Acta Math. Univ. Comenianae
Vol. LXIX, 2(2000), pp. 241–259

CLONES, COCLONES AND COCONNECTED SPACES

V. TRNKOVÁ

Abstract. Clones and coclones motivate this examination of coconnected spaces. A space X is **coconnected** if every continuous map $X \times X \to X$ depends only on one variable. We prove here that every monoid can be represented as the monoid of all nonconstant continuous selfmaps of a coconnected space and that, within the class of Hausdorff spaces, the coconnectedness is **not** expressible by a sentence of the first order language of the monoid theory: we construct two Hausdorff spaces with isomorphic monoids of all continuous selfmaps such that one of them is coconnected and the other is not.

I. Introduction And The Main Results

The topological results of the present paper are inspired by universal algebra, in which clones play an important role (see e.g. the monographs [**3**], [**11**]). Let us recall that **a clone on a set** P is a system of maps $P^n \to P^m$, $n, m \in \omega$ (where ω denotes, as usual, the set of all finite cardinal numbers), containing all the product projections $\pi_j^{(n)} : P^n \to P$, $j \in n$ $[= \{0, \dots, n-1\}]$, $n \in \omega$, and closed with respect to the composition $\circ$ of maps and with respect to fibered products, i.e. if $f_0, \dots, f_{m-1} : P^n \to P$ are in the system, then the map

$$f_0 \dot{\times} \dots \dot{\times} f_{m-1} : P^n \to P^m$$

sending each $z \in P^n$ to the m-tuple $(f_0(z), \dots, f_{m-1}(z))$ is also in the system. $\big[$Note that $\pi_0^{(n)} \dot{\times} \dots \dot{\times} \pi_{n-1}^{(n)}$ is just the identity map on P^n, so that the system of maps forms a category; viewing it as an abstract category, we get the notion of **abstract clone**, see [**11**], corresponding to **algebraic theory** in the sense of [**9**], [**10**].$\big]$

Another equivalent description of a clone on a set P, used in [**11**], is that it is a system of finitary operations $f : P^n \to P$ containing all the above projections

Received December 18, 2000.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 54C05, 08A10.
Key words and phrases. Clone, the first order language of clone theory, dual notions, the first order language of monoid theory, connected topological space, monoid of continuous selfmaps of a space, continuous binary operation.
Financial support of the Grant Agency of the Czech Republic under the grants no. 201/99/0310 and no. 201/00/1466 is gratefully acknowledged. Also supported by MSM 113200007.

$\pi_j^{(n)}\colon P^n \to P$ and closed with respect to the operations S_m^n, $m, n \in \omega$, defined as follows:

> if $g\colon P^m \to P$ and $f_0, \dots, f_{m-1}\colon P^n \to P$, then S_m^n replaces any z_i in $g(z_0, \dots, z_{m-1})$ by $f_i(x_0, \dots, x_{n-1})$, and hence it produces the map $P^n \to P$ given by the formula
>
> $$S_m^n(g; f_0, \dots, f_{m-1}) = g \circ (f_0 \dot{\times} \dots \dot{\times} f_{m-1}).$$

Each of these two descriptions of a clone on a set P can be easily transformed into the other one.

Given a topological space $X = (P, t)$, all continuous maps among the finite powers X^0, X, X^2, … of the space X form a clone on its underlying set P, called simply **the clone of the space** X.

The monograph [**15**] is devoted to examination of clones of topological spaces. Particular interest is paid to the possibility to describe some properties of a space X by sentences of the first order language of the clone theory [briefly: this language has ω sorts of variables, the variables of the n-th sort range over the continuous maps $X^n \to X$; in each n-th sort, there are n constant symbols, namely the product projections $\pi_j^{(n)}\colon X^n \to X$; there are no predicates other than the equality $=$; the above S_m^n are all operation symbols of this language; for a more detailed description see e.g. [**15**] or also [**13**], [**18**]].

Problem 1 in the monograph [**15**] asks whether, for topological spaces, the (first order) language of the clone theory has more expressive power than the (first order) language of the monoid theory [this language uses only the first sort of variables; they range over continuous maps $X \to X$; there is only one constant, namely the identity map; and there is only one operation symbol, namely S_1^1, which is just the composition of maps], i.e. whether there exist spaces X, Y with elementarily equivalent monoids (= satisfying precisely the same sentences of the language of the monoid theory) such that their clones are not elementarily equivalent (i.e. they do not satisfy the same sentences of the language of the clone theory). This is solved positively in [**18**]. In this paper, metric spaces X and Y are constructed such that

$$(\star) \begin{cases} \text{their monoids of all continuous selfmaps are isomorphic} \\ \text{but their clones are } \textbf{not} \text{ elementarily equivalent.} \end{cases}$$

(In fact, stronger results were proved in [**18**]; and these were strengthened and generalized still more in [**13**], [**14**]).

Coalgebras lead to dual notions: coclones on a set and coclones of topological spaces. We do not formulate explicitly the definitions of these dual notions which only reverse the arrows (hence also the order in the composition) and replace products and the product projections by coproducts and the coproduct injections. Also, the definition of the first order language of the coclone theory is just dual to the first order language of the clone theory $\big[$it has ω sorts variables, the variables of the n-th sort range over continuous maps $X \to nX$, where nX denotes the coproduct (= the sum) of n copies of X; in each n-th sort there are n constants, namely the coproduct injections $i_j^{(n)}\colon X \to nX$; there are no predicates other than $=$; and $\widetilde{S}^n_m$ are the only operation symbols, where

$$\widetilde{S}^n_m(f_0, \dots, f_{m-1}; g) = (f_0 \dotplus \cdots \dotplus f_{m-1}) \circ g$$

for $g\colon X \to mX$ and $f_0, \dots, f_{m-1}\colon X \to nX\big]$.

Let us compare, for topological spaces, the strength of the expressive power of the languages of the clone theory and of the coclone theory. The dual to the above statement ($\star$) is no longer true. One can see easily that if T_1-spaces X, Y have isomorphic monoids of all continuous selfmaps, then they have isomorphic (hence elementarily equivalent) coclones. But this does not solve the dual of the Problem 1 in [**15**]. If the monoids of all continuous selfmaps of spaces X, Y are only elementarily equivalent, are their coclones also elementarily equivalent? The affirmative answer is expected but the proof has not been done.

The "usual" topological properties, like regularity, normality, paracompactness, compactness, metrizability are expressible neither in the language of the clone theory nor in the language of the coclone theory. This can be seen by means of rigid spaces. Let us recall that a space X is **rigid** if every continuous selfmap $X \to X$ is either the identity or a constant. $\big[$A rigid metric continuum was constructed by H. Cook in [**4**].$\big]$ By [**6**], [**7**], every continuous map $f\colon X^n \to X$ with X rigid Hausdorff space is either a product projection or a constant map (and, if $n \in \omega$, then the assumption that X is a Hausdorff space can be replaced by card $X > 2$, see [**15**]). Hence if t_1, t_2 are rigid topologies on a set P (with card $P > 2$), then the spaces $X = (P, t_1)$ and $Y = (P, t_2)$ have the same clone. As noted in [**2**], we can choose the rigid topologies t_1 and t_2 on a set P with $\operatorname{card} P = 2^{\aleph_0}$ such that $X = (P, t_1)$ is a compact metrizable space and $Y = (P, t_2)$ is not a Hausdorff space. Hence no topological property between "being Hausdorff" and "being compact metrizable" can be expressed by a sentence in the language of the clone theory. And, since rigid spaces are always connected, the spaces $X = (P, t_1)$ and $Y = (P, t_2)$ have also the same coclone, so that these properties are also not expressible by a sentence in the language of the coclone theory.

On the other hand, the connectedness itself **can** be expressed in the language of the coclone theory. In fact, a space X is connected if and only if

> every continuous map $X \to X + X$ factors through a coproduct injection.

This **is** a sentence in the language of the coclone theory, formally stated as follows (where $x^{(n)}$, $y^{(n)}, \dots$ denote the variables of the n-th sort; we recall that the coproduct injections $i_j^{(n)}$, $j \in n$, are constants of the language):

$$(\forall x^{(2)})(\exists y^{(1)})((x^{(2)} = \widetilde{S}_1^2(i_0^{(2)}; y^{(1)})) \vee (x^{(2)} = \widetilde{S}_1^2(i_1^{(2)}; y^{(1)}))).$$

In [**19**], the dual sentence of the language of the clone theory, namely

$$(\forall x^{(2)})(\exists y^{(1)})((x^{(2)} = S_1^2(y^{(1)}; \pi_0^{(2)})) \vee (x^{(2)} = S_1^2(y^{(1)}; \pi_1^{(2)})))$$

is investigated (this is a duality distinct from that of [**1**], where the spaces dual to the connected spaces are precisely the totally disconnected spaces). The spaces satisfying the sentence, i.e. the spaces X such that every continuous map $f\colon X \times X \to X$ factors through a product projection $\big[$in other words, every continuous binary operation on X is essentially unary; by [**19**], every continuous finitary operation on such a space is essentially unary$\big]$ are called **coconnected**.

As mentioned above, every continuous $f\colon X \times X \to X$, with X rigid and card $X > 2$, is either a projection or constant, hence rigid spaces are coconnected. Are there also some other coconnected spaces? This problem is attacked in [**19**]. As proved in [**19**], every free monoid can be represented as the monoid of all nonconstant continuous selfmaps of a coconnected space. The problem stated in [**19**] asks which monoids have such a representation by means of coconnected spaces. The first result of the present paper strengthens considerably an answer to this question. We prove the following

Theorem 1. *For every triple of monoids $M_1 \subseteq M_2 \subseteq M_3$ (i.e. M_1 is a submonoid of M_2 and M_2 is a submonoid of M_3) there exists a metric coconnected space X such that all the nonconstant maps of X into itself which are*

nonexpanding form a monoid isomorphic to M_1,
uniformly continuous form a monoid isomorphic to M_2,
continuous form a monoid isomorphic to M_3.

Let us go back to [**15**]. In this monograph, some topological properties are described (within some classes of spaces) by sentences of the language of the clone theory using more sorts of variables than only the first sort. But this does not solve the Problem 1 of [**15**] because it is possible that a sentence of the language of the monoid theory, possibly more complicated, could give the same result (as mentioned above, the Problem 1 of [**15**] is solved later in [**18**]). For coclones, we can see such a situation concerning the connectedness of T_1-spaces. Though expressed by the above sentence of the language of the coclone theory, it can be expressed also as follows: a T_1-space X is connected if and only if

there exists no continuous $f\colon X \to X$ with card $f(X) = 2$.

This condition can be expressed by a sentence of the language of the monoid theory as follows $\big[$since there is only one sort of variables in this language, we write x, y,

$y_1, y_2, \ldots$ instead of $x^{(1)}, y^{(1)}, y_1^{(1)}, y_2^{(1)}, \ldots$; also we write simply $x \circ y$ instead of $S_1^1(x; y)$]: first, we introduce the predicate

$$C(x) \stackrel{\text{def}}{\equiv} (\forall y)(x \circ y = x)$$

describing constant maps (which, in turn, play the role of points). Then the existence of a continuous map $f\colon X \to X$ with card $f(X) = 2$ can be expressed by the following sentence s (in which $f(X) = \{y_1, y_2\}$):

$$\begin{aligned} s : (\exists f)(\exists y_1)(\exists y_2) \big[& C(y_1) \wedge C(y_2) \wedge (\neg(y_1 = y_2)) \\ & \wedge ((\forall z)(C(z) \Rightarrow ((f \circ z = y_1) \vee (f \circ z = y_2)))) \\ & \wedge ((\exists z)(f \circ z = y_1)) \ \wedge \ ((\exists z)(f \circ z = y_2)).\big] \end{aligned}$$

Hence the sentence of the language of the monoid theory expressing the connectedness of T_1-spaces is

$$\neg s.$$

There is also a sentence in the language of the monoid theory (mildly more complicated than the above $\neg s$) expressing the connectedness within the class of all T_0-spaces, but there exists no such sentence expressing the connectedness within the class of **all** topological spaces: the discrete and the indiscrete spaces on a two-point set have the same monoid of all continuous selfmaps but the later is connected and the former is not.

Is the coconnectedness also expressible, at least within the class of all T_1-spaces, by means of a sentence of the language of the monoid theory? The second result of the paper gives a negative answer to it. We prove the following

Theorem 2. *There exist Hausdorff spaces X and Y with isomorphic monoids of all continuous selfmaps such that Y is coconnected and X is not coconnected.*

The results stated in Theorem 1 and Theorem 2 were announced in the expository paper [**20**]. As stated explicitly in [**20**], the proofs have not yet been published. The author feels obliged to provide these proofs; they are contained in the present paper.

A metric space X which satisfies Theorem 1 was constructed already in [**17**], but its coconnectedness was neither proved nor even mentioned there. Hence we briefly review this construction in Part III of the present paper, and then we prove the coconnectedness of the resulting space X.

Using the ideas of [**18**], we prove Theorem 2 in Part II below.

The constructed space X which is **not** coconnected, is even metrizable, but the coconnected Y is not.

Problem. *Is coconnectedness expressible by a sentence of the language of the monoid theory within the class of all metrizable spaces? Or do there exist metrizable spaces X, Y with isomorphic (or at least elementarily equivalent) monoids of all continuous selfmaps such that Y is coconnected but X is not?*

II. Proof of Theorem 2

II.1. Let G_0 be a set with card $G_0 = 2^{\aleph_0}$. Let (P, b) be a free groupoid on G_0, i.e.

$$P = \bigcup_{k=0}^{\infty} G_k \qquad \text{where} \qquad G_k = G_0 \cup \bigcup_{j=1}^{k} B_j$$

and

$$b \colon P \times P \to P$$

maps bijectively $G_0 \times G_0$ onto B_1 and $(G_k \times G_k) \setminus (G_{k-1} \times G_{k-1})$ onto B_{k+1} for $k = 1, 2, \ldots$, and hence b maps $P \times P$ bijectively onto $B = \bigcup\limits_{j=1}^{\infty} B_j = P \setminus G_0$.

Let $M = \bigcup\limits_{k=0}^{\infty} M_k$ be the set of selfmaps of P obtained as follows:

$$M_0 = \{c_x | x \in G_0\} \cup \{\mathbb{I}\}$$

where $c_x \colon P \to P$ is the constant map with the value $x \in G_0$ and $\mathbb{I}$ is the identity map of P onto itself, and

$$M_{k+1} = M_k \cup \{b \circ (f_1 \dot{\times} f_2) \mid f_1,\, f_2 \in M_k\} \quad \text{for } k = 0,\, 1,\, \ldots$$

where $f_1 \dot{\times} f_2 \colon P \to P \times P$ is the map sending any $x \in P$ to $(f_1(x), f_2(x))$ and $\circ$ denotes the composition of maps. Note that

$$(\circ) \quad \text{every } f \in M \text{ is either constant or one-to-one.}$$

We are going to construct two Hausdorff topologies t_1, t_2 on the set P such that M is the monoid of all continuous selfmaps of both spaces $X = (P,\, t_1)$ and $Y = (P,\, t_2)$, and Y is coconnected but X is not.

II.2. For every $f_1,\, f_2 \in M$, let us denote

$$Z(f_1,\, f_2) = \{(f_1(x),\, f_2(x)) \,|\, x \in P\}$$

and put $\mathbb{Z} = \{Z(f_1,\, f_2) \,|\, f_1,\, f_2 \in M\}$. For every topology t on P let us denote by

$t \times t$ the product topology on the set $P \times P$,

and by

$\ulcorner t \urcorner$ the finest topology on $P \times P$ such that, for each $Z \in \mathbb{Z}$, the restriction $\ulcorner t \urcorner/_Z$ of $\ulcorner t \urcorner$ to Z is equal to the restriction ${}^{t \times t}/_Z$ of $t \times t$ to Z.

Clearly, if t is a Hausdorff topology, then $\ulcorner t \urcorner$ is also a Hausdorff topology.

In II.7–II.8 below, we construct topologies t_1 and t_2 on P such that

(1) **the spaces** $X = (P, t_1)$ **and** $Y = (P, t_2)$ **are** B**-semirigid** in the sense of [**18**], i.e. that every continuous selfmap $X \to X$ is either the identity

or a constant, or it sends the whole space into B ($= b(P \times P)$), and analogously for Y;

(2) **the map** $b\colon P \times P \to P$ is a homeomorphism of

$$(P \times P,\, t_1 \times t_1) \text{ onto } (B,\, {t_1}\!/_{B})$$

and it is also a homeomorphism of

$$(P \times P,\, t_2) \text{ onto } (B,\, {t_2}\!/_{B});$$

(3) G_0 **is a metrizable connected subset both in** X **and in** Y**.**

First, we show that such spaces already satisfy Theorem 2 (see II.3–II.6 below). Then (in II.7–II.8), a construction of spaces X, Y satisfying (1), (2) and (3) will be given.

II.3. First, we show that if X and Y satisfy (1) and (2) of II.2., then M is the monoid of all continuous selfmaps of both X and Y.

Every $f \in M$ is continuous as a map $X \to X$ and also as a map $Y \to Y$, evidently. We show the converse. The fact that any continuous $f\colon X \to X$ [or $Y \to Y$] is in M will be proved by induction on the smallest k such that the image $\mathrm{Im} f$ of f [i.e. $f(X)$ or $f(Y)$] intersects G_k. If $k = 0$, i.e. $\mathrm{Im} f$ intersects G_0, then f is either the identity or a constant because X [or Y] is B-semirigid, hence f is in M_0. If $k > 0$, then $\mathrm{Im} f$ is a subset of B, hence we can investigate the two continuous maps

$$f_1 = \pi_1 \circ b^{-1} \circ f, \quad f_2 = \pi_2 \circ b^{-1} \circ f$$

where π_1, $\pi_2\colon P \times P \to P$ are the first and second projections. By the form of $b\colon P \times P \to P$ in II.1, $\mathrm{Im} f_1$ intersects G_{k_1} with $k_1 < k$ and analogously for $\mathrm{Im} f_2$, hence the maps f_1, f_2 are in M, by the induction hypothesis. Hence $f = b \circ (f_1 \dot{\times} f_2)$ is also in M.

Clearly, the space $X = (P, t_1)$ is not coconnected because b as a map $X \times X \to X$ is continuous and it factorizes neither through π_1 nor through π_2. The proof that the space $Y = (P, t_2)$ satisfying (1), (2), (3) **is** coconnected is more subtle and it is given in II.4–II.6 below.

II.4. Lemma. *For every $x \in P$ there exist only finitely many f in M such that $x \in \mathrm{Im} f$.*

Proof. We proceed by induction in the smallest k such that $x \in G_k$.

$k = 0$: if $x \in G_0$, then $x \in \mathrm{Im} f$ if and only if $f = \mathbb{I}$ or $f = c_x$;

$k > 0$: if $x \in G_k$ with $k > 0$, then $x \in B$; denote $(x_1, x_2) = b^{-1}(x)$; then, by the form of b in II.1, $x_i \in G_{k_i}$ with $k_i < k$ for $i = 1, 2$; if $x \in \mathrm{Im} f$, $f \in M$, then either $f = \mathbb{I}$ or $f \in M \setminus M_0$, hence f is equal to some $b \circ (f_1 \dot{\times} f_2)$ with f_1, $f_2 \in M$. But, by the induction hypothesis, there exist only finitely many f_1 with $x_1 \in \mathrm{Im} f_1$ and only finitely many f_2 with $x_2 \in \mathrm{Im} f_2$, hence there exist only finitely many $f = b \circ (f_1 \dot{\times} f_2)$ with $x \in \mathrm{Im} f$. $\square$

II.5. Lemma. *Let $f, g \in M$ be nonconstant. Then the map*

$$f \times g : (P \times P,\, t_2 \times t_2) \to (P \times P, \lceil t \rceil_2)$$

sending each $(x, y) \in (P \times P)$ to $(f(x), g(y))$ is **not** *continuous.*

Proof. 1) Choose $x \in G_0$ and a one-to-one sequence $\{x_n |\, n \in \omega\}$ of elements of G_0 converging to x in (P, t_2) $x_n \neq x$ for all n. Let F be the set of all $(k, h) \in M \times M$ such that $(f(x), g(x)) = (k(p), h(p))$ for some $p \in P$. By II.4, F is finite. Since f is supposed to be nonconstant, it is one-to-one, by II.1 ($\circ$). If $n \in \omega$ and $(k, h) \in F$, put

$$Q(k, h, n) = \{p \in P \mid k(p) = f(x_n)\}.$$

If k is constant, then $Q(k, h, n) = \varnothing$ for all n because $k(p) = f(x) \neq f(x_n)$. If k is nonconstant, it is one-to-one, by II.1 ($\circ$), hence card $Q(k, h, n) \leq 1$ so that $\bigcup\limits_{(k,h)\in F} Q(k, h, n)$ is finite. We choose $y_n \in G_0$ such that $y_n \neq x$, the distance of x_n and y_n is less than $\frac{1}{n}$ and $g(y_n)$ is **not** in $\bigcup\limits_{(k,h)\in F} h(Q(k, h, n))$. This is possible because of (3) in II.2. Hence $\{y_n | n \in \omega\}$ is a sequence also converging to x so that $\{(f(x_n), g(y_n)) | n \in \omega\}$ converges to $(f(x), g(x))$ in $(P \times P,\, t_2 \times t_2)$. We show that $S = \{(f(x_n), g(y_n)) | n \in \omega\}$ does not converge to $(f(x), g(x))$ in $(P \times P, \lceil t \rceil_2)$.

2) Let $\tilde{t}$ be the topology on $P \times P$ such that, for $q \in P \times P \setminus \{(f(x), g(x))\}$, the $\tilde{t}$-neighborhoods of q are precisely its $(t_2 \times t_2)$-neighborhoods and a local $\tilde{t}$-basis of $(f(x), g(x))$ is formed by all sets $U \setminus S$ where U is a $(t_2 \times t_2)$-open neighborhood of $(f(x), g(x))$ and S is as above, i.e. $S = \{(f(x_n), g(y_n)) | n \in \omega\}$. It suffices to show that $\lceil t \rceil_2$ is finer than $\tilde{t}$. We show that

$$\tilde{t}/_{Z} = t_2 \times t_2/_{Z}$$

for all $Z \in \mathbb{Z}$. The case that $(f(x), g(x)) \notin Z$ is trivial. Thus, let us suppose that $Z = Z(k, h)$ and $(f(x), g(x)) = (k(p), h(p))$ for some $p \in P$. Then $(k, h) \in F$ where F is as in the part 1) of this proof. Then S does not intersect $Z(k, h)$, and hence $(U \setminus S) \cap Z(k, h) = U \cap Z(k, h)$. □

II.6. Lemma. *The space $Y = (P, t_2)$ is coconnected.*

Proof. Let $f\colon Y \times Y \to Y$ be a continuous map. We have to prove that either $f = \alpha \circ \pi_1$ of $f = \alpha \circ \pi_2$ for a continuous map $\alpha\colon Y \to Y$. We proceed by induction in the smallest k such that Imf intersects G_k.

$k = 0$:

Since Y is a semirigid Hausdorff space, every continuous map $Y \times Y \to Y$ is either a projection or a constant or sends the whole $Y \times Y$ into B, by Proposition II.5 in [**18**]; since Imf intersects $G_0 = P \setminus B$, necessarily f is a projection or a constant; in the latter case, f factorizes both through π_1 and through π_2.

$k > 0$:
Since b^{-1} as a map

$$(B,\, t_2/_B) \xrightarrow{b^{-1}} (P \times P,\, \sqcap t_2) \xrightarrow{\mathrm{id}} (P \times P,\, t_2 \times t_2)$$

is continuous, necessarily the maps

$$f_1 = \pi_1 \circ b^{-1} \circ f \qquad f_2 = \pi_2 \circ b^{-1} \circ f$$

are continuous. By the form of b in II.1, f_i intersects G_{k_i} with $k_i < k$ for $i = 1, 2$.

By the induction hypothesis, $f_1 = \alpha_1 \circ \pi_i$ and $f_2 = \alpha_2 \circ \pi_j$ for $i, j \in \{1, 2\}$ and continuous $\alpha_1, \alpha_2 \colon Y \to Y$. If $i = j$, then $f = b \circ (\alpha_1 \dot{\times} \alpha_2) \circ \pi_i$. If α_1 is constant map, then $f_1 = \alpha_1 \circ \pi_1 = \alpha_1 \circ \pi_2$, hence we may suppose that $i = j$. Analogously if α_2 is constant. The remaining case, in which both α_1 and α_2 are nonconstant and $i \neq j$, cannot occur. Indeed, if $f_1 = \alpha_1 \circ \pi_1$ and $f_2 = \alpha_2 \circ \pi_2$ and both α_1, α_2 are nonconstant, then the map

$$f \colon (P \times P,\, t_2 \times t_2) \xrightarrow{\alpha_1 \times \alpha_2} (P \times P,\, \sqcap t_2) \xrightarrow{b} (B,\, t_2) \subseteq (P,\, t_2)$$

is not continuous: the first arrow is discontinuous, by II.5 Lemma, and the second arrow is a homeomorphism. Analogously if $f_1 = \alpha_1 \circ \pi_2$ and $f_2 = \alpha_2 \circ \pi_1$. $\square$

II.7. To prove Theorem 2, it remains to construct the spaces $X = (P,\, t_1)$ and $Y = (P,\, t_2)$ satisfying (1), (2) and (3) in II.2. In fact, the space X is already constructed in [**18**]. We outline briefly its construction because we show in II.8 the modifications leading to the construction of the coconnected space Y.

We start from an extremally B-semirigid metric ρ on P in the sense of [**18**], i.e.

(α) diam $(P, \rho) = 1$ and $\rho(x, y) = 1$ for all $x, y \in B,\ x \neq y$;
(β) if t is a Hausdorff topology on P such that the topology t_ρ determined by ρ is finer than t and $t/_{G_0} = t_\rho/_{G_0}$, then $(P,\, t)$ is B-semirigid.

Such a metric ρ really does exist, it is constructed in [**18**]. Statements (α) and (β) imply that G_0 is a connected subset of (P, ρ), but this can also be seen easily from the construction in [**18**].

The space $X = (P,\, t_1)$ will be metrizable. We construct a chain of pseudometrics on P

$$\tau_0 \geq \tau_1 \geq \cdots \geq \tau_\alpha \geq \dots ,$$

where α ranges over all ordinals $\big[$this means that $\tau_\alpha(x, y) \geq \tau_{\alpha+1}(x, y)$ for all $x, y \in P\big]$ by means of transfinite induction as follows: $\tau_0 = \rho$;
if $\alpha = \beta + 1$, then $\tau_\alpha = u_\alpha * \rho$ is the pseudometric described below: u_α is **the** pseudometric on B, for which

$$b \colon P \times P \to P$$

is an isometry of $(P \times P,\, \tau_\beta \times \tau_\beta)$ onto $(B,\, u_\alpha)$ $\big[$where $\tau_\beta \times \tau_\beta$ is a pseudometric given by the usual formula

$$(\tau_\beta \times \tau_\beta)\,((x_1,\, x_2)(y_1,\, y_2)) = \max\{\tau_\beta(x_1,\, y_1),\, \tau_\beta(x_2,\, y_2)\}\big]$$

and $u_\alpha * \rho$ is the extension of u_α onto the whole P by the following rule (which we formulate generally because it will be used not only for this u_α):

> if u is an arbitrary pseudometric on B such that $u(x, y) \leq 1$ for all $x, y \in B$, we extend it by
>
> $$(u * \rho)(v, z) = \min\Big\{\rho(v, z),\ \inf_{x,y \in B}(\rho(v, x) + u(x, y) + \rho(y, z))\Big\}.$$

If α is a limit ordinal, then

$$\tau_\alpha = (\inf_{\beta < \alpha} u_\beta) * \rho.$$

The fact that $\tau_\alpha \geq \tau_{\alpha+1}$ for all α can be easily proved by transfinite induction. Hence necessarily there exists an ordinal γ such that $\tau_\gamma = \tau_{\gamma+1}$. Then we define t_1 as the topology determined by τ_γ and we put $X = (P,\, t_1)$. We outline why X has the properties (1), (2), (3). Since b is an isometry of $(P \times P,\, \tau_\gamma \times \tau_\gamma)$ onto $(B,\, u_{\gamma+1}) = (B,\, {\tau_{\gamma+1}}/_{B})$, the statement (2) is evident. The formula for $u * \rho$ implies that (3) is satisfied and that ${t_\rho}/_{G_0} = {t_1}/_{G_0}$. Since ρ is extremally B-semirigid, our space $X = (P,\, t_1)$ is B-semirigid, by (β), whenever X is a Hausdorff space. But, by Proposition III.7 in [**18**], all the above pseudometrics τ_α are metrics, hence X, being metrizable, is a Hausdorff space. For details of the above construction, see [**18**].

II.8. Our coconnected space $Y = (P, t_2)$ is no longer metrizable. We start from (P, ρ) as in II.7, however we modify the formula for the extension: if ξ is a topology on B, then

$$\xi * \rho$$

is the topology t on P defined as follows:

> for every $x \in G_0$, its t-neighborhoods are precisely all its t_ρ-neighborhoods; if $x \in B$, then its local base in (P, t) is formed by all the sets
>
> $$U(\varepsilon) = \big\{y \in P \mid \rho(y, U) < \varepsilon\big\}$$
>
> where $\varepsilon > 0$ and U is a ξ-open neighborhood of x in (B, ξ).

Now, we construct a transfinite chain of topologies on P

$$\xi_0,\, \xi_1,\, \ldots,\, \xi_\alpha,\, \ldots\, ,$$

where α ranges over all ordinals such that ξ_α is finer than $\xi_{\alpha+1}$ for all α (i.e. $\xi_{\alpha+1}$ is coarser than ξ_α; the usual notation $\xi_\alpha \leq \xi_{\alpha+1}$, of the fact that the identity map $(P,\, \xi_\alpha) \to (P,\, \xi_{\alpha+1})$ is continuous, is rather unfortunate here) as follows: $\xi_0 = t_\rho$,

if $\alpha = \beta + 1$, then $\xi_\alpha = \zeta_\alpha * \rho$, where ζ_α is **the** topology on B that b is a homeomorphism of $(P \times P,\, \ulcorner\xi_\beta)$ onto $(B,\, \zeta_\alpha)$ $\big[$where the operator $\ulcorner t$ is as in II.2$\big]$;

if α **is a limit ordinal**, we put $\xi_\alpha = (\sup\limits_{\beta<\alpha} \zeta_\beta) * \rho$ where $\sup \zeta_\beta$ means the finest topology coarser than every ζ_β with $\beta < \alpha$.

This transfinite induction has to stop, i.e. there exists γ such that $\xi_\gamma = \xi_{\gamma+1}$. We put $t_2 = \xi_\gamma$. Then, clearly, G_0 is a connected metrizable subset of $Y = (P,\, t_2)$ and b is a homeomorphism of $(P \times P,\, \ulcorner t_2)$ onto $(B,\, {t_2}/_{B})$. Since ρ is extremally B-semirigid, the space Y is B-semirigid, by (β) in II.7, once we show that it is a Hausdorff space. This follows easily from the comparison of the construction of Y to the construction of X: the topology ξ_α is always finer than the topology determined by the metric τ_α, hence all the spaces $(P,\, \xi_\alpha)$ are Hausdorff spaces. We conclude that Y satisfies (1), (2) and (3) in II.2.

III. Proof of Theorem 1

III.1. We start from the following statement which is a consequence of Lemma 2.4 in [**16**]:

> for every triple of monoids $M_1 \subseteq M_2 \subseteq M_3$ there exists a trigraph $G = (X, R_1, R_2, R_3)$ such that $End\,(X, R_3)$ is isomorphic to M_3, $End\,(X, R_2, R_3)$ is isomorphic to M_2 and $End\,(X, R_1, R_2, R_3)$ is isomorphic to M_1,

where (X, R_3) is a directed connected graph without loops (i.e. $R_3 \subseteq X \times X$, never $(x,x) \in R_3$ and, for every $x, y \in X$, there exist $x_0 = x, x_1, \ldots, x_n = y$ in X such that $(x_i, x_{i+1}) \in R_3 \cup R_3^{-1}$ for all $i = 0, \ldots, n-1$), $R_1 \subseteq R_2 \subseteq R_3$ and End denotes the corresponding monoid of endomorphisms (i.e. f is in $End\,(X, R_1, R_2, R_3)$ if and only if it is a map $X \to X$ such that, for $i = 1, 2, 3$, $(x, y) \in R_i$ implies $(f(x), f(y)) \in R_i$; analogously $End\,(X, R_2, R_3)$ and $End\,(X, R_3)$).

Then, using the idea of de Groot [**5**], an "arrow construction" is performed as follows: each arrow $r \in R_3$ of $G = (X, R_1, R_2, R_3)$ is replaced by a suitable space Q^r, in which three distinguished points t_1^r, t_2^r, t_3^r are specified, in the following way: in the coproduct (= disjoint sum) of the system $\{Q^r | r \in R_3\}$ we glue together the points t_i^r with $t_j^{r'}$, $i, j \in \{1, 2\}$, if and only if $r = (x_1, x_2)$, $r' = (x_1', x_2')$ and $x_i = x_j'$; moreover, we glue together all the points t_3^r for all $r \in R_3$. In our construction, all the spaces Q^r will be metric spaces (with a metric ρ^r) of diameter ≤ 1, $\rho^r(t_i, t_j) = 1$ for $i, j = \{1, 2, 3\}$, $i \neq j$, and the above coproduct and the gluing are performed in the category **Metr** of all metric spaces of diameter ≤ 1 and all nonexpanding maps (for a more detailed description of this arrow construction, see e.g. any of [**12, 17, 20**]). The metric space obtained will be denoted $T(G)$. Then, for every $r \in R_3$, we also have the isometric embedding

$$e^{(r)}\colon Q^r \to T(G)$$

which sends Q^r "identically" onto its copy in T(G). In III.2–III.3 below, we outline the construction of the spaces Q^r, $r \in R_3$, such that the space $X = T(G)$ has already all the properties required of it by Theorem 1.

III.2. The triangle construction, a basic stone in the construction of our spaces Q^r, begins with a countably infinite set $\mathcal{A}$ of pairwise-disjoint nondegenerate subcontinua of a Cook continuum $\mathcal{C}$ (we recall that a Cook continuum $\mathcal{C}$ is a nonempty metric continuum such that, for every subcontinuum K and every continuous map $f\colon K \to \mathcal{C}$, either f is constant or $f(x) = x$ for all $x \in K$; such a continuum was constructed by H. Cook in [**4**]; for a more detailed description see also [**12**]). We index them by elements of the set

$$Z = \{1, 2, 3\} \cup \big[\, N \times \{1, 2, 3\} \times \{1, 2, 3\} \,\big]$$

where N denotes the set of all positive integers, i.e. $\mathcal{A} = \{A_z \mid z \in Z\}$. For each A_z, we multiply its metric inherited form $\mathcal{C}$ by a suitable positive coefficient to get a metric space (denoted by A_z again) such that

$\operatorname{diam} A_i = \frac{1}{2}$ for $i \in \{1, 2, 3\}$,
$\operatorname{diam} A_{n,i,j} = \frac{1}{2^{n+1}}$ for all $n \in N$ and $i, j \in \{1, 2, 3\}$.

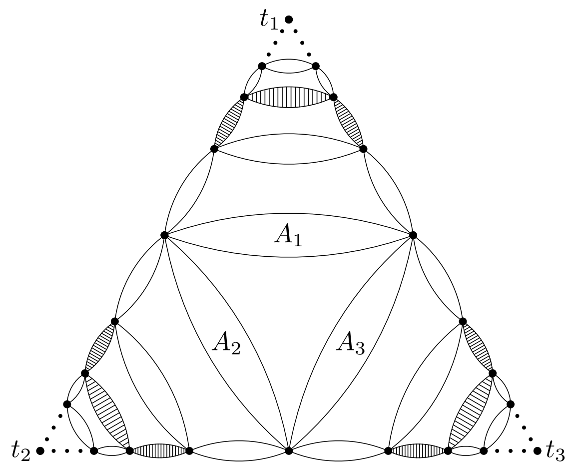
In each A_z, we choose points a_z and b_z whose distance equals to the diameter of A_z. In the coproduct of $\mathcal{A}$, we glue the metric spaces (with gluing points a_z and b_z) as visualized in the picture.

The coproduct and the gluing are done in the category Metr again. Finally, we form a completion T by adding the three points t_1, t_2, t_3. [The nine spaces $A_{2,i,j}$ (with $i, j \in \{1, 2, 3\}$) of the diameter $\frac{1}{8}$ are visualized by shading.] Clearly, $\operatorname{diam} T = 1$ and 1 is also the distance of t_i and t_j with $i \neq j$.

For each $z \in Z$, let us denote by

$$e_z\colon A_z \to T$$

the isometric "identical" embedding of A_z onto its copy in T. For subsequent use, let us denote by S (= the skelet of T) the subset of T consisting of t_1, t_2, t_3 and



all the gluing points, i.e. $e_z(a_z), e_z(b_z), z \in Z$. Clearly, S is totally disconnected and every pair of distinct $e_z(A_z), e_{\acute{z}}(A_{\acute{z}})$ can be "inserted in a circle", i.e.

$$(c) \begin{cases} \text{for } z, z' \text{ distinct there exist } z_1, \ldots, z_n \text{ in } Z \\ \text{such that } z = z_1,\ z' = z_k \text{ for some } k \le n \text{ and} \\ \text{card } (e_{z_i}(A_{z_i}) \cap e_{z_j}(A_{z_j}) = \begin{cases} 1 \text{ if } \{i, j\} = \{1, n\} \text{ or } |i - j| = 1 \\ 0 \text{ else.} \end{cases} \end{cases}$$

For a more detailed description of the construction see [**12**], [**17**].

III.3. Now, we construct already our space Q^r, $r \in R_3$. We start from two copies T_1 and T_2 of the triangle space constructed as in III.2 (everything is denoted as in III.2, and only the indices 1 and 2 are added, e.g. their distinguished points are $t_{1,j}, t_{2,j}, t_{3,j}$, $j = 1, 2$), but T_j is constructed from a set $\mathcal{A}_j = \{A_{z,j} \mid z \in Z\}$ where

$$\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$$

is a system of pairwise disjoint non-degenerate subcontinua of the Cook continuum. In the coproduct of T_1 and T_2, we glue together $t_{1,1}$ with $t_{1,2}$ (and we denote the resulting point simply by t_1) and $t_{2,1}$ with $t_{2,2}$ (and we denote the resulting point by t_2). Let us denote by $\tilde{Q}$ with a metric ρ the resulting space. To destroy its compactness, we remove the point $t_{3,2}$ from $\tilde{Q}$, and denote the resulting space by Q (the point $t_{3,1}$ is denoted simply by t_3).

Let us denote by

$$e_{z,j} \colon A_{z,j} \to Q$$

the "identical" embedding of $A_{z,j}$ onto its copy in Q. The **skelet** $\mathbb{S}$ of Q is

$$\{t_1, t_2, t_3\} \cup \{e_{z,j}(a_{z,j}),\, e_{z,j}(b_{z,j}) \,|\, z \in Z\,,\ j = 1, 2\}.$$

We define three metrics $\rho^{(1)}$, $\rho^{(2)}$, $\rho^{(3)}$ on Q by

$$\begin{aligned}\rho^{(1)}(x,y) &= \min\{1,\ \rho(x,y)\}\\ \rho^{(2)}(x,y) &= \min\{1,\ 2\rho(x,y)\}\\ \rho^{(3)}(x,y) &= \min\{1,\ 2\rho(x,y)+v(x,y)\}\end{aligned}$$

where

$$v(x,y)=\left|\frac{1}{\rho(x,t_{3,2})}-\frac{1}{\rho(y,t_{3,2})}\right|.$$

For our trigraph $G=(X,R_1,R_2,R_3)$, we put

$$\begin{aligned}Q^r &= (Q,\rho^{(1)}) \text{ for } r\in R_1,\\ Q^r &= (Q,\rho^{(2)}) \text{ for } r\in R_2\setminus R_1,\\ Q^r &= (Q,\rho^{(3)}) \text{ for } r\in R_3\setminus R_2.\end{aligned}$$

By means of **these** Q^r, we create $T(G)$ as described in III.1. The proof that $X=T(G)$ is coconnected is given in Lemmas III.4–8 below. $\big[$However, Lemmas III.4–6 also easily imply that X represents the given three monoids $End(X,R_1,R_2,R_3)\subseteq End(X,R_2,R_3)\subseteq End(X,R_3)$.$\big]$

III.4. Lemma. *Let $g\colon A_{z,j}\to T(G)$ be a nonconstant continuous map. Then there exists a unique $r\in R_3$ such that g is the composite*

$$A_{z,j}\xrightarrow{e_{z,j}} Q\xrightarrow{e^{(r)}} T(G).$$

Proof. Let us denote by $\mathbb{S}_G$ the union of all $e^{(r)}(\mathbb{S})$, $r\in R_3$, where $\mathbb{S}$ is the skelet of Q, and by $\mathbb{A}$ the union of all copies of $A_{z,j}$ in $T(G)$, i.e.

$$\mathbb{A}=\bigcup\big\{e^{(r)}(e_{z,j}(A_{z,j}))\,|\,r\in R_3\big\}\ .$$

We discuss the following possibilities:

1) $g(A_{z,j})$ **intersects** $T(G)\setminus(\mathbb{S}_G\cup\mathbb{A})$: hence there exists (z',j') distinct from (z,j) and $r\in R_3$ such that $g(A_{z,j})\cap G\neq\varnothing$ where $G=e^{(r)}(e_{z',j'}(A_{z',j'})\setminus\mathbb{S})$. Since G is open in $T(G), g^{-1}(G)$ is open subset of $A_{z,j}$. If $A_{z,j}\setminus g^{-1}(G)=\varnothing$, then g sends $A_{z,j}$ into the copy of $A_{z',j'}$ in $T(G)$, which is impossible because, for $(z,j)\neq(z',j')$, $A_{z,j}$ and $A_{z',j'}$ are disjoint subcontinua of the Cook continuum $\mathcal{C}$ so that g must be constant. Thus $A_{z,j}\setminus g^{-1}(G)$ must be non-empty. Then the closure $\overline{C}$ of the component C in $g^{-1}(G)$ of a point $x\in g^{-1}(G)$ intersects the boundary of $g^{-1}(G)$ (see e.g. [**8**] for this well-known fact), and hence $\overline{C}$ is a non-degenerate subcontinuum of $A_{z,j}$. But g sends it into $\overline{G}$, i.e. into the copy of $A_{z',j'}$ in $T(G)$ and g cannot be constant on $\overline{C}$ because $g(\overline{C})$ intersects the boundary

of G. However a non-degenerate subcontinuum of $A_{z,j}$ can be mapped into $A_{z',j'}$ only onto a point. We conclude that the case 1) cannot occur.

2) $g(A_{z,j}) \subset (\mathbb{S}_G \cup \mathbb{A})$: The skelet $\mathbb{S}$ is totally disconnected and so is $\mathbb{S}_G$. Since $g(A_{z,j})$ is a non-degenerate continuum, there exists precisely one $r \in R_3$ such that $g(A_{z,j}) \subseteq e^{(r)}(e_{z,j}(A_{z,j}))$. Then, by the properties of $\mathcal{C}$ again, necessarily $g = e^{(r)} \circ e_{z,j}$. □

III.5. Lemma. *Let $f\colon Q \to T(G)$ be a continuous map. For every (z,j), $j = 1, 2$, $z \in Z$, we denote*

$$g_{z,j}\colon A_{z,j} \xrightarrow{e_{z,j}} Q \xrightarrow{f} T(G).$$

Then

a) *if there exists (z,j) such that $g_{z,j}$ is constant, then f is constant;*
b) *if there exists (z,j) such that $g_{z,j}$ is nonconstant, then there exists a unique $r \in R_3$ such that $f = e^{(r)}$.*

Proof. By Lemma III.4, each $g_{z,j}$ is either a constant or $e^{(r)} \circ e_{z,j}$ for some (unique) $r \in R_3$.

1) Let us suppose that $g_{y,1}$ is constant for some $y \in Z$, with a value c. Then, by (c) in III.2, $g_{z,1}$ must be constant with the same value c for all $z \in Z$, hence f is constant on the closure of $\bigcup_{z\in Z} e_{z,1}(A_{z,1})$. In particular, $f(t_1) = f(t_2)$.

2) Let us suppose that there exists $y \in Z$ such that $g_{y,2}$ is nonconstant. Then $g_{z,2}$ is nonconstant for all $z \in Z$. It follows from 1) if we interchange 1 and 2 in it. Hence, by III.4, for every $(z,2)$ there exists a unique r such that $g_{z,2} = e^{(r)} \circ e_{z,2}$. The r could depend on $(z,2)$. In fact, it does not depend on $(z,2)$. This follows from (c) in III.2 again and from the fact that

$$e^{(r)}(Q \setminus \{t_1, t_2, t_3\}) \cap e^{(r')}(Q \setminus \{t_1, t_2, t_3\}) = \varnothing$$

whenever $r \neq r'$. We conclude that f is equal to $e^{(r)}$ on the closure of $\bigcup_{z\in Z} e_{z,2}(A_{z,2})$, particularly $f(t_1) \neq f(t_2)$.

3) By 1) and 2) (possibly interchanging 1 and 2), if some $g_{z,j}$ is constant, then all the $g_{z,j}$'s must be constant, and hence f must be constant. Otherwise there exist $r_1, r_2 \in R_3$ such that

$$f \text{ equals to } e^{(r_j)} \text{ on the closure of } \textstyle\bigcup_{z\in Z} e_{z,j}(A_{z,j}),\ j = 1, 2.$$

Consequently $e^{(r_1)}(t_1) = e^{(r_2)}(t_1)$ and $e^{(r_1)}(t_2) = e^{(r_2)}(t_2)$. This implies $r_1 = r_2$, and hence f equals $e^{(r_1)}$ on the whole Q. □

III.6. Lemma A. *Let* $f\colon (Q,\varrho^{(j)}) \to T(G)$ *be a nonconstant continuous map. Then there exists a unique* $r \in R_3$ *such that* $f = e^{(r)}$. $\big[$*Moreover, if* $j \in \{1,2\}$ *and* f *is uniformly continuous (or* $j=1$ *and* f *is nonexpanding) then* $r \in R_2$ *(or* $r \in R_1$*).*$\big]$

Proof. This follows immediately from III.5. □

Lemma B. *Let* $f\colon T(G) \to T(G)$ *be a continuous map. Then*

a) *if* $f \circ e^{(r)}$ *is constant for some* $r \in R_3$*, then* f *is constant;*
b) *if all the* $f \circ e^{(r)}$*'s are nonconstant, then there exists a* $g \in End(X,R_3)$ *such that* $f \circ e^{(r)} = e^{(\overline{r})}$ *where* $\overline{r} = [g \times g]\,(r)$ *for all* $r \in R_3$*.*

Proof. By III.5, each $f \circ e^{(r)}\colon Q \to T(G)$ is either constant or it equals to some $e^{(\tilde r)}$ for a unique $\tilde r \in R_3$.

a) If some $f \circ e^{(r)}$ is constant, then it glues together all the three points t_1, t_2, t_3; hence $f \circ e^{(r')}$ must be also constant for any arrow $r' \in R_3$ adjacent to r because, in this case, $e^{(r)}(Q)$ and $e^{(r')}(Q)$ have two distinct points in common (see III.1), namely $e^{(r)}(t_3)$ and either $e^{(r)}(t_1)$ or $e^{(r)}(t_2)$. Since (X,R_3) is connected, every $r' \in R_3$ can be reached by a finite chain of adjacent arrows. Thus f must be constant.

b) If all the maps $f \circ e^{(r)}$ are nonconstant, then, by Lemma A, for every $r \in R_3$ there exists $\tilde r$ such that

$$f \circ e^{(r)} = e^{(\tilde r)}.$$

However, if the arrows r and r' have a vertex in common, so do $\tilde r$ and $\tilde r'$. Hence there exists $g \in End(X,R_3)$ such that, for all $r = (x,y) \in R_3$, the arrow $\tilde r$ is precisely $(g(x), g(y))$. □

Remark. Lemma B implies immediately that the nonconstant continuous maps $f\colon T(G) \to T(G)$ are in one-to-one correspondence with elements g of $End(X,R_3)$. And the definition of the three metrics $\varrho^{(1)}, \varrho^{(2)}, \varrho^{(3)}$ in III.3 implies immediately that f is uniformly continuous or nonexpanding if and only if $g \in End(X,R_2,R_3)$ or $g \in End(X,R_1,R_2,R_3)$, respectively.

III.7. For $Q \times Q$, we denote by π_1 and π_2 the first and the second projection.

Lemma. *Let* $h : Q \times Q \to T(G)$ *be a continuous map. Then there exist unique* $r \in R_3$ *and* $s \in \{1,2\}$ *such that*

$$h = e^{(r)} \circ \pi_s.$$

Proof. Let us discuss the following cases:

1) **There exists** $x_0 \in Q \setminus \{t_1, t_2, t_3\}$ **such that**

$$h(x_0, -) : Q \to T(G)$$

is nonconstant. Then, by III.5, $h(x_0, -) = e^{(r_0)}$ for a unique $r_0 \in R_3$. We prove that in this case, $h(x, -) = e^{(r_0)}$ for all $x \in Q$ which implies $h = e^{(r_0)} \circ \pi_2$. Let us suppose the contrary, so let us suppose that there exists $x_1 \in Q$ such that $h(x_0, -) \neq h(x_1, -)$. Since h is continuous and $Q \setminus \{t_1, t_2, t_3\}$ is dense in Q, we may suppose that $x_1 \in Q \setminus \{t_1, t_2, t_3\}$. The following two cases have to be investigated.

1,1) $h(x_1, -)$ **is nonconstant:** then, by III.5 again, $h(x_1, -) = e^{(r_1)}$. Since $h(x_0, -) \neq h(x_1, -)$, necessarily $r_0 \neq r_1$. Choose $y \in Q \setminus \{t_1, t_2, t_3\}$ and put $g = h(-, y)$. Then $g(x_0) = h(x_0, y) = e^{(r_0)}(y)$ and $g(x_1) = e^{(r_1)}(y)$, so that $g(Q)$ intersects both $e^{(r_0)}(Q \setminus \{t_1, t_2, t_3\})$ and $e^{(r_1)}(Q \setminus \{t_1, t_2, t_3\})$; hence it is nonconstant, and hence it equals some $e^{(r)}$. Then $e^{(r)}(x_0) = g(x_0) = h(x_0, y) = e^{(r_0)}(y)$ and, analogously, $e^{(r)}(x_1) = e^{(r_1)}(y)$ so that $e^{(r)}(Q \setminus \{t_1, t_2, t_3\})$ intersects both $e^{(r_0)}(Q \setminus \{t_1, t_2, t_3\})$ and $e^{(r_1)}(Q \setminus \{t_1, t_2, t_3\})$, which is impossible.

1,2) $h(x_1, -)$ **is constant**, denote by c its value. For each $y \in Q$, denote $g_y = h(-, y)$. Then $g_y(x_1) = c$ and $g_y(x_0) = e^{(r_0)}(y)$. Since $e^{(r_0)}$ is one-to-one, $e^{(r_0)}(y) \neq c$ for all $y \in Q$ with possibly one exception, say y_0. For these y, g_y is nonconstant, so equal to some $e^{(r_y)}$. Since $e^{(r_y)}(x_1) = c = e^{(r_{y'})}(x_1)$ and $x_1 \in Q \setminus \{t_1, t_2, t_3\}$, necessarily $r_y = r_{y'}$ (by III.3) for all $y, y' \in Q \setminus \{y_0\}$, hence $g_y = g_{y'}$. But $g_y(x_0) = e^{(r_0)}(y) \neq e^{(r_0)}(y') = g_{y'}(x_0)$ for $y, y' \in Q \setminus \{y_0\}$, $y \neq y'$ which is a contradiction.

2) There exists $y_0 \in Q \setminus \{t_1, t_2, t_3\}$ such that $h(-, y_0)$ is nonconstant. We proceed as in 1) interchanging the coordinates only. We get that $h = e^{(r_0)} \circ \pi_1$ for a unique $r_0 \in R_3$.

3) **The cases sub 1) or 2) do not occur**, i.e. for each $x, y \in Q \setminus \{t_1, t_2, t_3\}$, $h(x, -)$ and $h(-, y)$ are constant. Then h is constant on the dense subspace $(Q \setminus \{t_1, t_2, t_3\}) \times (Q \setminus \{t_1, t_2, t_3\})$, and hence on the whole Q. □

III.8. Lemma. *Let $f : T(G) \times T(G) \to T(G)$ be a continuous map. Then f factors through a projection.*

Proof. Let $\pi_1, \pi_2 : T(G) \times T(G) \to T(G)$ be the first and the second projection. For every $r \in R_3$, we also denote by Q_r the subspace $e^{(r)}(Q)$ of $T(G)$. We discuss how f can look like in the following situations:

1) **There exist** r_1, r_2 **such that** $f/Q_{r_1} \times Q_{r_2}$ **is constant.** Then the map $f(x, -) : T(G) \to T(G)$ is constant on Q_{r_2} for each $x \in Q_{r_1}$, hence it is constant on the whole $T(G)$, by III.6. Hence f is constant on $Q_{r_1} \times T(G)$ so that

$f(-,y)\colon T(G)\to T(G)$ is constant on Q_{r_1} for each $y\in T(G)$, hence on the whole $T(G)$. We conclude that f is constant on the whole $T(G)\times T(G)$.

2) **The restriction of f to any $Q_{r_1}\times Q_{r_2}$ is nonconstant.** Choose $\tilde r_1$, $\tilde r_2\in R_3$. By III.7, $f/Q_{\tilde r_1}\times Q_{\tilde r_2}$ has the form $e^{(\tilde r)}\circ\pi_s$ for some $s\in\{1,2\}$, $\tilde r\in R_3$. We may suppose $s=1$, i.e. $f(x,-)\colon T(G)\to T(G)$ is constant on $Q_{\tilde r_2}$ hence on the whole $T(G)$, for each $x\in Q_{\tilde r_1}$. Hence f restricted to any $f/Q_{\tilde r_1}\times Q_{r_2}$ is equal to $e^{(\tilde r)}\circ\pi_1$. This implies that f restricted to any $Q_{r_1}\times Q_{r_2}$ must factorize through π_1. Let us suppose the contrary. Since $f/Q_{r_1}\times Q_{r_2}$ is always nonconstant, there exist r_1, r_2 such that $f/Q_{r_1}\times Q_{r_2}$ factorizes through π_2. Then for every $y\in Q_{r_2}$, $f(-,y)$ is constant on Q_{r_1} hence on the whole $T(G)$, by III.5 again, hence also on $Q_{\tilde r_1}\times Q_{r_2}$ which is a contradiction. □

Corollary. *$T(G)$ is a coconnected space.*

Remark. When inspecting the proofs of the Lemmas III.4–III.8, one can see that the space $X = T(G)$ has the following stronger property: every **separately** continuous map $f\colon X\times X\to X$ factors through a product projection. Such spaces are called SCFO-unary (an abbreviation of: **S**eparately **C**ontinuous **F**initary **O**perations are essentially unary) in [**20**]. The statements about SCFO-unary spaces, mentioned in [**20**], also follow from III.4–8 of the present paper. We have only to start from more general representation statements than merely the representation of the monoids $M_1\subseteq M_2\subseteq M_3$ as $End(X,R_1,R_2,R_3)\subseteq End(X,R_2,R_3)\subseteq End(X,R_3)$ (i.e. to use the Lemma 2.4 in [**16**] in its full generality).

Acknowledgement. The author would like to thank E. Murtinová and J. Sichler for their comments on an earlier version of this paper.

References

1. Archangelskij A. V. and Wiegandt R., *Connectedness and disconnectedness*, Gen. Top. Appl. **5** (1975), 9–33.
2. Barkhudaryan A., *On a characterization of unit interval in terms of clones*, Comment. Math. Univ. Carolin. **40** (1999), 153–164.
3. Cohn P. M., *Universal Algebra*, Harper and Row, New York, 1965.
4. Cook H., *Continua which admit only the identity mapping onto non-degenerate subcontinua*, Fund. Math. **60** (1967), 241–249.
5. de Groot J., *Groups represented by homeomorphism groups I*, Math. Annalen **138** (1959), 80–102.
6. Herrlich H., *On the concept of reflections in general topology*, Proc. Symp. on Extension Theory of Topological structures, Berlin, 1967.
7. Herrlich H., *Topologische Reflexionen und Coreflexionen*, Springer–Verlag Berlin – Heidelberg – New York, Lecture Notes in Math. **78** (1968).
8. Kuratowski C., *Topologie I, II*, Monografie Matematyczne, Warszaw, 1950.
9. Lawvere F. W., *Functorial semantic of algebraic theories*, Proc. Nat. Acad. Sci. U.S.A. **50** (1963), 869–872.

10. Lawvere F. W., *Some algebraic problems in context of functorial semantics of algebraic theories*, Springer – Verlag Berlin – Heidelberg – New York, Lecture Notes in Math. **61** (1968), 41–46.
11. McKenzie R. N., McNulty G. F. and Taylor W. F., *Algebras, Lattices, Varieties*, Vol. 1, Brooks/Cole, Monterey, California, 1978.
12. Pultr A. and Trnková V., *Combinatorial, Algebraic and Topological Representations of Groups, Semigroups and Categories*, North Holland and Academia, Praha, 1980.
13. Sichler J. and Trnková V., *Clones in topology and algebra*, Acta Math. Univ. Comenianae **66** (1997), 243–260.
14. ______, *Representations of algebraic theories by continuous maps*, J. Austral. Math. Soc. (Series A) **66** (1999), 255–286.
15. Taylor W., *The Clone of a Topological Space*, Research and Exposition Math., Vol. 13, Helderman Verlag, 1986.
16. Trnková V., *Simultaneous representations in discrete structures*, Comment. Math. Univ. Carolin. **27** (1986), 633–649.
17. ______, *Simultaneous representations by metric spaces*, Cahiers Topologie Géom. Différentielle **29** (1988), 217–239.
18. ______, *Semirigid spaces*, Trans. Amer. Math. Soc. **343** (1994), 305–329.
19. ______, *Co-connected spaces*, Serdica Math. J. **24** (1998), 25–36.
20. ______, *Amazingly extensive use of Cook continuum*, Math. Japonica **51** (2000), 499–549.

V. Trnková, Math. Institute of Charles University 186 75 Praha 8, Sokolovská 83, Czech Republic;
e-mail: trnkova@karlin.mff.cuni.cz