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ON C^j–CLOSENESS OF INVARIANT FOLIATIONS UNDER NUMERICS

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Abstract. In this paper we show that invariant center-unstable foliations are preserved in the C^j-topology under numerical approximations. Results on partial linearization are also given.

1. Introduction

In recent years there has been a considerable effort to understand the behavior of invariant objects of dynamical systems under discretization. The topic of the present paper fits well in the list of these works. We refer only to **[6]** (results on qualitative similarities between a flow and its discretization) **[7]** (results on various invariant manifolds around equilibria under numerics), **[8]** (results on C^j-closeness of global invariant manifolds), **[9]** (results on structural stability under numerics), **[11]** (a recent monograph on qualitative properties of numerical approximations). This list is not intended to be exhaustive or complete.

It is known that in the vicinity of a hyperbolic equilibrium point the discretization mapping conjugates to the time-h-map of the flow (h is the step-size), see **[7]**. (Related results in the case of delay differential equations can be found in **[5]**.) The proof goes via putting the problem into the general framework of the Hartman-Grobman theorem. If hyperbolicity is lost one would work with the generalized Hartman-Grobman theorem, see **[10]**, or with partial linearization, see **[1]**. Since the linearization procedure around nonhyperbolic equilibria goes via constructing invariant foliations, it is worth investigating these invariant foliations under numerical approximations. This is the core of the present work.

The generalized Hartman-Grobman theorem tells us that the crucial part of the dynamics is concentrated on the center-manifold. Indeed, the whole dynamics is

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topologically equivalent to the flow on the center-manifold times a linear saddle, see **[10]**. Thus, if we have conjugacy between the discretization and the time-h-map restricted to center-manifolds then we would obtain conjugacy between the discretization and the time-h-map. Using center-manifold reduction near a fold bifurcation point, it can be shown that this conjugacy exists, see **[4]**. In that case the conjugacy is $O(h^p)$ close to the identity (p is the order of the method), thanks to its construction on the center-manifolds and to the C^j-closeness of center-manifolds under numerics, see **[2]**.

The paper is organized as follows. Some general notations will be fixed in this section. Then Section 2 contains a result on partial linearization with a small parameter. Section 3 is devoted to the center-unstable foliations with a small parameter. We apply these results to the discretization problem in Section 4.

Let j, m_1, $m_2 \in \mathbf{N}$ and define

$$C^j(\mathbf{R}^{m_1}, \mathbf{R}^{m_2}) := \Big\{ w : \mathbf{R}^{m_1} \to \mathbf{R}^{m_2} \ : \ w \text{ is } j \text{ times continuously differentiable with bounded derivatives} \Big\}.$$

Equipped with the usual C^j-norm

$$|||w|||_j = \max\Big\{ \sup\{ |w^{(i)}(x)| \ : \ x \in \mathbf{R}^{m_1} \} \ : \ i = 0, \dots, j \Big\}$$

the space $C^j(\mathbf{R}^{m_1}, \mathbf{R}^{m_2})$ is a Banach space. We also need the following space

$$X^j(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \mathbf{R}^{m_1}) := \Big\{ w : \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \to \mathbf{R}^{m_1} \ : \ w \text{ is } j \text{ times continuously differentiable in its second variable and bounded} \Big\}.$$

Equipped with the norm

$$\|w\|_j := \max\Big\{ \sup\{ |w_y^{(i)}(x, y)| \ : \ (x, y) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2} \} \ : \ i = 0, \dots, j \Big\},$$

$X^j(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \mathbf{R}^{m_1})$ is a Banach space.

2. Partial Linearization

Let m_1, m_2 be two natural numbers and set $m = m_1 + m_2$. With some $h_0 > 0$ let $p^i : [0, h_0] \times \mathbf{R}^m \to \mathbf{R}^{m_1}$ and $q^i : [0, h_0] \times \mathbf{R}^m \to \mathbf{R}^{m_2}$ ($i = 1, 2$) be given mappings. For $A \in \mathbf{R}^{m_1 \times m_1}$ and $B \in \mathbf{R}^{m_2 \times m_2}$ we consider the following mappings

$$\begin{aligned} X &= e^{Ah} x + p^1(h, x, y), \\ Y &= e^{Bh} y + q^1(h, x, y), \end{aligned} \tag{1}$$

and

$$X = e^{Ah}x + p^2(h,x,y), \qquad (2)$$
$$Y = e^{Bh}y + q^2(h,x,y),$$

where $x, X \in \mathbf{R}^{m_1}$, $y, Y \in \mathbf{R}^{m_2}$ and $h \in [0, h_0]$.

We have the following assumptions.

(A1) $\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(B)\} < \beta < \alpha < \inf\{\operatorname{Re}\lambda \mid \lambda \in \sigma(A)\}$, and $\alpha > 0$.

Remark. From assumption (A1) it follows that, by passing to an equivalent norm,

$$|e^{-Ah}| \le 1 - h\alpha, \quad |e^{Bh}| \le 1 + h\beta, \quad |e^{-Ah}||e^{Bh}| \le 1 - h(\alpha - \beta).$$

From now on we fix this norm.

(A2) The functions $\xi = p^i, q^i$, $i = 1, 2$ are bounded and satisfy the following global Lipschitz property

$$|\xi(h,x,y) - \xi(h,\bar{x},\bar{y})| \le \rho h(|x - \bar{x}| + |y - \bar{y}|).$$

Moreover, ρ is so small such that

$$b_0 = (1 - h\alpha)(1 + 2\rho h) < 1$$

and

$$b_1 = (1 - h\alpha)(1 + \beta h + 4\rho h) < 1$$

hold for every $h \in (0, h_0]$.

Remark. Note that there is a constant $l > 0$ independent of h such that $b_0 < 1 - lh$ and $b_1 < 1 - lh$.

(A3) With some constant $K > 0$ (independent of (x, y) and h) and with some integer $p \ge 1$

$$|p^1(h,x,y) - p^2(h,x,y)| \le Kh^{p+1}$$

and

$$|q^1(h,x,y) - q^2(h,x,y)| \le Kh^{p+1}$$

hold true for all $h \in [0, h_0]$.

Theorem 1. *Assume* (A1)–(A3). *Then for all h small enough there are functions* $\gamma_h^i, \delta_h^i \colon \mathbf{R}^m \to \mathbf{R}^{m_1}$, $i = 1,2$, *such that* $H_h^i(x,y) = (\gamma_h^i(x,y), y)$ *and* $J_h^i(x,y) = (\delta_h^i(x,y), y)$ *are homeomorphisms,* $(H_h^i)^{-1} = J_h^i$ *and* $z = \gamma_h^i(x,y)$, $Z = \gamma_h^i(X,Y)$, $u = y$, $U = Y$ *transform* (1) *and* (2) *into*

$$
\begin{aligned}
Z &= e^{Ah} z \\
U &= e^{Bh} u + q^i(h, \delta_h^i(z,u), u),
\end{aligned}
\tag{3}
$$

respectively. Moreover, with some constant $K_1 > 0$ *(independent of* (x,y) *and* h*)*

$$|\delta_h^1(x,y) - \delta_h^2(x,y)| \le K_1 h^p$$

hold for all h small enough.

Proof. Let BC be the Banach space of bounded continuous mappings from $\mathbf{R}^m$ into $\mathbf{R}^{m_1}$ with the usual sup ($\|\cdot\|$) norm. Define the following function space V as

$$
\begin{aligned}
V := \big\{ v : \mathbf{R}^m \to \mathbf{R}^{m_1} : \text{there is a } \bar{v} \in BC, \\
v(x,y) = x + \bar{v}(x,y) \text{ for all } (x,y) \in \mathbf{R}^m \big\}.
\end{aligned}
$$

As in [**1**], we are looking for solutions in V of the following functional equations

$$\gamma_h^i(e^{Ah}x + p^i(h,x,y), e^{Bh} + q^i(h,x,y)) = e^{Ah}\gamma_h^i(x,y) \tag{4}$$

and

$$\delta_h^i(e^{Ah}x, e^{Bh}y + q^i(h, \delta_h^i(x,y), y)) = e^{Ah}\delta_h^i(x,y) + p^i(h, \delta_h^i(x,y), y) \tag{5}$$

where $i = 1,2$.

First we claim that (4) has a unique solution in V. By setting $v_h^i(x,y) = \gamma_h^i(x,y) - x$, $v_h^i \in BC$ and (4) has the form

$$v_h^i(x,y) = e^{-Ah} v_h^i(e^{Ah}x + p^i(h,x,y), e^{Bh}y + q^i(h,x,y)) + e^{-Ah}p^i(h,x,y). \tag{6}$$

For $v \in BC$ we define

$$F_h^i(v)(x,y) =: e^{-Ah} v(e^{Ah}x + p^i(h,x,y), e^{Bh}y + q^i(h,x,y)) + e^{-Ah}p^i(h,x,y).$$

Then (6) is equivalent to the fixed point setting $F_h^i(v_h^i)(x,y) = v_h^i(x,y)$. It is easy to see that $F_h^i \colon BC \to BC$ is a contraction with Lipschitz constant $\operatorname{Lip} F_h^i \le 1 - h\alpha < 1$, and the claim follows.

Next we claim that (5) has a solution in V. By setting $w_h^i(x,y) = \delta_h^i(x,y) - x$, $w_h^i \in BC$ and (5) has the form

$$
\begin{aligned}
w_h^i(x,y) = {} & e^{-Ah} w_h^i(e^{Ah}x, e^{Bh}y + q^i(h, x + w_h^i(x,y), y)) \\
& - e^{-Ah} p^i(h, x + w_h^i(x,y), y).
\end{aligned} \tag{7}
$$

We define the following function space

$$
W := \big\{ w \in BC \ : \ |w(x,y) - w(x,\bar{y})| \le |y - \bar{y}| \ \text{ for all } (x,y),(x,\bar{y}) \in \mathbf{R}^m \big\}.
$$

Endowed with the metric inherited from the sup norm, the space W is a complete metric space. For $w \in W$ define

$$
\begin{aligned}
G_h^i(w)(x,y) := {} & e^{-Ah} w(e^{Ah}x, e^{Bh}y + q^i(h, x + w(x,y), y)) \\
& - e^{-Ah} p^i(h, x + w(x,y), y).
\end{aligned} \tag{8}
$$

Thus we have a fixed point setting $G_h^i(w_h^i)(x,y) = w_h^i(x,y)$. In what follows we show that $G_h^i \colon W \to W$ is a contraction with Lipschitz constant $\operatorname{Lip} G_h^i \le b_0 < 1$ which proves that (7) (and thus (5)) has at least one solution in V.

On one hand

$$
\begin{aligned}
|G_h^i(w)(x,y) - G_h^i(w)(x,\bar{y})| \le {} & |e^{-Ah}| \Big(|e^{Bh}||y - \bar{y}| \\
& + |q^i(h, x + w(x,y), y) - q^i(h, x + w(x,\bar{y}), \bar{y})| \\
& + |p^i(h, x + w(x,y), y) - p^i(h, x + w(x,\bar{y}), \bar{y}))| \Big) \\
\le {} & (1 - h\alpha)((1 + h\beta) + 4\rho h)|y - \bar{y}| = b_1|y - \bar{y}| \le |y - \bar{y}|
\end{aligned}
$$

which proves that $G_h^i : W \to W$. On the other hand

$$
\begin{aligned}
& |G_h^i(w)(x,y) - G_h^i(\bar{w})(x,y)| \le |e^{-Ah}| \\
& \times \Big(|w(e^{Ah}x, e^{Bh}y + q^i(h, x + w(x,y), y)) - \bar{w}(e^{Ah}x, e^{Bh}y + q^i(h, x + w(x,y), y))| \\
& + |\bar{w}(e^{Ah}x, e^{Bh}y + q^i(h, x + w(x,y), y)) - \bar{w}(e^{Ah}x, e^{Bh}y + q^i(h, x + \bar{w}(x,y), y))| \\
& + \rho h|w(x,y) - \bar{w}(x,y)| \Big) \\
& \le |e^{-Ah}|(\|w - \bar{w}\| + |q^i(h, x + w(x,y), y) - q^i(h, x + \bar{w}(x,y), y)| + \rho h\|w - \bar{w}\|) \\
& \le (1 - h\alpha)(1 + 2\rho h)\|w - \bar{w}\| = b_0\|w - \bar{w}\|
\end{aligned}
$$

which proves the desired contraction property.

Now let $\gamma_h^i \in V$ be the unique solution of (4) and let $\delta_h^i \in V$ be an arbitrary solution of (5). We claim that $\gamma_h^i(\delta_h^i(x,y), y) = x$ for all $(x,y) \in \mathbf{R}^m$.

Set $\psi_h^i(x,y) := \gamma_h^i(\delta_h^i(x,y),y)$. Then $\psi_h^i \in V$ and since

$$\begin{aligned}&\gamma_h^i(\delta_h^i(e^{Ah}x, e^{Bh}y + q^i(h,\delta_h^i(x,y),y)), e^{Bh}y + q^i(h,\delta_h^i(x,y),y))\\ &\quad = \gamma_h^i(e^{Ah}\delta_h^i(x,y) + p^i(h,\delta_h^i(x,y),y), e^{Bh}y + q^i(h,\delta_h^i(x,y),y))\\ &\quad = e^{Ah}\gamma_h^i(\delta_h^i(x,y),y)\end{aligned}$$

the function ψ_h^i is a solution of

$$\psi_h^i(e^{Ah}x, e^{Bh} + q^i(h,\delta_h^i(x,y),y)) = e^{Ah}\psi_h^i(x,y).$$

Set $\phi_h^i(x,y) := \psi_h^i(x,y) - x$. Then $\phi_h^i \in BC$ and

$$\phi_h^i(x,y) = e^{-Ah}\phi_h^i(e^{Ah}x, e^{Bh}y + q^i(h,\delta_h^i(x,y),y)).$$

By taking supremum of the norm in the right-hand side we have

$$|\phi_h^i(x,y)| \le |e^{-Ah}|\|\phi_h^i\|$$

and thus

$$\|\phi_h^i\| \le (1-h\alpha)\|\phi_h^i\|$$

which shows that $\phi_h^i = 0$. Note that we have proved that (5) has a unique solution in V as well.

Finally we claim that $\operatorname{Range}(\delta_h^i(\cdot,y_0)) = \mathbf{R}^{m_1}$ for all $y_0 \in \mathbf{R}^{m_2}$. But this is a simple consequence of the homotopy property of the degree applied to $\delta_t(x,y_0) = x + tw_h^i(x,y_0)$. By using $\delta_h^i(x,y) = \delta_h^i(\gamma_h^i(\delta_h^i(x,y),y),y)$ we obtain that $\delta_h^i(\gamma_h^i(x,y),y) = x$ which proves $(H_h^i)^{-1} = J_h^i$.

It remains to prove the closeness result. With $w \in W$ consider the following estimates

$$\begin{aligned}|G_h^1(w)(x,y) - G_h^2(w)(x,y)| &\le |e^{-Ah}|\Big(|q^1(h,x+w(x,y),y) - q^2(h,x+w(x,y),y)|\\ &\qquad + |p^1(h,x+w(x,y),y) - p^2(h,x+w(x,y),y)|\Big)\\ &\le 2Kh^{p+1}.\end{aligned}$$

Now we compare w_h^1 and w_h^2 as

$$\begin{aligned}\|w_h^1 - w_h^2\| &= \|G_h^1(w_h^1) - G_h^2(w_h^2)\|\\ &\le \|G_h^1(w_h^1) - G_h^1(w_h^2)\| + \|G_h^1(w_h^2) - G_h^2(w_h^2)\|\\ &\le b_0\|w_h^1 - w_h^2\| + 2Kh^{p+1}.\end{aligned}$$

Thus

$$\|w_h^1 - w_h^2\| \le 2Kh^{p+1}/(1-b_0) = (2K/l)h^p$$

and we are done. $\square$

3. Invariant Foliations

Let $n \in \mathbf{N}$ and assume that $p^i(h,\cdot,\cdot) \in C^{n+p+1}(\mathbf{R}^m,\mathbf{R}^{m_1})$, $q^i(h,\cdot,\cdot) \in C^{n+p+1}(\mathbf{R}^m,\mathbf{R}^{m_2})$. Furthermore, we impose conditions:

(H1) $(p+n+1)\beta < \alpha$.

(H2) The functions $\xi = p^i, q^i$, $i = 1,2$ are bounded and satisfy the following global Lipschitz property

$$|\xi(h,x,y) - \xi(h,\bar{x},\bar{y})| \le \rho h(|x-\bar{x}| + |y-\bar{y}|).$$

Moreover, ρ is so small such that

$$b_k := (1-h\alpha)((1+\beta h + 2\rho h)^k + 2\rho) < 1$$

for all $k = 0,1,\dots,n+p+1$.

(H3) With some constant $K > 0$ (independent of $z = (x,y)$ and h)

$$\begin{gathered} |(p^1)_z^{(k)}(h,x,y) - (p^2)_z^{(k)}(h,x,y)| \le Kh^{p+1} \quad k = 0,1,\dots,n \\ |(p^1)_z^{(n+k)}(h,x,y) - (p^2)_z^{(n+k)}(h,x,y)| \le Kh^{p+1-k} \quad k = 0,1,\dots,p+1 \end{gathered}$$

and

$$\begin{gathered} |(q^1)_z^{(k)}(h,x,y) - (q^2)_z^{(k)}(h,x,y)| \le Kh^{p+1} \quad k = 0,1,\dots,n \\ |(q^1)_z^{(n+k)}(h,x,y) - (q^2)_z^{(n+k)}(h,x,y)| \le Kh^{p+1-k} \quad k = 0,1,\dots,p+1. \end{gathered}$$

(H4) With some constant $K_2 > 0$ (independent of $z = (x,y)$ and h)

$$|(\xi)_z^{(k)}(h,x,y)| \le K_2 h \quad k = 0,1,\dots,n+p$$

where $\xi = p^i, q^i$, $i = 1,2$.

Remark. There is a constant $l > 0$ such that $b_k < 1 - lh$ for all $k = 0,1,\dots,n+p+1$.

Since (H1)–(H3) implies (A1)–(A3) we can apply Theorem 1 in this situation. As a result we obtain functions $\delta_h^i \in W$ $(i = 1,2)$. With these functions we define the invariant foliations as follows: Let $c \in \mathbf{R}^{m_1}$ and set $S_h^i(c) := \{(x,y) \in \mathbf{R}^m : x = \delta_h^i(c,y)\}$. We call the set $S_h^i(c)$ the leaf of the foliation corresponding to c. It is easy to see that mappings (1) and (2) send one leaf onto another, thus the family of sets (manifolds) $\{S_h^i(c)\}_{c\in\mathbf{R}^{m_1}}$ form invariant foliations. Only one leaf remains fixed (the one corresponding to $0 \in \mathbf{R}^{m_1}$) which is called the center-unstable manifold.

In what follows we prove that the fibers of foliations are smooth (i.e. functions $\delta_h^i(x,y)$ are smooth in y) and are close in the X^j-topology. Namely, we have

Thoeorem 2. *Assume* (H1)–(H4). *Then for all* $c \in \mathbf{R}^{m_1}$ *we have that*

$$\delta_h^i(c,\cdot) \in C^{n+p+1}(\mathbf{R}^{m_2},\mathbf{R}^{m_1})$$

and

$$\|\delta_h^1(c,\cdot)-\delta_h^2(c,\cdot)\|_{n+k} \le K_3 h^{p-k}, \quad k=0,1,\dots,p$$

with some constant $K_3>0$ *independent of* c *and* h.

Proof. Set $c(0)=1$, $W_0^{c(0)}=W$. Given a finite sequence of positive numbers $\{c(j)\}_{j=1}^{n+p}$ we inductively define, for $j=1,2,\dots,n+p$,

$$W_j^{c(j)} = \Big\{ w \in W_{j-1}^{c(j-1)} \ : \ w \text{ is } j \text{ times continuously differentiable in } y, \\ |w_y^{(j)}(x,y)-w_y^{(j)}(x,\bar y)| \le c(j)|y-\bar y| \Big\}.$$

Note that if $w \in W_{j-1}^{c(j-1)}$ and w is j times continuously differentiable in y then $|w_y^{(j)}(x,y)| \le c(j-1)$. Further, an inductive application of Arzela-Ascoli theorem shows that $W_{n+p}^{c(n+p)} \subset W$ is a closed subset.

If $w \in W_1^{c(1)}$ then $G_h^i(w)$ is continuously differentiable in y and

$$(G_h^i(w))'_y = e^{-Ah}\tilde w'_y(e^{Bh}+(q^i)'_y+(q^i)'_x w'_y) \\ - e^{-Ah}((p^i)'_x w'_y + (p^i)'_y),$$

where $\tilde w$ means w with argument $(e^{Ah}x, e^{Bh}y + q^i(h, x+w(x,y),y))$. Recall that $|w'_y(x,y)| \le 1$ for all $(x,y) \in \mathbf{R}^m$. A simple calculation shows that $(G_h^i(w))'_y$ is globally Lipschitzian in y with Lipschitz constant $c(1)b_2 + r_1$, where r_1 is a polynomial in the variables $c(0), c(1)$ and the coefficient of each term is a nonconstant polynomial of $|(p^i)'_z|$, $|(p^i)''_z|$, $|(q^i)'_z|$ and $|(q^i)''_z|$. Now set $c(1)=r_1/(1-b_2)$. Then $G_h^i(W_1^{c(1)}) \subset W_1^{c(1)}$. Since $c(1) \le r_1/(lh)$ and $|(p^i)'_z|$, $|(p^i)''_z|$, $|(q^i)'_z|$ and $|(q^i)''_z|$ are of order h (see (H4)), we obtain that $c(1)$ can be choosen independently of h (and $i=1,2$).

We proceed by induction. If $w \in W_j^{c(j)}$ $(j=2,\dots,n+p)$ then $G_h^i(w)$ is j times continuously differentiable in y and

$$(G_h^i(w))_y^{(j)} = e^{-Ah}\tilde w_y^{(j)}(e^{Bh}+(q^i)'_y+(q^i)'_x w'_y)^j + e^{-Ah} w_y^{(j)}(\tilde w'_y (q^i)'_x - (p^i)'_x) + R_j,$$

where R_j is a polynomial function in the variables $w'_y, \ \dots, \ w_y^{(j-1)}, \ (p^i)'_x, \ \dots,$ $(p^i)_x^{(j)}$, $(p^i)''_{xy}, \dots, (q^i)_y^{(j)}$.

The global Lipschitz property of $(G_h^i(w))^{(j)}$ with respect to y easily follows with Lipschitz constant $c(j)b_{j+1}+r_j$, where r_j is a polynomial in the variables $c(0),\dots,c(j-1)$ and the coefficient of each term is a nonconstant polynomial of

$|(p^i)'_x|, \ldots, |(p^i)^{(j+1)}_x|, |(p^i)''_{xy}|, \ldots, |(q^i)^{(j+1)}_y|$. Now set $c(j) = r_j/(1-b_{j+1})$. Then $G^i_h(W^{c(j)}_j) \subset W^{c(j)}_j$, $j = 1, 2, \ldots, n+p$. Since $c(j) \le r_j/(lh)$ and (by (H4)) r_j is of order h for $j = 0, 1, \ldots, n+p-1$, we obtain that $\{c(j)\}_{j=0}^{n+p-1}$ can be choosen independently of h (and $i = 1, 2$).

From this construction we see that the fixed points of G^i_h are in $W^{c(n+p)}_{n+p}$. For a proof of existence and continuity of the remaining $(n+p+1)$th derivative we refer to **[1]**, **[3]**, **[12]**.

In order to compare the derivatives of δ^i_h we build up fixed points settings. To this end we pass to an equivalent norm $|\cdot|_j$ on $X^j(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \mathbf{R}^{m_1})$.

First observe that

$$\|(G^i_h(w))^{(j)}_y - (G^i_h(\bar{w}))^{(j)}_y\| \le \sum_{k=0}^{j} L^k_j \|w^{(k)}_y - (\bar{w})^{(k)}_y\|$$

whenever $w, \bar{w} \in W^{c(j)}_j$, $j = 0, 1, \ldots, n+p$. It is readily checked that L^0_0 can be choosen for b_0. Moreover, L^j_j can be choosen for b_j, $j = 1, 2, \ldots, n+p$. Finally, L^k_j (for $j = 1, 2, \ldots, n+p$, $k = 0, 1, \ldots, j-1$) can be taken as a polynomial in the variables $c(0), \ldots, c(j-1)$ where the coefficient of each term is a nonconstant polynomial of $|(p^i)'_x|, \ldots, |(p^i)^{(j+1)}_x|, |(p^i)''_{xy}|, \ldots, |(q^i)^{(j+1)}_y|$.

For $w \in X^j(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \mathbf{R}^{m_1})$ we set

$$|w|_j := \sum_{k=0}^{j} d(k) \|w^{(k)}_y\|, \quad j = 0, 1, \ldots, n+p,$$

where $d(0) = 1$ and $\{d(k)\}_{k=0}^{n+p}$ is a finite sequence of positive constants specified later. It is easy to see that $\|\cdot\|_j$ and $|\cdot|_j$ are equivalent norms on $X^j(\mathbf{R}^{m_1} \times \mathbf{R}^{m_2}, \mathbf{R}^{m_1})$. On one hand

$$\begin{aligned}
|G^i_h(w) - G^i_h(\bar{w})|_j &= \sum_{k=0}^{j} d(k) \|(G^i_h(w))^{(k)}_y - (G^i_h(\bar{w}))^{(k)}_y\| \\
&\le \sum_{k=0}^{j} \sum_{l=k}^{j} d(l) L^k_l \|w^{(k)}_y - \bar{w}^{(k)}_y\| \\
&\le \sum_{k=0}^{j} (d(k) b_k + \sum_{l=k+1}^{j} d(l) L^k_l) \|w^{(k)}_y - \bar{w}^{(k)}_y\|.
\end{aligned}$$

On the other hand

$$|w - \bar{w}|_j = \sum_{k=0}^{j} d(k) \|w^{(k)}_y - \bar{w}^{(k)}_y\|.$$

Comparing the coefficients of $\|w_y^{(k)} - \bar{w}_y^{(k)}\|$ and using (H2) we obtain that (with a suitable choice of $\{d(k)\}_{k=1}^{n+p}$)

$$d(k)b_k + \sum_{l=k+1}^{j} d(l)L_l^k \le (1+b_k)d(k)/2$$

for all $j = 0, 1, \dots, n+p$ and $k = 0, 1, \dots, j$. Thus

$$|G_h^i(w) - G_h^i(\bar{w})|_j \le \max\{(1+b_k)/2 \ : \ k = 0, 1, \dots, j\}|w - \bar{w}|_j$$

for all $w, \bar{w} \in W_j^{c(j)}$, $j = 0, 1, \dots, n+p$.

Now we claim that $\{d(k)\}_{k=0}^{n+p-1}$ can be chosen independently of h. Recall that $d(0) = 1$. As before, it is enough to prove that L_j^k is of order h for $k = 0, 1, \dots, n+p-1$, $j = k+1, k+2, \dots, n+p-1$. Since $\{c(j)\}_{j=0}^{n+p-1}$ is independent of h, L_j^k is a nonconstant polynomial in the variables $|(p^i)'_x|, \dots, |(p^i)_x^{(j+1)}|$, $|(p^i)''_{xy}|$, $\dots$, $|(q^i)_y^{(j+1)}|$, thus the desired result follows from (H4).

Finally, for $w \in W_{n+k}^{c(n+k)}$, consider the estimates

$$\begin{aligned}\|(G_h^1(w))_y^{(j)} - (G_h^2(w))_y^{(j)}\| \le K_4 \sum_{k=0}^{j} \Big(&\|(p^1)_z^{(k)}(h,x,y) - (p^2)_z^{(k)}(h,x,y)\| \\ &+ \|(q^1)_z^{(k)}(h,x,y) - (q^2)_z^{(k)}(h,x,y)\| \Big)\end{aligned}$$

with some constant $K_4 > 0$ and $j = 0, 1, \dots, n+p-1$. Using the above estimates, (H3) and the definition of $|\cdot|_j$ we have that

$$|G_h^1(w) - G_h^2(w)|_{n+k} \le K_5 h^{p+1-k}, \quad k = 0, 1, \dots, p-1.$$

Now we are in a position to prove the closeness of the invariant foliations. First, the $k = p$ case follows from the facts that $c(n+p-1)$ is independent of h and $\delta_h^i \in X^{n+p}$.

If $k \ne p$ then

$$\begin{aligned}|w_h^1 - w_h^2|_{n+k} &= |G_h^1(w_h^1) - G_h^2(w_h^2)|_{n+k} \\ &\le |G_h^1(w_h^1) - G_h^1(w_h^2)|_{n+k} + |G_h^1(w_h^2) - G_h^2(w_h^2)|_{n+k} \\ &\le (1 - (l/2)h)|w_h^1 - w_h^2|_{n+k} + K_5 h^{p+1-k}.\end{aligned}$$

Thus

$$|w_h^1 - w_h^2|_{n+k} \le (2K_5/l)h^{p-k}$$

and we are done. □

4. Applications

In this section we show that Theorems 1 and 2 can be applied to the problem of discretization.

Let $f\colon \mathbf{R}^m \to \mathbf{R}^m$ be a globally Lipschitzian mapping and consider the differential equation

$$\dot{z} = f(z). \tag{9}$$

By its h-discretized equation we mean equation

$$Z = \varphi(h, z), \quad (z, Z \in \mathbf{R}^m,\ h > 0)$$

where φ is a fixed one-step method with stepsize h. Assume that φ is of order $p \geq 1$, i.e. there exist constants h_0 and K_6 such that

$$|\Phi(h,z) - \varphi(h,z)| \leq K_6 h^{p+1} \text{ for all } h \in (0, h_0],\ z \in \mathbf{R}^m \tag{10}$$

where $\Phi(h, \cdot)$ is the time-h-map of the induced solution flow of (9).

If we assume that $f,\ \varphi \in C^{n+p+1}(\mathbf{R}^m, \mathbf{R}^m)$ then (for details see **[7]**) there is a constant $K_7 > 0$ such that

$$\begin{aligned} |\Phi_z^{(j)}(h,z) - \varphi_z^{(j)}(h,z)| &\leq K_7 h^{p+1}, \quad j = 0, \dots, n \\ |\Phi_z^{(n+j)}(h,z) - \varphi_z^{(n+j)}(h,z)| &\leq K_7 h^{p+1-j}, \quad j = 0, \dots, p+1 \end{aligned} \tag{11}$$

for all $h \in (0, h_0]$ and $z \in \mathbf{R}^m$.

Consider a globally Lipschitzian C^∞ cut-off function μ with $\mu(z) = 0$ whenever $|z| \geq 2$ and $\mu(z) = 1$ whenever $|z| \leq 1$.

From now on we assume that $f, \varphi \in C^1(\mathbf{R}^m, \mathbf{R}^m)$, $f(z) = Cz + g(z)$, where $C \in \mathbf{R}^{m \times m}$ and $g(0) = 0$, $g'(0) = 0$. Let $g(z; \varepsilon) := \mu(z/\varepsilon) g(z)$, $z \in \mathbf{R}^m$, $\varepsilon > 0$. Consider the differential equation

$$\dot{z} = Cz + g(z; \varepsilon). \tag{12}$$

Denote the h-discretized equation of (12) by $Z = \varphi(h, z; \varepsilon)$. Write the flow induced by (12) as

$$\Phi(t, z; \varepsilon) = e^{Ct} z + s^1(t, z; \varepsilon), \quad t \in \mathbf{R},\ z \in \mathbf{R}^m,\ \varepsilon > 0. \tag{13}$$

We consider a modified h-discretization equation of (12) as follows

$$Z = e^{Ch} + s^2(h, z; \varepsilon) \tag{14}$$

where

$$s^2(h, z; \varepsilon) = \mu(z)(\varphi(h, z; \varepsilon) - \Phi(h, z; \varepsilon)) + s^1(h, z; \varepsilon), \quad h > 0,\ z \in \mathbf{R}^m,\ \varepsilon > 0.$$

Notice that (14) coincides with the one-step method for $|z| \le \varepsilon$ and with the flow (13) for $|z| \ge 2\varepsilon$.

It is known, see Prop. 1.2 and 1.3 in **[7]**, that there exist a bounded continuous function $\Omega : (0,\infty) \to \mathbf{R}^+$ with $\Omega(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that (with $i = 1, 2$)

$$\begin{aligned} |s^i(h,\cdot;\varepsilon)| &\le \Omega(\varepsilon)\varepsilon h \\ \operatorname{Lip}(s^i(h,\cdot;\varepsilon)) &\le \Omega(\varepsilon) h \end{aligned} \tag{15}$$

whenever $h \in (0, h(\varepsilon)]$, $\varepsilon > 0$. For sake of simplicity, set $s^2(0,z;\varepsilon) = 0$. (We note that although the one-step method is defined only for positive h we can set $\varphi(0,z) = z$ by continuity thanks to (10).)

Assume that C admits a splitting $C = \operatorname{diag}(A,B)$ such that (A1) holds.

Now we want to apply Theorem 1 with mappings (13) and (14) (with $(p^i(h,x,y), q^i(h,x,y)) = s^i(h,(x,y);\varepsilon)$). Property (A2) is direct consequence of (15) (with ε small enough) while (A3) follows from (10). Thus our Theorem 1 applies to maps (13) and (14).

Secondly assume that $f, \varphi \in C^{n+p+1}(\mathbf{R}^m, \mathbf{R}^m)$ and that C admits a splitting $C = \operatorname{diag}(A,B)$ such that (H1) holds. Now we want to apply Theorem 2 with mappings (13) and (14) (with $(p^i(h,x,y), q^i(h,x,y)) = s^i(h,(x,y);\varepsilon)$). Property (H2) is a direct consequence of (15) (with ε small enough) while (H3) follows from (11). Finally (H4) holds because of (11) and the fact that $(s^1)_z^{(j)}$ is continuously differentiable in t and $s^1(0,z;\varepsilon) = 0$. Thus our Theorem 2 applies to maps (13) and (14).

Remark. It is known that invariant foliations (manifolds) constructed via the time-h-map of a flow are independent of h and are the invariant foliations (manifolds) for the flow as well, see e.g. **[10]**.

Remark. Concerning the C^j closeness of the leaf corresponding to $0 \in \mathbf{R}^{m_1}$ we get Corollary 3.7. in **[7]**.

Remark. By reversing time the results show the C^j closeness of center-stable foliations as well.

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