Acta Math. Univ. Comenianae
Vol. LXIX, 2(2000), pp. 145–149

THE BOREL STRUCTURE OF THE COLLECTIONS OF SUB–SELF–SIMILAR SETS AND SUPER–SELF–SIMILAR SETS

M. MCCLURE AND R. W. VALLIN

Abstract. We show that the sets of sub-self-similar sets and super-self-similar sets are both dense, first category, F_σ subsets of $\mathcal{K}(\mathbb{R}^d)$, the Hausdorff metric space of non-empty compact, subsets of $\mathbb{R}^d$. We also investigate the set of self-similar sets as a subset of the sub-self-similar sets and the super-self-similar sets.

1. Introduction

In [**Fal1**], Falconer introduced the notion of sub-self-similarity as a generalization of self-similarity and showed that sub-self-similar sets retain many of the nice properties of self-similar sets. Later in [**Fal2**] we find the notion of a super-self-similar set. The question arises as to how strong a generalization are these new concepts. In this paper, we quantify this question using topological notions in $\mathcal{K}(\mathbb{R}^d)$, the Hausdorff metric space of non-empty compact subsets of $\mathbb{R}^d$. In particular, we show that the sets of sub-self-similar sets and super-self-similar sets are both dense, first category, F_σ subsets of $\mathcal{K}(\mathbb{R}^d)$. The fact that these sets are dense could be interpreted as meaning that we have an understanding of many compact subsets of $\mathbb{R}^d$. The fact that these sets are first category indicates that most compact sets are not encompassed in these definitions. We also consider the set of self-similar sets as a subset of the sub-self-similar sets and the super-self-similar sets. In particular, we show that the sub-self-similar sets which are not self-similar are dense in the set of sub-self-similar sets, and similarly for the super-self-similar sets. This indicates that Falconer's new concepts are a considerable generalization over the self-similar sets.

2. Definitions

We work in a fixed Euclidean space $\mathbb{R}^d$. Let $\mathcal{K}(\mathbb{R}^d)$ be the set of non-empty, compact subsets of $\mathbb{R}^d$. The Hausdorff metric ρ on $\mathcal{K}(\mathbb{R}^d)$ is defined by

$$\rho(A,B) = \max\{\sup_{x\in A}\{\operatorname{dist}(x,B)\}, \sup_{y\in B}\{\operatorname{dist}(y,A)\}\}.$$

Received February 23, 1999.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 54B20.

A discussion of the Hausdorff metric may be found in [**Ed**] Section 2.4. Of particular interest is Theorem 2.4.4, which states that $\mathcal{K}(\mathbb{R}^d)$ is complete. This allows us to appeal to Baire category type arguments in $\mathcal{K}(\mathbb{R}^d)$. Also of note is Exercise 2.4.2, which characterizes the limit of a sequence of sets in the Hausdorff metric as follows: If $A_n \to A$ in the Hausdorff metric, then

$$A = \{x : \exists \{x_n\}_{n=1}^{\infty} \text{ with } x_n \in A_n \text{ and } x_n \to x\}.$$

A function $T\colon \mathbb{R}^d \to \mathbb{R}^d$ is a similarity with ratio $r = r(T) > 0$ if

$$|T(x) - T(y)| = r|x - y| \text{ for all } x, y \in \mathbb{R}^d.$$

If $r < 1$, then T is called contractive. A fundamental result ([**Ed**], Thm. 4.1.3) states that if $T_i : \mathbb{R}^d \to \mathbb{R}^d$ is a contractive similarity for each $i \in \{1, \dots, m\}$, then there is a unique, nonempty, compact set $E \subseteq \mathbb{R}^d$ such that

$$E = \cup_{i=1}^{m} T_i(E).$$

The set E is called self-similar.

Sub-self-similar sets are obtained by relaxing the equality to inclusion. Thus, the compact set E is sub-self-similar if there are contractive similarities $T_i\colon \mathbb{R}^d \to \mathbb{R}^d$ for $i \in \{1, \dots, m\}$ such that

$$E \subseteq \cup_{i=1}^{m} T_i(E).$$

Clearly any self-similar set is sub-self-similar. [**Fal1**] contains many other examples of sub-self-similar sets and describes their basic properties. The following lemma provides an example of a non-sub-self-similar set.

Lemma 1. *Let $E = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$. Then E is not a sub-self-similar set.*

Proof. Assume that $\{T_i\}_{i=1}^{m}$ are contractive similarities. We will show that $E \not\subseteq \cup_{i=1}^{m} T_i(E)$.

Suppose first that no T_i has 0 as a fixed point. Then there is a neighborhood U of 0 such that $T_i(0) \notin U$ for every $i \in \{1, \dots, m\}$. Since 0 is the only cluster point of E, it follows that $U \cap \cup_{i=1}^{m} T_i(E)$ can contain only finitely many points. But $U \cap E$ is infinite, so $E \not\subseteq \cup_{i=1}^{m} T_i(E)$.

Now, by reordering the set $\{T_i\}_{i=1}^{m}$ if necessary, choose $n \leq m$ such that $\{T_i\}_{i=1}^{n}$ are those similarities with 0 as a fixed point. We will show that $E \setminus \cup_{i=1}^{n} T_i(E)$ is infinite. Note that $T_i(E) \cap E = \{0\}$ unless $r(T_i)$ is of the specific form $\frac{p_i}{q_i}$ where $q_i, p_i \in \mathbb{N}$ and $q_i \geq 2$. Thus if p is a prime larger than q_i for each $i \in \{1, \dots, n\}$, then $\cup_{i=1}^{n} T_i(E)$ will contain no number of the form $\frac{1}{kp}$, where $k \in \mathbb{N}$. Now the remaining portion $\cup_{i=n+1}^{m} T_i(E)$ may contain only finitely many points of E for the reasons outlined above. Thus we again have $E \not\subseteq \cup_{i=1}^{m} T_i(E)$. □

The above argument may clearly be embedded in $\mathbb{R}^d$ by associating $\mathbb{R}$ with just one of the coordinates of $\mathbb{R}^d$. Furthermore, if E is the set in the lemma, we may obtain other non-sub-self-similar sets by scaling and translating E. Finally, the union of such a set with any finite set will be non-sub-self-similar. Using this fact together with the fact that the finite sets are dense in $\mathcal{K}(\mathbb{R}^d)$, we obtain the following important corollary.

Corollary 1. *The set of non-sub-self-similar sets is dense in* $\mathcal{K}(\mathbb{R}^d)$.

The super-self-similar sets were introduced in [**Fal2**] by reversing the inclusion. Thus, the compact set E is super-self-similar if there are contractive similarities $T_i : \mathbb{R}^d \to \mathbb{R}^d$ for $i \in \{1, \dots, m\}$ such that

$$E \supseteq \cup_{i=1}^m T_i(E).$$

It again turns out that the super-self-similar sets retain some nice properties of the self-similar sets, although some additional assumption may need to be added (see [**Fal2**], Cor. 3.4). As with the sub-self-similar sets, we will need the fact that the set of non-super-self-similar sets is dense in $\mathcal{K}(\mathbb{R}^d)$.

Lemma 2. *No finite set with more than one element is super-self-similar.*

Proof. Let F be a finite set with more than one element and let $\{T_i\}_{i=1}^m$ be contractive similarities. We will show that $F \not\supseteq \cup_{i=1}^m T_i(F)$. Let x be the fixed point of T_1 and let $y \in F$ satisfy $|x - y| = \mathrm{dist}(x, F \setminus \{x\})$. Then clearly $T_1(y) \notin F$ so $F \not\supseteq \cup_{i=1}^m T_i(F)$. □

As the finite sets are dense in $\mathcal{K}(\mathbb{R}^d)$, we obtain the following corollary immediately.

Corollary 2. *The set of non-super-self-similar sets is dense in* $\mathcal{K}(\mathbb{R}^d)$.

Note that the finite sets are all sub-self-similar while the set E from Lemma 1 is super-self-similar for the set of transformations $\{T_1(x) = \frac{1}{2}x, T_2(x) = \frac{1}{3}x\}$.

As a notational convenience, we will denote the set of self-similar sets by ss, the set of sub-self-similar sets by sss and the set of super-self-similar sets by Sss.

3. The Main Results

In this section, we prove our main results. Theorem 1 states that sss is a first category, F_σ subset of $\mathcal{K}(\mathbb{R}^d)$.

Theorem 1. *The set of sub-self-similar sets may be expressed as the countable union of closed, nowhere dense subsets of* $\mathcal{K}(\mathbb{R}^d)$.

Proof. For $m, n \in \mathbb{N}$, define $sss_{m,n}$ to be the set of all those sub-self-similar sets E such that there exists contractive similarities $\{T_i\}_{i=1}^m$ with $\frac{1}{n} \le r(T_i) \le 1 - \frac{1}{n}$,

$E \subseteq \cup_{i=1}^m T_i(E)$, and $|T_i(0)| \leq n$ for every $i \in \{1, \dots, m\}$. Clearly, $\cup_{m=1}^\infty \cup_{n=1}^\infty sss_{m,n}$ is precisely the set of sub-self-similar sets.

We first prove that $sss_{m,n}$ is closed for every $m, n \in \mathbb{N}$. Suppose that $E_k \to E$ in the Hausdorff metric, where $E_k \in sss_{m,n}$ for every $k \in \mathbb{N}$. To each E_k corresponds $\{T_i^k\}_{i=1}^m$ such that $1/n \leq r(T_i^k) \leq 1 - 1/n$, $E_k \subseteq \cup_{i=1}^m T_i^k(E_k)$, and $|T_i^k(0)| \leq n$. Using the standard matrix, vector representation of an affine transformation, each T_i^k may be associated with a point, x_i^k, in $\mathbb{R}^{d^2+d}$. The conditions on each T_i^k ensure that the set of all such points, K, is compact. By recursively choosing successively finer subsequences, we may assume that each sequence $\{x_i^k\}_{k=1}^\infty$ is convergent to say $x_i \in K$. Each point x_i in turn defines a contractive similarity $T_i : \mathbb{R}^d \to \mathbb{R}^d$ satisfying $\frac{1}{n} \leq r(T_i) \leq 1 - \frac{1}{n}$ and $|T_i(0)| \leq n$ for every $i \in \{1, \dots, m\}$. The correspondence between affine transformations on $\mathbb{R}^d$ and points in $\mathbb{R}^{d^2+d}$, along with the continuity of the algebraic operations, implies that $T_i^k \to T_i$ pointwise as $k \to \infty$. We must now show that $E \subseteq \cup_{i=1}^m T_i(E)$. Let $x \in E$. Then for every $k \in \mathbb{N}$, there is an $x_k \in E_k$ such that the sequence $\{x_k\}_{k=1}^\infty$ converges to x. Since $E_k \subseteq \cup_{i=1}^m T_i^k(E_k)$, there is an $i_k \in \{1, \dots, m\}$ such that $x_k \in T_{i_k}^k(E_k)$. Since there are only finitely many choices for i_k, at least one must occur infinitely often. Thus we have a subsequence $\{k_j\}_{j=1}^\infty$ and a fixed $i \in \{1, \dots, m\}$ such that $i_{k_j} = i$ for every j. Along this subsequence we have

$$T_{i_{k_j}}^{k_j}(E_{k_j}) = T_i^{k_j}(E_{k_j}) \to T_i(E)$$

as $j \to \infty$, since $T_i^{k_j} \to T_i$ pointwise and $E_{k_j} \to E$ in the Hausdorff metric. Thus $x \in T_i(E)$ since $x_{k_j} \to x$ and $x_{k_j} \in T_i^{k_j}(E_{k_j})$ for all j.

Finally, we prove that $sss_{m,n}$ is nowhere dense in $\mathcal{K}(\mathbb{R}^d)$ for all $m, n \in \mathbb{N}$. Since $sss_{m,n}$ is closed, we must simply show that it contains no open set. But this is immediate since its complement is dense in $\mathcal{K}(\mathbb{R}^d)$ by Corollary 1. □

The next theorem states a similar result for Sss.

Theorem 2. *The set of super-self-similar sets may be expressed as the countable union of closed, nowhere dense subsets of* $\mathcal{K}(\mathbb{R}^d)$.

Proof. The proof of this theorem is very similar to the proof of Theorem 1. For $m, n \in \mathbb{N}$, define $Sss_{m,n}$ to be the set of all those super-self-similar sets E such that there exists contractive similarities $\{T_i\}_{i=1}^m$ with $\frac{1}{n} \leq r(T_i) \leq 1 - \frac{1}{n}$, $E \supseteq \cup_{i=1}^m T_i(E)$, and $|T_i(0)| \leq n$ for every $i \in \{1, \dots, m\}$. Using the exact construction from Theorem 1, we obtain a sequence of sets $E_k \to E$ and a sequence of transformations $\{T_i^k\}_{k=1}^\infty$, for each $i \in \{1, \dots, m\}$ satisfying $1/n \leq r(T_i^k) \leq 1 - 1/n$, $E_k \supseteq \cup_{i=1}^m T_i^k(E_k)$, and $|T_i^k(0)| \leq n$. As before, there are transformations $\{T_i\}_{i=1}^m$ which are the pointwise limits as $k \to \infty$ of $\{T_i^k\}_{k=1}^\infty$, for each $i \in \{1, \dots, m\}$ and which satisfy $\frac{1}{n} \leq r(T_i) \leq 1 - \frac{1}{n}$ and $|T_i(0)| \leq n$. We must now show that $E \supseteq \cup_{i=1}^m T_i(E)$. Suppose that $x \in T_i(E)$ for some $i \in \{1, \dots, m\}$. Since $E_k \to E$

in the Hausdorff metric, $T_i(E_k) \to T_i(E)$ by the continuity of T_i. Thus for each k we may choose $x_k \in E_k$ such that $T_i(x_k) \to T_i(x)$. Thus $x_k \to x$ by the continuity of T_i^{-1} and $x \in E$.

In order to show that $Sss_{m,n}$ is nowhere dense in $\mathcal{K}(\mathbb{R}^d)$ it again suffices to show that it contains no open set, since it is closed. But this follows immediately from Corollary 2. $\square$

The above theorems may be somewhat improved. By allowing more general affine contractions, rather than strict similarities, we obtain the notions of sub-self-affinity and super-self-affinity. The above proofs clearly apply to the larger sets of sub-self-affine sets and super-self-affine sets.

We now turn our attention to the set of self-similar sets. It is well known that ss is dense in $\mathcal{K}(\mathbb{R}^d)$. This is essentially the content of the collage theorem (see [**Ba**] section 3.10, Theorem 1). This implies that sss and Sss are dense in $\mathcal{K}(\mathbb{R}^d)$ as they both contain ss. In fact, $ss = sss \cap Sss$. This implies that ss is a first category, F_σ subset of $\mathcal{K}(\mathbb{R}^d)$. Finally, we are interested in the size of ss compared to sss and Sss. As sss and Sss are not G_δ subsets of $\mathcal{K}(\mathbb{R}^d)$, it makes no sense to consider the Baire category of their subsets (see [**Ox**], chapter 12). Thus we content ourselves with the following theorem which states that ss is a small subset of both sss and Sss.

Theorem 3. *$sss \setminus ss$ is dense in sss and $Sss \setminus ss$ is dense in Sss.*

Proof. The first part is quite simple since any finite set is sub-self-similar. The finite sets are dense in $\mathcal{K}(\mathbb{R}^d)$ and, therefore, dense in $sss \setminus ss$.

The second part is slightly more difficult. It suffices to find a class of super-self-similar sets which are not self-similar, but are dense in $\mathcal{K}(\mathbb{R}^d)$. Since ss is dense in $\mathcal{K}(\mathbb{R}^d)$, we show how to approximate any self-similar set with a super-self-similar set which is not self-similar. Let E be self-similar for the transformations $\{T_i\}_{i=1}^m$. Choose $R > 0$ such that $T_i(\overline{B_R(0)}) \subseteq \overline{B_R(0)}$ for each $i \in \{1, \ldots, m\}$. Let $E_1 = \cup_{i=1}^m T_i(\overline{B_R(0)})$ and for $n > 1$ let $E_n = \cup_{i=1}^m T_i(E_{n-1})$. Then each E_n is super-self-similar, but not self-similar and $E_n \to E$ in the Hausdorff metric. $\square$

References

[**Ba**] Barnsley M., *Fractals Everywhere*, Academic Press, 1988.

[**Ed**] Edgar G. A., *Measure, Topology, and Fractal Geometry*, Springer-Verlag, 1990.

[**Fal1**] Falconer K. J., *Sub-Self-Similar Sets*, Trans. Amer. Math. Soc. **347** (1995), 3121–3129.

[**Fal2**] ———, *Techniques in Fractal Geometry*, John Wiley & Sons, 1997.

[**Ox**] Oxtoby J. C., *Measure and Category, 2nd Edition*, Springer-Verlag, 1980.

M. McClure, Department of Mathematics, University of North Carolina at Asheville, Asheville, North Carolina, USA; *e-mail*: mcmcclur@bulldog.unca.edu

R. W. Vallin, Department of Mathematics, Slippery Rock University, Slippery Rock, Pennsylvania, USA; *e-mail*: robert.vallin@sru.edu