Acta Math. Univ. Comenianae
Vol. LXIX, 2(2000), pp. 137–144

ON LARGE RANDOM ALMOST EUCLIDEAN BASES

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Abstract. A new class of random proportional embeddings of l_2^n into certain Banach spaces is found. Let $(\xi_i)_{i=1}^n$ be i.i.d. mean zero Cramèr random variables. Suppose $(x_i)_{i=1}^n$ is a sequence in the unit ball of a Banach space with $\mathbf{E}\|\sum_i \varepsilon_i x_i\| \geq \delta n$. Then the system of $\lceil cn \rceil$ independent random vectors distributed as $\sum_i \xi_i x_i$ is well equivalent to the euclidean basis with high probability (c depends on ξ_1 and δ). A connection with combinatorial discrepancy theory is presented.

1. Sign Embeddings And Short Films

G. Schechtman proved that in l_1^n a certain random choise of cn vectors is well equivalent to the euclidean basis ([**Sch1**], see also [**M-S**, 7.15]). More precisely, by ε_i we denote the Rademacher random variables, i.e. independent random variables taking values -1 and 1 with probability 1/2, by e_i the canonical vectors in $\mathbf{R}^n$, and by $c_1, c_2, \dots$ absolute constants. A system $(z_i)_{i=1}^k$ of vectors in a Banach space is said to be **c-equivalent to the euclidean basis** if there is a linear operator $T\colon \operatorname{span}(z_i) \to l_2^k$ sending each z_i to e_i, with $\|T\|\|T^{-1}\| \leq c$. Then Schechtman's theorem says the following. Every system of $\lceil c_1 n \rceil$ independent random vectors in l_1^n distributed as $\sum_{j=1}^n \varepsilon_j e_j$ is c_2-equivalent to the euclidean basis with probability $\geq 1 - \exp(-c_3 n)$.

This result is generalized here in two directions. Instead of the canonical vector basis of l_1^n, we work with arbitrary sequence $(x_j)_{j=1}^n$ of vectors in the unit ball $B(X)$ of a Banach space X satisfying

$$\mathbf{E}\Big\|\sum_{j=1}^n \varepsilon_j x_j\Big\| \geq \delta n \tag{1}$$

for some $\delta > 0$. This estimate is known as the **random δ-sign embedding from l_1^n** condition [**F-J-S**]. In [**Sch1**] it was considered in spaces with a good cotype; our proof does not require cotype restrictions.

Moreover, instead of the Bernoullian distribution of each coordinate, we consider arbitrary distribution of a mean zero r.v. ξ having moment generating function, that is $\mathbf{E}e^{\alpha|\xi|} < \infty$ for some $\alpha > 0$. This is called the **Cramèr condition**,

Received December 2, 1998; revised December 20, 2000.
1980 *Mathematics Subject Classification* (1991 *Revision*). Primary 46B09, 05B20, 41A28.

and is equivalent to the following: there are constants a, $\alpha > 0$ so that

$$\mathbf{P}\{|\xi| > t\} \le a e^{-\alpha t} \quad \text{for all } t \tag{2}$$

(see [**P**, Lemma III.5]).

Theorem 1.1. *Let $(\xi_j)_{j=1}^n$ be independent copies of a mean zero r.v. ξ satisfying* (2); *set $\alpha_1 = \mathbf{E}|\xi|$. Suppose $(x_j)_{j=1}^n$ is a sequence in $B(X)$ satisfying* (1), *and set $s = \sqrt{a}/\alpha_1\alpha\delta$. There is a $c = c(s) > 0$ so that the system of $\lceil cn \rceil$ independent random vectors distributed as $\sum_{j=1}^n \xi_j x_j$ is $(c_1 s)$-equivalent to the euclidean basis with probability $\ge 1 - 2\exp(-c_2 s^{-2} n)$.*

Remarks. 1. One can set $c(s) = c_2/s^2 \log(c_1 s)$. We see that Theorem 1.1 is controlled by the only parameter s.

2. Actually, we prove that the operator T realizing the equivalence satisfies $\|T\| \le c_3(\alpha_1 \delta n)^{-1}$, and $\|T^{-1}\| \le c_4 \sqrt{a} \alpha^{-1} n$.

J. Elton [**E**] proved that (1) yields the existence of a subset $A \subset \{1, \dots, n\}$, $|A| \ge c(\delta) n$, such that the sequence $(x_j)_{j \in A}$ is $c'(\delta)$-equivalent to the canonical vector basis of $l_1^{|A|}$. If combined with Schechtman's theorem, this gives another form of proportional euclidean sections of X (however, with a worse dependence on δ: $c(\delta) \sim \delta^2/\log^2(4/\delta)$, $c'(\delta) \sim \delta^{-3}$).

For convenience, we restricted ourselves to identically distributed random variables ξ, but the main result can easily be modified to handle the case when ξ have different distributions.

Theorem 1.1 admits an immediate application to random matrices. The next corolary says that the unit cube in $\mathbf{R}^n$ under the action of a random $k \times n$ matrix (with k proportional to n) is close to the euclidean ball B_2^k. We denote the unit euclidean ball in $\mathbf{R}^k$ by B_2^k.

Corollary 1.2. *Suppose ξ is a random variable satisfying* (1), *then there exist $c, \mu, \nu > 0$ such that we have the following. Let A be the $k \times n$ matrix whose entries are independent random variables distributed as ξ. If $k \le cn$ then with probability $\ge 1 - 2\exp(-cn)$*

$$\mu B_2^k \subset n^{-1} A([-1,1]^n) \subset \nu B_2^k.$$

Proof. Pass to the dual setting and apply Theorem 1.1 together with Remark 2. □

Now we discuss a relation between almost euclidean bases in l_1^n and combinatorial discrepancies. Given a two-coloring χ, say White and Black, of a finite set Ω, the **discrepancy** $\operatorname{disc}(A, \chi)$ of a set $A \subset \Omega$ is the number of White points in A minus the number of Black points in A (cf. [**A-S**], [**B-S**]). A family $\overline{\chi} = (\chi_j)_{j=1}^n$ of two-colorings on Ω is called a **film** of length n. We define the **film discrepancy** $\operatorname{fdisc}(A, \overline{\chi})$ of a set $A \subset \Omega$ as the average $\frac{1}{n}\sum_{j=1}^n |\operatorname{disc}(A, \chi_j)|$.

The problem is to make a short **homogeneous** film, so that the film discrepancies of any two sets $A, B \subset \Omega$ of equal size be nearly the same: $\operatorname{fdisc}(A, \overline{\chi}) \approx \operatorname{fdisc}(B, \overline{\chi})$ (the relation $x \approx y$ means $c_1 x \le y \le c_2 x$ for some absolute constants $c_1, c_2 > 0$). Since nobody wants to watch a monochromatic film, we require it to be **balanced**, that is the density of each shot be nontrivial: $|\operatorname{disc}(\Omega, \chi_j)| \le (1-c_3)|\Omega|$ for all j and some absolute constant $c_3 > 0$.

One might think that balanced homogeneous films must be fairly long comparing with $|\Omega|$, but this is unjustified.

Theorem 1.3. *Given a finite set Ω, there is a balanced homogeneous film on Ω of length $c_1|\Omega|$.*

Proof. We begin with a geometrical interpretation of the problem, as in [**Sp**]. Let $k = |\Omega|$. A coloring χ on Ω is regarded as a sequence $(\varepsilon_i) \in \{-1, 1\}^k$, assigning 1 to White and -1 to Black. A set $A \subset \Omega$ is identified with its incidence vector $(a_i) \in \{0, 1\}^k$. Then $\operatorname{disc}(A, \chi) = \sum_{i=1}^k \varepsilon_i a_i$.

Now we clarify a relation to Schechtman's result, that is Theorem 1.1 with $\xi_j = \varepsilon_j$ and $(x_j) =$ the canonical vectors in $X = l_1^n$. Let n be the minimal integer such that $\lceil cn \rceil \ge k$. In this case we get with probability $\ge 1 - 2\exp(-c_2 n)$

$$\frac{1}{n}\sum_{j=1}^{n}\Big|\sum_{i=1}^{k} a_i \varepsilon_{ij}\Big| \approx \Big(\sum_{i=1}^{k} |a_i|^2\Big)^{1/2} \quad \text{for all scalars } (a_i), \tag{3}$$

where ε_{ij} are Rademacher random variables (see Remark 2 following Theorem 1.1). Let $\overline{\chi}$ be a random film of length n, so that $\chi_j = (\varepsilon_{ij})_{i=1}^k$. Then (3) yields that, with probability $\ge 1 - 2\exp(-c_2 n)$, every set $A \subset \Omega$ satisfies $\operatorname{fdisc}(A, \overline{\chi}) \approx \sqrt{|A|}$. Hence most films are homogeneous.

It suffices to show that most films are also balanced. Consider a random coloring $\chi = (\varepsilon_i)$ on Ω. Using a subgaussian tail estimate for Rademacher sums (see [**L**] or apply Theorem), we have

$$\mathbf{P}\Big\{|\operatorname{disc}(\Omega, \chi)| \le \frac{1}{2}|\Omega|\Big\} = \mathbf{P}\Big\{\Big|\sum_{i=1}^{k} \varepsilon_i\Big| \le k/2\Big\} \ge 1 - 2\exp(-k/8).$$

Then the probability that $|\operatorname{disc}(\Omega, \chi_j)| \le \frac{1}{2}|\Omega|$ for all $j = 1, \ldots, n$ is at least $1 - 2n\exp(-k/8)$. Since $n \le c_1 k$, this probability tends to 1 as $k \to \infty$. This completes the proof. $\square$

I do not know whether there are asymptotically shorter balanced homogeneous films.

To prove the main result, we will apply a deviation inequality for sums of independent Banach space valued random variables.

Theorem 1.4. *Let $X_1, \dots, X_n$ be independent Banach space valued random variables with $\mathbf{P}\{\|X_i\| > t\} \le ae^{-\alpha_i t}$ for all t and i. Let $d \ge \max_{i \le n} \alpha_i^{-1}$ and $b \ge a \sum_{i=1}^n \alpha_i^{-2}$. Then setting $S_n = \sum_{i=1}^n X_i$ we have*

$$\mathbf{P}\big\{\big|\|S_n\| - \mathbf{E}\|S_n\|\big| > t\big\} \le \begin{cases} 2\exp(-t^2/32b) & \text{for } 0 \le t \le 4b/d \\ 2\exp(-t/8d) & \text{for } t \ge 4b/d. \end{cases}$$

This result can be derived by truncation from known deviation inequalities for sums of bounded random variables (see e.g. [**Le-Ta**, Section 6.2]). However, it is more convenient and more instructive to give a direct proof based on martingales, as in [**Yu**, Sec. 3.3]. A rather short instructive proof is given in §2.

§3 consists of the proof of Theorem 1.1.

This work was supported by C.N.R. (Italy). I am grateful to P. Terenzi and to V. Kadets for discussions.

2. Deviations of Sums

In this section we prove Theorem 1.4.

First we recall that problems about Banach space valued independent random variables can often be reduced to a **real valued martingale** case, see [**Le-Ta**, Ch. 6.3].

Let $\mathcal{A}_i$ be the σ-algebra generated by the random variables $X_1, \dots, X_i$, $i \le n$, and $\mathcal{A}_0$ be the trivial σ-algebra. The conditional expectation with respect to $\mathcal{A}_i$ is denoted by $\mathbf{E}^{\mathcal{A}_i}$. Set, for each i, $d_i = \mathbf{E}^{\mathcal{A}_i}\|S_n\| - \mathbf{E}^{\mathcal{A}_{i-1}}\|S_n\|$. Then $(d_i)_{i=1}^n$ forms a real valued martingale difference sequence, and $\sum_{i=1}^n d_i = \|S_n\| - \mathbf{E}\|S_n\|$.

Lemma 2.1. *For every i and every $p \ge 1$*

$$\mathbf{E}^{\mathcal{A}_{i-1}}|d_i|^p \le 2^p \mathbf{E}\|X_i\|^p$$

almost surely.

Proof. Yurinskii's inequality states that $|d_i| \le \|X_i\| + \mathbf{E}\|X_i\|$ almost surely (see [**Le-Ta**, Lemma 6.16]). Then $|d_i|^p \le 2^{p-1}\big(\|X_i\|^p + (\mathbf{E}\|X_i\|)^p\big)$. Hence

$$\begin{aligned} \mathbf{E}^{\mathcal{A}_{i-1}}|d_i|^p &\le 2^{p-1}\big(\mathbf{E}^{\mathcal{A}_{i-1}}\|X_i\|^p + (\mathbf{E}\|X_i\|)^p\big) \\ &= 2^{p-1}\big(\mathbf{E}\|X_i\|^p + (\mathbf{E}\|X_i\|)^p\big) \le 2^p \mathbf{E}\|X_i\|^p, \end{aligned}$$

and we are done. □

Proof of Theorem 1.4. Apply Chebyshev's inequality. For every $\lambda \ge 0$

$$P := \mathbf{P}\Big\{\|S_n\| - \mathbf{E}\|S_n\| > t\Big\} = \mathbf{P}\Big\{\sum_{i=1}^n d_i > t\Big\} \le e^{-\lambda t}\mathbf{E}\exp\Big(\lambda \sum_{i=1}^n d_i\Big). \tag{4}$$

But

$$
\begin{aligned}
\mathbf{E}\exp(\lambda\sum_{i=1}^{n} d_i) &= \mathbf{E}\Big(\mathbf{E}^{\mathcal{A}_{n-1}}\exp(\lambda\sum_{i=1}^{n} d_i)\Big) = \mathbf{E}\Big(\exp(\lambda\sum_{i=1}^{n-1} d_i)\mathbf{E}^{\mathcal{A}_{n-1}}\exp(\lambda d_n)\Big) \\
&\le \|\mathbf{E}^{\mathcal{A}_{n-1}}\exp(\lambda d_n)\|_\infty \mathbf{E}\exp(\lambda\sum_{i=1}^{n-1} d_i) = \cdots \qquad (5)\\
&= \prod_{i=1}^{n}\|\mathbf{E}^{\mathcal{A}_{i-1}}\exp(\lambda d_i)\|_\infty .
\end{aligned}
$$

So we are to evaluate

$$
\begin{aligned}
\mathbf{E}^{\mathcal{A}_{i-1}}\exp(\lambda d_i) &= 1+\sum_{p=2}^{\infty}\frac{\lambda^p\mathbf{E}^{\mathcal{A}_{i-1}}d_i^p}{p!} \qquad (\text{since } \mathbf{E}^{\mathcal{A}_{i-1}}d_i=0)\\
&\le 1+\sum_{p=2}^{\infty}\frac{\lambda^p 2^p\mathbf{E}\|X_i\|^p}{p!} \qquad (\text{by Lemma 2.1}).
\end{aligned}
$$

Note that

$$
\mathbf{E}\|X_i\|^p = \int_0^\infty \mathbf{P}\{\|X_i\|>t\}\,dt^p \le \int_0^\infty ae^{-\alpha_i t}\,dt^p = a\alpha_i^{-p}p! \tag{6}
$$

Then for $0\le\lambda\le\alpha_i/4$

$$
\mathbf{E}^{\mathcal{A}_{i-1}}\exp(\lambda d_i) \le 1+a(2\lambda/\alpha_i)^2\sum_{p=2}^{\infty}(2\lambda/\alpha_i)^{p-2} \le 1+a(2\lambda/\alpha_i)^2 2 \le \exp(8\lambda^2 a\alpha_i^{-2}).
$$

Combining this estimate, (5), and (4), we obtain for $0\le\lambda\le 1/4d$

$$
P\le e^{-\lambda t}\prod_{i=1}^{n}\exp(8\lambda^2 a\alpha_i^{-2}) \le \exp(-\lambda t+8\lambda^2 b).
$$

The minimum here is attained for $\lambda = t/16b$. If $t\le 4b/d$, then the condition $\lambda\le 1/4d$ is satisfied, and $P\le\exp(-t^2/32b)$. If $t\ge 4b/d$, then we take $\lambda := 1/4d$, and get $P\le\exp(-t/8d)$.

Similarly, one obtains the same estimates on $\mathbf{P}\{\|S_n\|-\mathbf{E}\|S_n\|<-t\}$. □

3. Random Euclidean Embeddings

In this section Theorem 1.1 is proved.

We will use a simple symmetrization lemma, see [**Le-Ta**, Lemma 6.3].

Lemma 3.1. *For every finite sequence (X_i) of Banach space valued mean zero random variables*

$$\frac{1}{2}\mathbf{E}\Big\|\sum_i \varepsilon_i X_i\Big\| \le \mathbf{E}\Big\|\sum_i X_i\Big\| \le 2\mathbf{E}\Big\|\sum_i \varepsilon_i X_i\Big\|.$$

Next, we need a known generalization of the Khinchine inequality.

Proposition 3.2. *Let (ξ_i) be a sequence of real valued i.i.d. mean zero random variables. Then for every finite sequence of numbers (a_i)*

$$\frac{1}{2}A_p\|\xi_1\|_{\min(2,p)}\Big(\sum_i |a_i|^2\Big)^{1/2} \le \Big\|\sum_i a_i\xi_i\Big\|_p \le 2B_p\|\xi_1\|_{\max(2,p)}\Big(\sum_i |a_i|^2\Big)^{1/2},$$

where A_p and B_p are the constants from the classical Khinchine inequality.

Actually, we will use the following particular case of the inequality, and give a proof only for this case:

$$\frac{1}{2\sqrt{2}}\|\xi_1\|_1\Big(\sum_i |a_i|^2\Big)^{1/2} \le \Big\|\sum_i a_i\xi_i\Big\|_1 \le \|\xi_1\|_2\Big(\sum_i |a_i|^2\Big)^{1/2}. \tag{7}$$

Proof (sketch). To prove the left-hand side observe that, by Lemma 3.1, $\mathbf{E}|\sum_i a_i\xi_i|$ is nearly the same as $\mathbf{E}|\sum_i \varepsilon_i a_i\xi_i|$. Now it is enough to apply partial integration and use the classical Khinchine inequality (note that $A_1 = 1/\sqrt{2}$ [**Sz**]). Since $\|\sum_i a_i\xi_i\|_1 \le \|\sum_i a_i\xi_i\|_2$, the right-hand side of (7) follows from the orthogonality of (ξ_i) in $L_2(\Omega)$, due to the independentness. □

Another simple consequence of the symmetrization is this.

Lemma 3.3. *Let (η_i) be a finite sequence of real valued i.i.d. mean zero random variables. Then, for any sequence (x_i) in a Banach space,*

$$\mathbf{E}\Big\|\sum_i \eta_i x_i\Big\| \ge \frac{1}{2}\|\eta_1\|_1\mathbf{E}\Big\|\sum_i \varepsilon_i x_i\Big\|.$$

Proof. By the symmetry, $\varepsilon_i|\eta_i|$ has the same distribution as $\varepsilon_i\eta_i$. Using partial integration and the triangle inequality, we have

$$\mathbf{E}\Big\|\sum_i \varepsilon_i\eta_i x_i\Big\| = \mathbf{E}\Big\|\sum_i \varepsilon_i|\eta_i|x_i\Big\| \ge \mathbf{E}\Big\|\sum_i \varepsilon_i\|\eta_i\|_1 x_i\Big\|.$$

Now it is enough to apply Lemma 3.1 with $X_i = \eta_i x_i$. □

Finally, recall a standard approximation lemma (see [**M-S**, 4.1]).

Lemma 3.4. *Let X be a Banach space, and $F\colon X \to \mathbf{R}$ be a non-negative convex homogeneous function. Suppose for some θ-net $\mathcal{N}$ of $S(X)$ one has $a \le F(x) \le b$ for every $x \in \mathcal{N}$. Then*

$$a - \frac{\theta}{1-\theta} b \le F(x) \le \frac{1}{1-\theta} b$$

for every $x \in S(X)$.

In particular, if $\theta \le a/3b$, then $\frac{1}{2}a \le F(x) \le \frac{3}{2}b$ for every $x \in S(X)$.

Proof of Theorem 1.1. Let (ξ_{ij}) be independent copies of ξ. Let $k \le cn$. We are to show that the random vectors $y_i = n^{-1}\sum_{j=1}^n \xi_{ij} x_j$, $i = 1, \dots, k$, are well equivalent to the euclidean basis.

Fix $\overline{a} = (a_i)_{i=1}^k$ in the unit sphere $S(l_2^k)$. Consider a sequence of independent random variables

$$X_{ij} = n^{-1} a_i \xi_{ij} x_j, \quad i = 1, \dots, k, \quad j = 1, \dots, n,$$

and their sum $S(\overline{a}) = \sum_{i=1}^k \sum_{j=1}^n X_{ij}$. We will prove that, with high probability, $\|S(\overline{a})\|$ is bounded from above and below for every $\overline{a}$.

Theorem 1.4 applied to the sum of X_{ij} helps here. Note that $\mathbf{P}\{\|X_{ij}\| > t\} = \mathbf{P}\{n^{-1}|a_i||\xi| > t\} \le a \exp(-\alpha n |a_i|^{-1} t)$, thus we take

$$d = \alpha^{-1} n^{-1} \quad \text{and} \quad b = a \sum_{i=1}^k \sum_{j=1}^n \alpha^{-2} n^{-2} |a_i|^2 = a\alpha^{-2} n^{-1}.$$

Furthermore,

$$\begin{aligned}
\mathbf{E}\|S(\overline{a})\| &= n^{-1}\mathbf{E}\Big\| \sum_{j=1}^n \Big(\sum_{i=1}^k a_i \xi_{ij} \Big) x_j \Big\| \\
&\ge n^{-1} \frac{1}{2} \Big\| \sum_{i=1}^k a_i \xi_{ij} \Big\|_1 \mathbf{E} \Big\| \sum_{j=1}^n \varepsilon_j x_j \Big\| \quad \text{(by Lemma 3.3)} \\
&\ge \frac{1}{4\sqrt{2}} \alpha_1 \delta \quad \text{(by (7) and the condition on } (x_j)).
\end{aligned}$$

Conversely, let $\alpha_2 = \|\xi\|_2$. Note that $\alpha_2 \le \sqrt{2}\sqrt{a}\alpha^{-1}$, as in (6). Then by the triangle inequality and (7)

$$\mathbf{E}\|S(\overline{a})\| \le n^{-1} \sum_{j=1}^n \mathbf{E} \Big| \sum_{i=1}^k a_i \xi_{ij} \Big| \le \alpha_2 \le \sqrt{2}\sqrt{a}\alpha^{-1}.$$

Now set $t := \frac{1}{8\sqrt{2}}\alpha_1\delta \leq \frac{1}{2}\mathbf{E}\|S(\overline{a})\|$ and apply Theorem 1.4. Clearly, $t \leq 4b/d$ is the case, because $\delta \leq 1$ and $\alpha_1 \leq a\alpha^{-1}$ as in (6). Thus

$$\mathbf{P}\{d_1 \leq \|S(\overline{a})\| \leq d_2\} \geq 1 - 2\exp(-d_3 n),$$

where $d_1 = \frac{1}{8\sqrt{2}}\delta\alpha_1$, $d_2 = \frac{3}{2}\sqrt{2}\sqrt{a}\alpha^{-1}$, and $d_3 = c_4(\alpha\alpha_1\delta/\sqrt{a})^2 = c_4 t^{-2}$.

The preceding observations hold for **fixed** $\overline{a}$. Now let $\overline{a}$ run over a θ-net $\mathcal{N}$ in $S(l_2^k)$, $|\mathcal{N}| \leq \exp(k\log 3/\theta)$, where $\theta = d_1/3d_2$ (there is such a net, cf. [**M-S**, 2.6]). Then

$$\mathbf{P}\{\forall\overline{a} \in \mathcal{N}, \quad d_1 \leq \|S(\overline{a})\| \leq d_2\} \geq 1 - 2\exp(k\log 3/\theta - d_3 n).$$

We conclude by Lemma 3.4,

$$\mathbf{P}\{\forall\overline{a} \in S(l_2^k), \quad d_1/2 \leq \|S(\overline{a})\| \leq 3d_2/2\} \geq 1 - 2\exp(k\log 3/\theta - d_3 n).$$

If c was chosen small enough, then $k\log 3/\theta \leq \frac{1}{2}d_3 n$.

It remains to note that $d_2/d_1 = c_5 s$, and the required $(c_1 s)$-equivalence to the euclidean basis follows. $\square$

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