maybe not commuting) on $L^1(X)$. We suppose that $T_1,\dots,T_d$ are contractions (or power bounded) in $L^\infty(X)$ and T_d is a contraction in $L^1(X)$ and the other operators verifying that: For all countable sets Δ, for all families $(f_\alpha)_{\alpha\in\Delta}$

$$(*)\qquad \int\sup_{\alpha\in\Delta}\|T_jf_\alpha\|\,d\mu\le\int\sup_{\alpha\in\Delta}\|f_\alpha\|\,d\mu$$

for $j=1,\dots,d-1$. Under this hypothesis, we prove the almost everywhere convergence of the multiparameter weighted averages

$$B(n_1,\dots,n_d,T_1,\dots,T_d)f=\frac{1}{n_1\dots n_d}\sum_{0\le k_1,\dots k_d<n_j}\alpha^1_{k_1}\dots\alpha^d_{k_d}T_1^{k_1}\dots T_d^{k_d}f$$

as $n_j\to\infty$, for $j=1,\dots,d$, where $\left\{\left(\alpha^j_n\right)\right\}$ are bounded Besicovitch sequences, for all $f\in L^p(X)$, $1<p<\infty$.

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Recall that a sequence $a(k)$ is called Besicovitch sequence if for $\varepsilon > 0$, there is a trigonometric polynomial ψ_ε such that

$$\limsup_n \frac{1}{n} \sum_{k=0}^{n-1} |a(k) - \psi_\varepsilon(k)| < \varepsilon$$

For $d = 2$, we prove the convergence a.e. of these averages for $f \in L(X) \log L(X)$ which is more general than $f \in L^1(X)$. The general case ($d > 2$) can be obtained by assuming that $\sup_{n_j} \|B(n_1, \dots, n_d, T_1, \dots, T_d)f\| \in L^1$.

Using linear modulus, J. Olsen [**7**] proved the almost everywhere convergence of the averages

$$B(n_1, \dots, n_d, T_1, \dots, T_d)f = \frac{1}{n_1 \dots n_d} \sum_{0 \le k_1, \dots k_d < n_j} \alpha^1_{k_1} \dots \alpha^d_{k_d} T_1^{k_1} \dots T_d^{k_d} f$$

for all $f \in L^p(\mathbf{R})$, $1 < p < \infty$ and $\left(\alpha^j_n\right)$ are bounded Besicovitch sequences $(j = 1, \dots, d)$. The difficulty in obtaining the result in the vector case is the absence of the analog to linear modulus. In [**4**] it was shown that the existence of a vector operator on $L^1(X)$ does not admit a linear modulus which means that for the vector case the results cannot be obtained by Olsen's method.

Naturally, one asks if there is another argument does not require the use of linear modulus. For this reason, we introduce the above hypothesis $(*)$ on the operators $T_1, \dots, T_{d-1}$. Our result will generalize Chacon's Theorem [**1**] to multiparameter operators satisfying the condition $(*)$.

Remark 1. Any operator satisfying the condition $(*)$ is a contraction in $L^1(X)$.

2. Main Result

Before stating our main results, the following two Theorems are needed for which the proofs can be found in [**1**] and [**5**] respectively.

Theorem 2 (Chacon [**1**])**.** *Let X be a reflexive Banach space, T be a linear contraction in both $L^1(\Omega, X)$ and in $L^\infty(\Omega, X)$. If $f \in L^1(\Omega, X)$, then*

$$a\mu \left\{ \sup \left\| \frac{1}{n} \sum_{i=0}^{n} T^i f \right\| > a \right\} \le \int_{\{f^* > a\}} \|f\|_X \, d\mu$$

and the averages $\frac{1}{n} \sum_{i=0}^{n} T^i f$ converge a.e.

The next theorem is in fact, an extension of Theorem 2 for the weighted averages.

Theorem 3. *Let X be a reflexive Banach space, T be a linear contraction in both $L^1(\Omega, X)$ and in $L^\infty(\Omega, X)$. If $f \in L^1(\Omega, X)$, and α_n is a bounded real sequence, then*

$$a\mu\left\{\sup\left\|\frac{1}{n}\sum_{i=0}^{n}\alpha_i T^i f\right\| > a\right\} \le \int_{\{f^* > a\}} \|f\|_X \, d\mu$$

and if α_n is Besicovitch sequence the averages $\frac{1}{n}\sum_{i=0}^{n}\alpha_i T^i f$ converge a.e.

Now, we are ready to state our main results.

Theorem 4. *Let $T_1, \ldots, T_d$ be linear operators (maybe not commuting) on $L^1(X)$ power bounded in $L^\infty(X)$. T_d is a contraction in $L^1(X)$ and the other operators $T_1, \ldots, T_{d-1}$ satisfying the condition $(*)$. Let $\left(\alpha_n^j\right)$ be bounded Besicovitch sequences for $j = 1, \ldots, d$. Then:*

(i) For $f \in L^p(X)$, $1 < p < \infty$, and $\alpha = \sup_{k=1,\ldots,d}\sup_j \left|\alpha_j^k\right|$ we have

$$\left\|\sup_{n_j}\|B(n_1, \ldots, n_d, T_1, \ldots, T_d)f\|\right\|_p \le \alpha\left(\frac{p}{p-1}\right)^d \|f\|_p$$

and $\lim_{n_j} B(n_1, \ldots, n_d, T_1, \ldots, T_d)f$ exists a.e.

(ii) For $f \in L(X)\log L(X)$, $\lim_{n_1,n_2} B_{N_2}(T_1, T_2)f$ exists a.e.

Example 5. Operators verifying the condition $(*)$:

1) Any positive linear contraction in $L^1(\mathbf{R})$.

2) Any vector operator defined on $L^1(X)$ which is dominated by a positive contraction in $L^1(\mathbf{R})$. (Note that the measure preserving transformation (m.p.t) $Tf = fo\theta \in L^1(X)$ verifies $\|Tf\| = \|fo\theta\| = \|f\|\, o\theta = \tau(\|f\|)$ where τ is m.p.t. defined on $L^1(\mathbf{R})$. It follows that every m.p.t. satisfies the condition $(*)$).

3) The surjective isometries on $L^1(\Omega, X)$ (of course, they are not given by m.p.t.) also are dominated by a positive contraction on $L^1(\mathbf{R})$ (see [**4**]).

The next proposition proves that the class of real operators acting in $L^1(\mathbf{R})$ and satisfying the condition $(*)$ is the same as the class of the contractions in $L^1(\mathbf{R})$.

Proposition 6. *Let T be a linear operator on $L^1(\mathbf{R})$ the following assertions are equivalent.*

(i) T verifies the condition $(*)$.

(ii) T is a contraction in $L^1(\mathbf{R})$.

Proof. If T satisfies $(*)$, then it suffices to take $f_n = f$ to prove that T is contraction in $L^1(X)$. Conversely, let τ be the linear modulus of T (see [**8**]), τ is a linear positive operator on $L^1(\mathbf{R})$ and verifies that for all $f \in L^1(\mathbf{R})$; $|Tf| \le \tau(|f|)$.

Moreover, if T is $L^1(X)$-contraction then, τ is a $L^1(\mathbf{R})$-contraction, we write

$$\begin{aligned}\int \sup_{\alpha\in\Delta} |Tf_\alpha| \, d\mu &\leq \int \sup_{\alpha\in\Delta} \tau\left(|f_\alpha|\right) d\mu \\ &\leq \int \tau\left(\sup_{\alpha\in\Delta} \left(|f_\alpha|\right)\right) d\mu \qquad (\tau \geq 0) \\ &\leq \int \left(\sup_{\alpha\in\Delta} \left(|f_\alpha|\right)\right) d\mu \qquad (\|\tau\|_1 \leq 1)\end{aligned}$$

which ends the proof of the proposition. □

Proof of Theorem 4. We prove this theorem in two steps.

First, we prove the case when the weight of average are one. The second case is the general weighted averages.

1. The case $\alpha_n^j = 1$, **for every** $n \in N$. First, we study the case $d = 2$: Let T_1 and T_2 be two linear operators on $L^1(X)$ such that T_1 is a contraction in $L^\infty(X)$ and verifying: For all $f_n \in L^p(X)$, $1 < p < \infty$

$$(**) \qquad \int \sup_n \|T_1 f_n\| \, d\mu \leq \int \sup_n \|f_n\| \, d\mu$$

and and T_2 is a contraction in $L^1(X)$ and a contraction (or power bounded) in $L^\infty(X)$.

Consider the Banach space $\mathcal{X}_1 = l^\infty(X) = \{(x_n); x_n \in X \text{ and } \sup_n \|x_n\| < \infty\}$. Define the norm $\|x\|_{\mathcal{X}_1} = \sup_n \|x_n\|$, let U the operator on $L^1(\mathcal{X}_1)$ defined by $Uf = (T_1 f_n)$ for every $f = (f_n) \in L^1(\mathcal{X}_1)$. By $(**)$ we have $Uf \in L^1(\mathcal{X}_1)$, and

$$\begin{aligned}\|Uf\|_{L^1(\mathcal{X}_1)} &= \int \|Uf(\omega)\|_{\mathcal{X}_1} \, d\mu(\omega) = \int \sup_n \|T_1 f_n(\omega)\| \, d\mu(\omega) \\ &\leq \int \sup_n \|f_n(\omega)\| \, d\mu(\omega) \quad (\text{by } (**)) \\ &= \int \|f(\omega)\|_{\mathcal{X}_1} \, d\mu(\omega) = \|f\|_{L^1(\mathcal{X}_1)}\end{aligned}$$

which proves that U is a contraction in $L^1(\mathcal{X}_1)$. On the other hand, we have

$$\begin{aligned}\|Uf(\omega)\|_{\mathcal{X}_1} &= \sup_n \|T_1 f_n(\omega)\| \leq \sup_\omega \sup_n \|T_1 f_n(\omega)\| \\ &= \sup_n \|T_1 f_n\|_\infty \leq \sup_n \|f_n\|_\infty \qquad (\|T_1\|_\infty \leq 1) \\ &= \sup_\omega \sup_n \|f_n(\omega)\| = \sup_\omega \|f_n\|_{\mathcal{X}_1} = \|f\|_{L^\infty(\mathcal{X}_1)}\end{aligned}$$

it follows that $\|Uf\|_{L^\infty(\mathcal{X}_1)} \leq \|f\|_{L^\infty(\mathcal{X}_1)}$, so U is a contraction in $L^\infty(\mathcal{X}_1)$. Now, form the Cesaro average

$$\frac{1}{m}\sum_{j=0}^{m} U^j F = \frac{1}{m}\sum_{j=0}^{m} U^j \left(\frac{1}{n}\sum_{j=0}^{n} T_2^j f\right) = \left(\frac{1}{mn}\sum_{j=0}^{m}\sum_{j=0}^{n} T_1^j T_2^j f\right)_n$$

where $F = \left(\frac{1}{n}\sum_{j=0}^{n} T_2^j f\right)$. Although $\mathcal{X}_1$ is not reflexive (the reflexivity is not needed to prove the weak estimate in Chacon's theorem), it follows from Theorem 2 that for $f \in L^p(X)$, $1 < p < \infty$, we have $F \in L^p(\mathcal{X}_1)$ and

$$\begin{aligned}\left\|\sup_m \sup_n \left\|\frac{1}{mn}\sum_{j=0}^{m}\sum_{j=0}^{n} T_1^j T_2^j f\right\|_X\right\|_{L^p} &= \left\|\sup_m \left\|\frac{1}{m}\sum_{i=0}^{m} U^j F\right\|_{\mathcal{X}_1}\right\|_{L^p} \\ &\le \frac{p}{p-1}\|F\|_{L^p(\mathcal{X}_1)} \\ &= \frac{p}{p-1}\left\|\sup_n \left\|\frac{1}{n}\sum_{j=0}^{n} T_2^j f\right\|_X\right\|_{L^p} \\ &\le \left(\frac{p}{p-1}\right)^2 \|f\|_{L^p}.\end{aligned}$$

By a standard way we can prove that the averages $\frac{1}{mn}\sum_{j=0}^{m}\sum_{j=0}^{n} T_1^j T_2^j f$ converge in a dense subset of L^p. The Banach principle ends the proof of part (i) for the case $d = 2$.

The case $d > 2$ can be done by induction on d. For, suppose that the theorem is true for $d-1$ operators and prove it for d operators. Consider the Banach space

$$\mathcal{X}_{d-1} = \left\{(x_{(n_2,\dots,n_d)});\ x_{(n_2,\dots,n_d)} \in X \text{ and } \sup_{n_2}\dots\sup_{n_d}\left\|x_{(n_2,\dots,n_d)}\right\| < \infty\right\}$$

normed with $\|x\|_{\mathcal{X}_{d-1}} = \sup_{n_2}\dots\sup_{n_d}\left\|x_{(n_2,\dots,n_d)}\right\|$. Now, define $Uf = (T_1 f_{(n_2,\dots,n_d)})$. Then by $(*)$, U acts on $L^1(\mathcal{X}_{d-1})$. Next, by the induction hypothesis, we have

$$\sup_{n_2}\dots\sup_{n_d}\|A(n_2,\dots,n_d,T_2,\dots,T_d)f\| \in L^p(X)$$

therefore

$$F = \left(f_{(n_2,\dots,n_d)}\right) = (A(n_2,\dots,n_d,T_2,\dots,T_d)f) \in L^p(\mathcal{X}_{d-1}) \subset L^1(\mathcal{X}_{d-1}).$$

Now, apply on F the Cesaro average of U then we get

$$\begin{aligned}A(n_1,U)f_{(n_2,\dots,n_d)} &= A(n_1,T_1)\left[A(n_2,\dots,n_d,T_2,\dots,T_d)f\right] \\ &= A(n_1,\dots,n_d,T_1,\dots,T_d)f\end{aligned}$$

the rest of the proof is virtually identical to that of the case $d = 2$. This completes the proof part (i).

Now, we turn to the proof of (ii). It was shown in Lemma 6.2 [**8**, p. 52] that if $f \in L(X)\log L(X)$ then $f^*_{T_2} = \sup_n \left\| \frac{1}{n}\sum_{j=0}^n T_2^j f \right\| \in L^1$ which means that $F = \left(\frac{1}{n}\sum_{j=0}^n T_2^j f\right)_n \in L^1(\mathcal{X}_{d-1})$. Applying the weak estimate on the operators U and T_2 (which are contraction in $L^1(\mathcal{X}_{d-1})$ and $L^\infty(\mathcal{X}_{d-1})$ (resp. in $L^1(X)$ and in $L^\infty(X)$), we obtain

$$\begin{aligned} a\mu\left\{\sup_m \sup_n \|A(m,n,T_1,T_2)f\| > a\right\} &= a\mu\left\{\sup_m \sup_n \left\| \frac{1}{mn}\sum_{j=0}^m \sum_{j=0}^n T_1^j T_2^j f \right\| > a\right\} \\ &= a\mu\left\{\sup_m \left\| \frac{1}{m}\sum_{i=0}^m U^j F\right\|_{\mathcal{X}_1} > a\right\} \\ &\le \int_{\{F_U^* > a\}} \|F\|_{\mathcal{X}_1}\, d\mu \\ &= \int_{\{F_U^* > a\}} \sup_n \left\| \frac{1}{n}\sum_{j=0}^n T_2^j f\right\| d\mu \\ &= \int_{\{F_U^* > a\}} \sup_n \|A(n,T_2)\|\, d\mu < \infty \end{aligned}$$

Finally, the Banach principle yields the required results.

2. General case. ($\alpha_n^j \neq 1$,for every $n \in N$)

The proof of the general case can be done by a similar way but with minor changes, namely the Cesàro average $A(n,f) = \frac{1}{n}\sum_{j=0}^{n-1} T^j f$ is replaced by the weighted averages $B(n,f) = \frac{1}{n}\sum_{j=0}^{n-1}\alpha_j T^j f$ and then Theorem 3 completes the proof.

By assuming the integrabilite of the supremum of $B(n_i,\dots,n_d,T_i,\dots,T_d)f$ for $i = 1,\dots,d$, we obtain the a.e. convergence of the averages $B(n_1,\dots,n_d, T_1,\dots,T_d)f$.

Theorem 7. *If* $f^*_{i,d} = \sup_{n_i}\dots\sup_{n_d}\|B(n_i,\dots,n_d,T_i,\dots,T_d)f\| \in L^1$, $i = 1,\dots,d$, *then for* $f \in L(X)\log^d L(X)$ *we have*

1. $a\mu\left\{f^*_{i,d} > a\right\} \le \int_{\left\{f^*_{i,d} > a\right\}} f^*_{i+1,d}\, d\mu$,
2. $f^*_{i+1,d} \in L(X)\log^{d-1} L(X)$ *and* $f^*_{1,d} \in L(X)\log L(X)$,
3. $\lim_{n_1,\dots,n_d} B(n_1,\dots,n_d,T_1,\dots,T_d)f$ *exists a.e.*

Proof. To prove 1. we apply Chacon's inequality to the corresponding U

$$\begin{aligned} a\mu\left\{f^*_{i,d} > a\right\} &= a\mu\left\{\sup_{n_i}\dots\sup_{n_d}\|B(n_i,\dots,n_d,T_i,\dots,T_d)f\| > a\right\} \\ &= a\mu\left\{\sup_{n_i}\left\|B(n_i,U)f_{(n_{i+1},\dots,n_d)}\right\|_{\mathcal{X}_{d-i}} > a\right\} \end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\left\{f_{(n_{i+1},\ldots,n_d)}>a\right\}} \left\|f_{(n_{i+1},\ldots,n_d)}\right\|_{\mathcal{X}_{d-i}}\, d\mu \quad \text{(by Theorem 3)}\\
&= \int_{\left\{f_{(n_{i+1},\ldots,n_d)}>a\right\}} \left\|(B(n_{i+1},\ldots,n_d,T_i,\ldots,T_d)f)_{(n_{i+1},\ldots,n_d)}\right\|_{\mathcal{X}_{d-i}}\, d\mu\\
&= \int_{\left\{f_{(n_{i+1},\ldots,n_d)}>a\right\}} \sup_{n_{i+1}}\ldots\sup_{n_d}\left\|(B(n_{i+1},\ldots,n_d,T_i,\ldots,T_d)f)\right\|_{\mathcal{X}_{d-i}}\, d\mu\\
&= \int_{\left\{f_{(n_{i+1},\ldots,n_d)}>a\right\}} f^*_{i+1,d}\, d\mu.
\end{aligned}
$$

Since the functions $f^*_{i,d}$ and $f^*_{i+1,d}$ are in a weak maximal type relation then by [**8**, p. 54], we have for $i=1,\ldots,d$

$$f^*_{i+1,d}\in L(X)\log^d L(X) \Rightarrow f^*_{i,d}\in L(X)\log^{d-1} L(X).$$

For $f\in L(X)\log^d L(X)$, we have by induction on i:

$$
\begin{aligned}
f^*_{d,d} &= \sup_{n_d}\|B(n_i,T_d)f\| \in L(X)\log^d L(X)\\
&\Rightarrow f^*_{d-1,d}\in L(X)\log^{d-1}L(X)\\
&\Rightarrow f^*_{d-2,d}\in L(X)\log^{d-2}L(X)\\
&\Rightarrow \ldots\\
&\Rightarrow f^*_{1,d} = \sup_{n_1}\ldots\sup_{n_d}\|B(n_1,\ldots,n_d,T_1,\ldots,T_d)f\|\in L(X)\log L(X).
\end{aligned}
$$

The limit of multiparameter averages can be obtained by combining 2. and Banach principle, which ends the proof. □

Remark 8. The condition $(**)$ can be replaced by another weaker one:

$$(***)\qquad \int \sup_n \|T_1A(n,T_2)f\|\, d\mu \leq \int \sup_n \|A(n,T_2)f\|\, d\mu.$$

Under this condition, we assume the commutation of operators. The proof of Theorem 4 remains true if we replace the space $L^1(\mathcal{X}_{d-1})$ by the subspace

$$H(X)=\left\{(A(n,T_2)f)_n;\ n\in N \text{ and } f\in L^1(X)\right\}$$

and the operator U will be defined by

$$U\left(A(n,T_2)f\right)=[T_1A(n,T_2)f]_n \quad (=A(n,T_2)T_1f)\,.$$

Since $T_1T_2=T_2T_1$ then U take values in $H(X)$. By Chacon's inequality (which remains true for a subspace of $L^1(\mathcal{X}_{d-1})$) we have the required result.

Remark 9. Note that the condition $(***)$ does not necessarily mean that the operator T is a contraction in $L^1(X)$.

3. Applications

In this section we apply Theorem 4 to obtain the following versions of Zygmund-Fava's theorem [**8**, p. 196]:

Real version.

Theorem 10. *Let $T_1, \dots, T_d$ be linear $L^1 - L^\infty$ contractions in L^1 of a σ-finite measure space. Then* $\lim_{n_1,\dots,n_d} B(n_1, \dots, n_d, T_1, \dots, T_d)f$ *exists a.e. for* $f \in L \log^{d-1} L$.

Proof. In order to prove this theorem, it suffices to notice that positive contractions verify the condition $(*)$. □

Vector version. Similarly, we obtain the following:

Theorem 11. *Let X be reflexive Banach space, Ω a σ-finite measure space and $\theta_1, \dots, \theta_d$ measure preserving transformations on Ω, if $T_j f = fo\theta_j$ then* $\lim_{n_1,\dots,n_d} B(n_1, \dots, n_d, T_1, \dots, T_d)f$ *exists a.e. for* $f \in L(X) \log^{d-1} L(X)$.

Remark 12. It's worth mentioning here that in the new proof of Theorem of Zygmund-Fava (which is real), we have applied Chacon's theorem (vectorial) on the (non-relexive) Banach space $\mathcal{X}_d = l^\infty$ or $\mathcal{X}_d = l^\infty(X)$.

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