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HOMOGENEOUS ESTIMATES FOR OSCILLATORY INTEGRALS

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Abstract. Let $u(x,t)$ be the solution to the free time-dependent Schrödinger equation at the point (x,t) in space-time $\mathbf{R}^{n+1}$ with initial data f. We characterize the size of u in terms $L^p(L^q)$-estimates with power weights. Our bounds are given by norms of f in homogeneous Sobolev spaces $\dot{H}^s(\mathbf{R}^n)$.

Our methods include use of spherical harmonics, uniformity properties of Bessel functions and interpolation of vector valued weighted Lebesgue spaces.

1. Introduction

1.1. In this paper we shall assume that the space-dimension n is greater than or equal to 2. For a function f in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ we denote the Fourier transform by $\widehat{f}$. If u is the solution to the free time-dependent Schrödinger equation $\Delta_x u = i\partial_t u$ with initial data f, then

$$u(x,t) = \frac{1}{(2\pi)^n}\int_{\mathbf{R}^n} e^{i(x\xi+t|\xi|^2)}\,\widehat{f}(\xi)\,d\xi.$$

The Fourier transform is by duality extended to the space of tempered distributions $\mathcal{S}'(\mathbf{R}^n)$. Let the Sobolev space $H^s(\mathbf{R}^n)$ consist of all tempered distributions F such that the function $\xi \mapsto (1+|\xi|^2)^s|\widehat{F}(\xi)|^2$ is integrable. In Vega **[25]** the following is claimed: **For any b greater than 1 there exists a number C independent of f such that**

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u(x,t)|^2\,\frac{dtdx}{(1+|x|)^b} \le C\,\|f\|^2_{H^{-1+b/2}(\mathbf{R}^n)}. \tag{1.1}$$

See [**25**, Theorem 3, p. 874]. In particular, the claim applies to cases when b is greater than but arbitrarily close to 1. In S. L. Wang **[31]** there is for b greater than but arbitrarily close to 1 an example of a function f in $H^{-1+b/2}(\mathbf{R}^n)$ such

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that the left hand side of (1.1) is infinite. See [**31**, Theorem 1, p. 88]. From this contradiction arises the problem of finding those values of b and s for which there exists a number C independent of f such that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u(x,t)|^2 \,\frac{dtdx}{(1+|x|)^b} \,\le\, C\,\|f\|^2_{H^s(\mathbf{R}^n)}\,.$$

Theorem A. (Ben-Artzi, Klainerman [**3**, Corollary 2, p. 28], Kato, Yajima [**12**, (1.5), p. 482] and [**29**, Theorem 2.1(a) and 2.2(a), p. 385]) *There exists a number C independent of f such that*

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u(x,t)|^2 \,\frac{dtdx}{(1+|x|)^b} \,\le\, C\,\|f\|^2_{H^{-1/2}(\mathbf{R}^n)}\,,$$

if $b > n = 2$ or $n > b = 2$.

Theorem B. ([**29**, Theorem 2.1(b) and 2.2(b), p. 385]) *Let B^n be the open unit ball in $\mathbf{R}^n$. Assume that there exists a number C independent of f such that*

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u(x,t)|^2 \,\frac{dtdx}{(1+|x|)^b} \,\le\, C\,\|f\|^2_{L^2(\mathbf{R}^n)}\,, \quad \operatorname{supp}\widehat{f} \subseteq B^n.$$

Then $b > n = 2$ or $b \ge 2$.

Theorem C. (Sjögren, Sjölin [**16**]) *Let B^{n+1} be the open unit ball in $\mathbf{R}^{n+1}$. Assume that there exists a number C independent of f such that*

$$\|u\|_{L^2(B^{n+1})} \,\le\, C\,\|f\|_{H^s(\mathbf{R}^n)}\,.$$

Then $s \ge -1/2$.

Similar sharp results hold for u being the solution to the pseudo-differential equation $-|\Delta_x|^{a/2}\,u = i\partial_t u$, $a > 1$. See e.g. [**28**, Theorem 14.7, p. 222 and Theorem 14.8, p. 227] and [**30**, Theorem 2.1].

1.2. Let B denote the open unit ball in $\mathbf{R}$ and let ψ be an infinitely differentiable even function on $\mathbf{R}$ such that

$$\psi(B) \,=\, \{0\}\,, \quad \psi(\mathbf{R}) \,\subseteq\, [0,1] \quad \text{and} \quad \psi(\mathbf{R}\setminus 2B) \,=\, \{1\}\,.$$

Although claim (1.1) found in Vega [**25**] is contradicted by S. L. Wang [**31**] there is for every $b > 1$ a number C independent of f such that

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u_\psi(x,t)|^2 \,\frac{dtdx}{(1+|x|)^b} \,\le\, C\,\|f_\psi\|^2_{H^{-1+b/2}(\mathbf{R}^n)}\,. \tag{1.2}$$

This follows from Theorem 2.14 below. Here u_ψ is the solution to the free time-dependent Schrödinger equation with initial data f_ψ, $\widehat{f_\psi}(\xi) = \psi(|\xi|)\widehat{f}(\xi)$. Note however that the expression

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u_\psi(x,t)|^2 \frac{dtdx}{(1+|x|)^b}$$

for fixed f decreases to 0 as b increases to infinity whereas the expression

$$\|f_\psi\|^2_{H^{-1+b/2}(\mathbf{R}^n)}$$

increases to infinity as b increases to infinity.

1.3 The purpose of this paper. In the definition of $H^s(\mathbf{R}^n)$ let us replace the weight $(1+|\xi|^2)^s$ by $|\xi|^{2s}$. In this way we obtain the **homogeneous** Sobolev space $\dot{H}^s(\mathbf{R}^n)$. As f varies the expression $\|f_\psi\|_{H^{-1+b/2}(\mathbf{R}^n)}$ in the right hand side of (1.2) is equivalent to the expression $\|f_\psi\|_{\dot{H}^{-1+b/2}(\mathbf{R}^n)}$. As the title indicates, in this paper we seek bounds for u in terms of homogeneous norms. To make our inequalities scaling invariant we should replace $(1+|x|)^{-b}$ in the left hand side of (1.2) by a homogeneity. The following fact is easily established.

Proposition. *Assume that there exists a number C independent of f such that*

$$\int_{\mathbf{R}^n}\int_{\mathbf{R}} |u(x,t)|^2 \frac{dtdx}{|x|^b} \le C\ \|f\|^2_{\dot{H}^s(\mathbf{R}^n)}$$

Then $s = (b-2)/2$.

In the proof of this proposition one uses dilations of f, i.e. one replaces $\widehat{f}(\xi)$ by $\widehat{f}(\varepsilon\xi)$. See §2.11 for the details.

The purpose of this paper is to establish a family of **homogeneous** $L^p(L^q)$-estimates

$$\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} |u(x,t)|^q \frac{dt}{|t|^{\gamma q}}\right)^{p/q} \frac{dx}{|x|^b}\right)^{1/p} \le C\ \|f\|_{\dot{H}^s(\mathbf{R}^n)}$$

for u as in §1.1 and related oscillatory expressions $u = S^a f$ where C is independent of f. The $L^2(L^q)$-members of this family of estimates can easily be reduced to a **uniformity property** of Bessel functions which is classical. See Lemma 4.7. We shall indicate an alternate proof of this property.

Other members of the family of $L^p(L^q)$-estimates will be obtained by interpolation between the $L^2(L^q)$-members and Strichartz estimates.

Our main result in the case of u as in §1.1 is stated in the following theorem.

Theorem. *Let $q_1 \geq 2$, $0 \leq \gamma < 1/q_1$ and $0 \leq \theta \leq 1$. Then there exists a number C independent of f, x' and t' such that*

$$\left(\int_{\mathbf{R}^n} \left(\int_{\mathbf{R}} |u(x,t)|^{q_\theta} \frac{dt}{|t-t'|^{\gamma q_\theta \theta}} \right)^{p_\theta / q_\theta} \frac{dx}{|x-x'|^{b p_\theta \theta / 2}} \right)^{1/p_\theta} \leq C \, \|f\|_{\dot{H}^{s_\theta}(\mathbf{R}^n)}$$

if

$$\frac{1}{p_\theta} = \frac{n(1-\theta)}{2(n+2)} + \frac{\theta}{2}$$
$$\frac{1}{q_\theta} = \frac{n(1-\theta)}{2(n+2)} + \frac{\theta}{q_1}$$

and

$$s_\theta = \left[2 \left(\gamma - \frac{1}{q_1} \right) + \frac{b}{2} \right] \theta.$$

1.5 Earlier results. The estimates in this paper are versions of regularity and decay estimates. Such estimates has been studied in several papers during the last years. In this subsection we give an incomplete list of references.

We have already mentioned the work of Ben-Artzi, Klainerman [**3**], Kato, Yajima [**12**], Sjögren, Sjölin [**16**], Vega [**25**], Wang [**31**] and the present author [**28**], [**29**], [**30**]. Sjölin [**17**] proved and applied local smoothing expressed by the inequality

$$\|u\|_{L^2(B^{n+1})} \leq C \, \|f\|_{H^{-1/2}(\mathbf{R}^n)}$$

to improve results on the local integrability of the Schrödinger maximal function. Subsequently Constantin, Saut [**5**], [**6**] and Ben-Artzi, Devinatz [**2**] also proved smoothing estimates for solutions to the Schrödinger equation.

The following large time decay and regularity estimate can be found in Simon [**15**]:

$$\int_{\mathbf{R}^n} \int_{\mathbf{R}} |u(x,t)|^2 \frac{dtdx}{(1+|x|)^2} \leq \frac{\pi}{2} \, \|f\|^2_{\dot{H}^{-1/2}(\mathbf{R}^n)}, \quad n \geq 3. \tag{1.3}$$

See [**15**, (2), p. 66]. Note the similarity and difference with Theorem A. In [**15**] it is also shown that if $\varepsilon > 0$ is given the inequality (1.3) with $\pi/2$ replaced by $\pi/2 - \varepsilon$ cannot hold for all f.

Kenig, Ponce, Vega [**13**] proved the space-local time-global high energy estimate

$$\left(\int_{|x| \leq R} \|u[x]\|^2_{L^2(\mathbf{R})} \, dx \right)^{1/2} \leq C \, R \, \|f\|_{\dot{H}^{-1/2}(\mathbf{R}^n)}$$

for expressions u which generalize the solutions to the time-dependent Schrödinger equation. See [**13**, Theorem 4.1, p. 54].

Vilelas work [**26**] contains results for the non-homogeneous time-dependent Schrödinger equation $(\Delta_x - i\partial_t)\, u = F$ which are in accordance with those in this paper.

Results in the case of variable coefficients may be found in Craig, Kappeler, Strauss [**7**].

Other references. A useful introduction to space-time estimates may be found in Strauss [**22**] as well as in Stein [**19**, §5.16, 5.18, 5.19]. Interesting material concerning (local) smoothing of Fourier integrals is given in Sogge [**18**]. Iosevich and Sawyer has interesting work on oscillatory integrals although in a slightly different direction than ours. See e.g. [**11**].

1.6 The plan of this paper. In Section 2 we introduce notation and state our theorems. In Section 3 we collect two well-known inequalities: Pitt's inequality and Strichartz' inequality. We also state and prove a result in interpolation theory regarding weighted Lebesgue spaces of vector-valued functions. In Section 4 we give the prepartion needed regarding Bessel functions. Finally, in Section 5 we prove our theorems.

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2. Notation and Theorems

2.1 The Fourier transform. For x and ξ in $\mathbf{R}^n$ we let $x\xi = x_1\xi_1 + \cdots + x_n\xi_n$. If f is in the Schwartz class $\mathcal{S}(\mathbf{R}^n)$ we define the Fourier transform of f, denoted by $\widehat{f}$, by the formula

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{-ix\xi} f(x)\, dx.$$

The Fourier **transformation**, i.e. the linear operator on $\mathcal{S}(\mathbf{R}^n)$ taking f to $\widehat{f}$, extends by duality to an isomorphism on $\mathcal{S}'(\mathbf{R}^n)$.

With our normalization of the Fourier transform the inversion formula valid for all $f \in \mathcal{S}(\mathbf{R}^n)$ reads

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} e^{ix\xi} \widehat{f}(x)\, dx.$$

2.2 Oscillatory Integrals. With the Fourier transform at hand we can now define the oscillatory integrals we are interested in. For a bounded and measurable function m we set

$$(S_m^a f)[x](t) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} m(x, |\xi|)\, e^{i(x\xi + t|\xi|^a)} \widehat{f}(\xi)\, d\xi, \quad t \in \mathbf{R}, \quad a \neq 0.$$

Whenever $m = 1$ we will write S^a instead of S_1^a.

Note that if $u(x,t) = (S^2 f)[x](t)$, then $\Delta_x u = i\partial_t u$, i.e. u is a solution to the free time-dependent Schrödinger equation. If instead $u(x,t) = (S^1 f)[x](t)$ then u is a solution to the classical wave-equation $\Delta_x u = \partial_t^2 u$.

2.3 Sobolev spaces. We introduce the fractional Sobolev spaces

$$\dot{H}^s(\mathbf{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{\dot{H}^s(\mathbf{R}^n)}^2 = \int_{\mathbf{R}^n} |\xi|^{2s} \left| \widehat{f}(\xi) \right|^2 d\xi < \infty \right\}$$

and

$$H^s(\mathbf{R}^n) = \left\{ f \in \mathcal{S}'(\mathbf{R}^n) : \|f\|_{H^s(\mathbf{R}^n)}^2 = \int_{\mathbf{R}^n} \left(1 + |\xi|^2\right)^s \left| \widehat{f}(\xi) \right|^2 d\xi < \infty \right\}.$$

The Sobolev space $\dot{H}^s(\mathbf{R}^n)$ is called **homogeneous**. This stems from the fact that $\xi \mapsto |\xi|^{2s}$ is a **homogeneous function** (of degree $2s$). In this paper it is important to use homogeneous Sobolev spaces since we want to determine the size of $S_m^a f$ in terms of weighted L^p-spaces with **homogeneous** weight functions $x \mapsto |x|^{-b}$ for some number $b > 0$.

2.4 Auxiliary notation. By B^n and Σ^{n-1} we denote the open unit ball and the unit sphere in $\mathbf{R}^n$ respectively. (B^1 will be denoted by B.)

Numbers denoted by C may be different at each occurrence.

Unless otherwise explicitly stated all functions f are supposed to belong to $\mathcal{S}(\mathbf{R}^n)$.

2.5. We now have everything at hand to state our theorems.

2.6 Theorem. *Let* $q_1 \geq 2$, $0 \leq \gamma < 1/q_1$, $1 < b < n$ *and* $a \neq 0$. *Then there exists a number* C *independent of* f *such that*

$$\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} |(S_m^a f)[x](t)|^{q_1} \frac{dt}{|t|^{\gamma q_1}}\right)^{2/q_1} \frac{dx}{|x|^b}\right)^{1/2} \leq C \, \|f\|_{\dot{H}^{s_1}(\mathbf{R}^n)},$$

if

$$s_1 = a\left(\gamma - \frac{1}{q_1}\right) + \frac{b}{2}.$$

2.7 Remark. In the case $a = q_1 = 2$, $\gamma = 0$ and $m = 1$ Theorem 2.6 has been proved before by Ben-Artzi, Klainerman [**3**, Theorem 1(a), p. 26], Kato, Yajima [**12**, Theorem 1, p. 482] and Vilela [**26**, Theorem 2, p. 4].

2.8 Theorem.

(a) *Let* $q_1 \geq 2$, $0 \leq \gamma < 1/q_1$, $1 < b < n$, $a > 0$ *and* $0 \leq \theta \leq 1$. *Then there exists a number* C *independent of* f *such that*

$$\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} |(S^a f)[x](t)|^{q_\theta} \frac{dt}{|t|^{\gamma q_\theta \theta}}\right)^{p_\theta/q_\theta} \frac{dx}{|x|^{b p_\theta \theta/2}}\right)^{1/p_\theta} \leq C \, \|f\|_{\dot{H}^{s_\theta}(\mathbf{R}^n)},$$

if

$$\frac{1}{p_\theta} = \frac{n(1-\theta)}{2(n+a)} + \frac{\theta}{2}$$
$$\frac{1}{q_\theta} = \frac{n(1-\theta)}{2(n+a)} + \frac{\theta}{q_1}$$

and

$$s_\theta = \left[a\left(\gamma - \frac{1}{q_1}\right) + \frac{b}{2}\right]\theta.$$

(b) *Assume that there exists a number* C *independent of* f *such that*

$$\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} |(S^a f)[x](t)|^{q} \frac{dt}{|t|^{\gamma}}\right)^{p/q} \frac{dx}{|x|^b}\right)^{1/p} \leq C \, \|f\|_{\dot{H}^{s}(\mathbf{R}^n)}.$$

Then

$$s = \frac{n}{2} + \frac{a(\gamma - 1)}{q} + \frac{b-n}{p}. \tag{2.1}$$

2.9 Remark. If we replace γ, b, q and p by $\gamma q_\theta \theta$, $bp_\theta\theta/2$, q_θ and p_θ respectively in (2.1) it is easy to verify that $s = s_\theta$ as expected.

2.10 Corollary. *Let $q_1 \geq 2$, $0 \leq \gamma < 1/q_1$, $1 < b < 2$, $a > 0$ and $0 \leq \theta \leq 1$. Then there exists a number C independent of f, x' and t' such that*

$$\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} |(S^a f)[x](t)|^{q_\theta}\,\frac{dt}{|t-t'|^{\gamma q_\theta \theta}}\right)^{p_\theta/q_\theta}\frac{dx}{|x-x'|^{bp_\theta\theta/2}}\right)^{1/p_\theta} \leq C\,\|f\|_{\dot{H}^{s_\theta}(\mathbf{R}^n)},$$

if p_θ, q_θ and s_θ fulfills the same assumptions as in Theorem 2.8.

Proof. Since $|(S^a f)[x-x'](t-t')| = |(S^a g)[x](t)|$ where

$$\widehat{g}(\xi) = e^{-i(x'\xi + t'|\xi|^a)}\,\widehat{f}(\xi)$$

and since $\|g\|_{\dot{H}^s(\mathbf{R}^n)} = \|f\|_{\dot{H}^s(\mathbf{R}^n)}$ the corollary follows from Theorem 2.8. $\square$

2.11 Proof of Theorem 2.8(b). For $\varepsilon > 0$ let us write $f_\varepsilon(\xi) = f(\varepsilon\xi)$. If $\|f\|_{L^2(\mathbf{R}^n)} = 1$, then

$$\|f_\varepsilon\|_{L^2(\mathbf{R}^n)} = \varepsilon^{-n/2}.$$

Define

$$\left(\widetilde{S^a} f\right)[x](t) = \frac{1}{|t|^\gamma |x|^{b/p}}\int_{\mathbf{R}^n} e^{i(x\xi + t|\xi|^a)}|\xi|^{-s} f(\xi)\,d\xi, \quad t \in \mathbf{R}.$$

Then

$$\left(\widetilde{S^a} f_\varepsilon\right)[x](t) = \varepsilon^{s-n-a\gamma-b/p}\left(\widetilde{S^a} f\right)\left[\frac{x}{\varepsilon}\right]\left(\frac{t}{\varepsilon^a}\right).$$

For a function w on $\mathbf{R}^{n+1}$ with values denoted by $w[x](t)$, $x \in \mathbf{R}^n$, $t \in \mathbf{R}$ set

$$\|w\|_{L^p(\mathbf{R}^n, L^q(\mathbf{R}))} = \left(\int_{\mathbf{R}^n} \|w[x]\|^p_{L^q(\mathbf{R})}\,dx\right)^{1/p}.$$

The assumption of the theorem says that there exists a number C independent of f such that

$$\left\|\widetilde{S^a} f\right\|_{L^p(\mathbf{R}^n, L^q(\mathbf{R}))} \leq C\,\|f\|_{L^2(\mathbf{R}^n)}.$$

Hence, if we fix f there exists a number C independent of ε such that

$$\begin{aligned}
&\left\|\widetilde{S^a} f_\varepsilon\right\|_{L^p(\mathbf{R}^n, L^q(\mathbf{R}))}\\
&= \varepsilon^{s-n-a\gamma-b/p+a/q+n/p}\left(\int_{\mathbf{R}^n}\left(\int_{\mathbf{R}} |(S^a f)[x](t)|^q\,\frac{dt}{|t|^{\gamma q}}\right)^{p/q}\frac{dx}{|x|^b}\right)^{1/p}\\
&\leq C\,\varepsilon^{-n/2}.
\end{aligned}$$

Since ε is arbitrary we must have $s = n/2 + a(\gamma-1)/q + (b-n)/p$. $\square$

2.12. Theorem 2.6 and 2.8 (a) will be proved in Section 4.

2.13. Although our next theorem concerns an inhomogeneous estimate we include it since it improves on the estimate (1.2). Recall our definition of f_ψ from §1.2.

2.14 Theorem.

(a) *If $b > 1$ there exists a number C independent of f such that*

$$\int_{\mathbf{R}^n} \|(S^a_m f_\psi)[x]\|^2_{L^2(\mathbf{R})} \frac{dx}{(1+|x|)^b} \le C \, \|f_\psi\|^2_{H^{(1-a)/2}(\mathbf{R}^n)} .$$

(b) *Assume that there exists a number C independent of f such that*

$$\int_{\mathbf{R}^n} \|(S^a f)[x]\|^2_{L^2(\mathbf{R})} \frac{dx}{(1+|x|)^b} \le C \, \|f\|^2_{L^2(\mathbf{R}^n)} , \quad \operatorname{supp} \widehat{f} \subseteq 4B^n \setminus \overline{B^n}.$$

Then $b > 1$.

3. General preparation

3.1. In this section we collect some results which will be used in the proofs of our theorems.

3.2 Theorem (Pitt's Inequality)**.** (Muckenhoupt [**14**, p. 729]) *Assume that $q \ge p$, $0 \le \alpha < 1 - 1/p$, $0 \le \gamma < 1/q$ and $\gamma = \alpha + 1/p + 1/q - 1$. Then there exists a number C independent of f such that*

$$\left(\int_{\mathbf{R}} |\widehat{f}(\xi)|^q \frac{d\xi}{|\xi|^{\gamma q}} \right)^{1/q} \le C \left(\int_{\mathbf{R}} |f(x)|^p |x|^{\alpha p} \, dx \right)^{1/p} .$$

3.3 Theorem. (Strichartz [**23**]. Cf. also Stein [**19**, §5.19(b), p. 369].) *For $a > 0$ let $q = 2 + 2a/n$. Then there exists a number C independent of f such that*

$$\|S^a f\|_{L^q(\mathbf{R}^{n+1})} \le C \, \|f\|_{L^2(\mathbf{R}^n)} . \tag{3.1}$$

3.4 Remark. The condition $q = 2 + 2a/n$ is necessary for (3.1) to hold with a number C independent of f. This can be seen by choosing $s = \gamma = b = 0$ and $p = q$ in Theorem 2.8(b).

3.5 Weighted Lebesgue spaces. We will use weighted Lebesgue spaces for power weights and we will interpolate not only between Lebesgue exponents but also between weights and range spaces. For that purpose we need to define

$L^p_w(M,A)$ where M is a measure space with measure μ, w is a non-negative measurable function and A is a Banach space. We have

$$\|f\|_{L^p_w(M,A)} = \left(\int_M \|f\|_A^p \; w \, d\mu\right)^{1/p}$$

and

$$L^p_w(M,A) = \Big\{ f : f \text{ is strongly measurable } M \longrightarrow A \text{ and } \|f\|_{L^p_w(M,A)} < \infty \Big\}.$$

Here and at similar instances we denote the function $x \mapsto \|f(x)\|_A$ by $\|f\|_A$. To simplify our notation we will write $L^p_w(A)$ instead of $L^p_w(M,A)$.

3.6. For the following theorem the author of this paper has not succeeded in finding a proper reference. The proof is an adaption of material found in Bergh, Löfström [**4**, pp. 107–108].

3.7 Theorem. *Assume that A_0 and A_1 are Banach spaces, that $1 \le p_0 < \infty$, $1 \le p_1 < \infty$ and that $0 \le \theta \le 1$. Then (with the notation from complex interpolation theory* [**4**, p. 88]*)*

$$\left(L^{p_0}_{w_0}(A_0), L^{p_1}_{w_1}(A_1)\right)_{[\theta]} = L^{p_\theta}_{w_\theta}((A_0,A_1)_{[\theta]})$$

where

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad 0 \le \theta \le 1$$

and

$$w_\theta{}^{1/p_\theta} = w_0{}^{(1-\theta)/p_0}\, w_1{}^{\theta/p_1}.$$

Proof. As already stated, the proof follows closely the proof of [**4**, Theorem 5.1.2, pp. 107–108].

Let S denote the space of simple functions on M with values in $A_1 \cap A_2$. S is dense in $L^{p_0}_{w_0}(A_0) \cap L^{p_1}_{w_1}(A_1)$. By [**4**, Theorem 4.2.2, p. 91] S is dense also in $\left(L^{p_0}_{w_0}(A_0), L^{p_1}_{w_1}(A_1)\right)_{[\theta]}$ and in $L^{p_\theta}_{w_\theta}((A_0,A_1)_{[\theta]})$. Hence it is enough to consider functions a in S only.

3.7.1 The complex interpolation method. Recall that for any couple of Banach spaces (Ξ_0, Ξ_1)

$$\|\xi\|_{(\Xi_0,\Xi_1)_{[\theta]}} = \inf_{g(\theta)=\xi} \|g\|_{\mathcal{F}(\Xi_0,\Xi_1)} \tag{3.2}$$

where

$$\|g\|_{\mathcal{F}(\Xi_0,\Xi_1)} = \sup\left\{\|g(j+it)\|_{\Xi_j} : t\in\mathbf{R},\ j\in\{0,1\}\right\}. \tag{3.3}$$

Cf. [**4**, pp. 87–88]. The space $\mathcal{F}(\Xi_0,\Xi_1)$ consists of all functions g with values in $\Xi_0+\Xi_1$ which are bounded and continuous on the strip $S=\{z\in\mathbf{C}: 0\le \operatorname{Re} z\le 1\}$, analytic in its interior and such that the functions $t\mapsto g(j+it)$, $j\in\{0,1\}$ are continuous $\mathbf{R}\longrightarrow\Xi_j$ and tend to 0 as $|t|$ tend to infinity.

3.7.2 The direct inequality. We want to derive the inequality

$$\|a\|_{\left(L^{p_0}_{w_0}(A_0),L^{p_1}_{w_1}(A_1)\right)_{[\theta]}} \le \|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}.$$

Let $\varepsilon>0$ and $x\in M$ be given. By (3.2) and by the assumption $a\in\mathsf{S}$ there is a function $g[x]\in\mathcal{F}(A_0,A_1)$ such that $\|g[x]\|_{\mathcal{F}(A_0,A_1)}\le(1+\varepsilon)\ \|a(x)\|_{(A_0,A_1)_{[\theta]}}$ and $g[x](\theta)=a(x)$. Put

$$f(z) = \left(\frac{\|a\|_{(A_0,A_1)_{[\theta]}}}{\|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}}\right)^{p_\theta(1/p_1-1/p_0)(z-\theta)} g(z)\left(\frac{w_\theta}{w_0}\right)^{(1-z)/p_0}\left(\frac{w_\theta}{w_1}\right)^{z/p_1}.$$

$f(z)$ and $g(z)$ are for fixed z functions on M whose values at x are denoted by $f[x](z)$ and $g[x](z)$ respectively. Note that

$$\begin{aligned}\|f(it)\|^{p_0}_{A_0} &\le \left(\frac{\|a\|_{(A_0,A_1)_{[\theta]}}}{\|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}}\right)^{p_\theta-p_0} \|g\|^{p_0}_{\mathcal{F}(A_0,A_1)}\ \frac{w_\theta}{w_0}\\ &\le \left(\frac{\|a\|_{(A_0,A_1)_{[\theta]}}}{\|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}}\right)^{p_\theta-p_0} (1+\varepsilon)^{p_0}\ \|a\|^{p_0}_{(A_0,A_1)_{[\theta]}}\ \frac{w_\theta}{w_0}.\end{aligned}$$

Integrating over M yields

$$\|f(it)\|_{L^{p_0}_{w_0}(A_0)} = \left(\int_M \|f(it)\|^{p_0}_{A_0}\ w_0\,d\mu\right)^{1/p_0} \le (1+\varepsilon)\ \|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}.$$

Similarly,

$$\|f(1+it)\|_{L^{p_1}_{w_1}(A_1)} \le (1+\varepsilon)\ \|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}.$$

Since ε was arbitrary it follows from (3.3) that

$$\|f\|_{\mathcal{F}\left(L^{p_0}_{w_0}(A_0),L^{p_1}_{w_1}(A_1)\right)} \le \|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}.$$

By (3.2) with $\Xi_j = L^{p_j}_{w_j}(A_j)\,,\ j \in \{0,1\}$

$$\|a\|_{\left(L^{p_0}_{w_0}(A_0),L^{p_1}_{w_1}(A_1)\right)_{[\theta]}} \le \|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)}\,.$$

3.7.3 The reversed inequality. Now we want to derive the inequality

$$\|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)} \le \|a\|_{\left(L^{p_0}_{w_0}(A_0),L^{p_1}_{w_1}(A_1)\right)_{[\theta]}}\,.$$

Let

$$P_j(s+it,\tau) = \frac{e^{-\pi(\tau-t)}\,\sin(\pi s)}{\sin(\pi s)^2 + \left(\cos(\pi s) - e^{ij\pi-\pi(\tau-t)}\right)^2},\quad j\in\{0,1\}\,.$$

Note that

$$\frac{1}{1-\theta}\int_{\mathbf{R}} P_0(\theta,\tau)\,d\tau = \frac{1}{\theta}\int_{\mathbf{R}} P_1(\theta,\tau)\,d\tau = 1.$$

The functions P_j are the Poisson kernels for S. Cf. [**4**, p. 93]. If $f[x]\in\mathcal{F}(A_0,A_1)$ and $f(\theta)=a$ then by [**4**, Lemma 4.3.2(ii), p. 93]

$$\begin{aligned}
\|a\|^{p_\theta}_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)} &= \int_M \|a\|^{p_\theta}_{(A_0,A_1)_{[\theta]}}\ w_\theta\,d\mu \\
&\le \int_M \left(\frac{1}{1-\theta}\int_{\mathbf{R}} \|f(i\tau)\|_{A_0}\ P_0(\theta,\tau)\,d\tau\right)^{p_\theta(1-\theta)} {w_0}^{p_\theta(1-\theta)/p_0} \\
&\quad\times \left(\frac{1}{\theta}\int_{\mathbf{R}} \|f(1+i\tau)\|_{A_1}\ P_1(\theta,\tau)\,d\tau\right)^{p_\theta\theta} {w_1}^{p_\theta\theta/p_1}\,d\mu.
\end{aligned}$$

Since

$$\frac{p_0}{p_\theta(1-\theta)} = 1 + \frac{p_0\,\theta}{p_1\,(1-\theta)} > 1$$

we may apply Hölder's inequality with the dual exponents $p_0/p_\theta(1-\theta)$ and $p_1/p_\theta\theta$ to the integral with respect to μ to get

$$\begin{aligned}
&\|a\|^{p_\theta}_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)} \\
&\quad\le \left[\int_M \left(\frac{1}{1-\theta}\int_{\mathbf{R}} \|f(i\tau)\|_{A_0}\ P_0(\theta,\tau)\,d\tau\right)^{p_0} w_0\,d\mu\right]^{p_\theta(1-\theta)/p_0} \\
&\qquad\times \left[\int_M \left(\frac{1}{\theta}\int_{\mathbf{R}} \|f(1+i\tau)\|_{A_1}\ P_1(\theta,\tau)\,d\tau\right)^{p_1} w_1\,d\mu\right]^{p_\theta\theta/p_1}.
\end{aligned}$$

We now apply Minkowski's inequality to get

$$\begin{aligned}
\|a\|^{p_\theta}_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)} &\le \left(\frac{1}{1-\theta}\int_{\mathbf{R}} \|f(i\tau)\|_{L^{p_0}_{w_0}(A_0)}\ P_0(\theta,\tau)\,d\tau\right)^{p_\theta(1-\theta)} \\
&\quad\times \left(\frac{1}{\theta}\int_{\mathbf{R}} \|f(i\tau)\|_{L^{p_1}_{w_1}(A_1)}\ P_1(\theta,\tau)\,d\tau\right)^{p_\theta\theta} \\
&\le \sup_{\tau\in\mathbf{R}} \|f(i\tau)\|^{p_\theta(1-\theta)}_{L^{p_0}_{w_0}(A_0)} \times \sup_{\tau\in\mathbf{R}} \|f(1+i\tau)\|^{p_\theta\theta}_{L^{p_1}_{w_1}(A_1)} \le \|f\|^{p_\theta}_{\mathcal{F}\left(L^{p_0}_{w_0}(A_0),L^{p_1}_{w_1}(A_1)\right)}\,.
\end{aligned}$$

Now we take the infimum over f subject to $f(\theta) = a$ and use (3.2) with $\Xi_j = L^{p_j}_{w_j}(A_j)$, $j \in \{0,1\}$. This gives

$$\|a\|_{L^{p_\theta}_{w_\theta}\left((A_0,A_1)_{[\theta]}\right)} \le \|a\|_{\left(L^{p_0}_{w_0}(A_0),L^{p_1}_{w_1}(A_1)\right)_{\theta]}}$$

as desired. □

3.8 Theorem (Interpolation of Bessel Potential Spaces)**.** ([**4**, Theorem 5.4.1(7), p. 153]) *Let* $0 \le \theta \le 1$. *Put*

$$s_\theta = s_0\,(1-\theta) + s_1\,\theta.$$

Then we have

$$\left(\dot H^{s_0}(\mathbf{R}^n), \dot H^{s_1}(\mathbf{R}^n)\right)_{[\theta]} = \dot H^{s_\theta}(\mathbf{R}^n).$$

4. Preparation: Bessel functions

4.1. In this section we introduce some notation and give some results on Bessel functions which will be used in the proofs of our theorems. We also indicate an alternate proof of a classical uniformity property (see Watson [**32**, (2), p. 403]) for such functions.

4.2 Bessel functions as oscillatory integrals. Poisson's representation formula. For integers l we define the Bessel function of order l by

$$J_l(\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(\rho\sin\omega - l\omega)}\,d\omega \tag{4.1}$$

and for real numbers $\lambda > -1/2$ the Bessel function of order λ by

$$J_\lambda(\rho) = \frac{\rho^\lambda}{2^\lambda\Gamma(\lambda+1/2)\Gamma(1/2)}\int_{-1}^{1} e^{ir\rho}\left(1-r^2\right)^{\lambda-1/2}\,dr. \tag{4.2}$$

Here Γ is the well-known Gamma function. (4.1) is consistent with (4.2). See e.g. Stein, Weiss [**20**, Lemma 3.1, p. 153]. (4.2) is called **Poisson's representation**. We will also refer to **Schläfli's generalisation of Bessel's integral**

$$J_\lambda(\rho) = \frac{1}{2\pi}\int_0^{2\pi} e^{i(\rho\sin\omega-\lambda\omega)}\,d\omega - \frac{\sin(\lambda\pi)}{\pi}\int_0^\infty e^{-\lambda\tau-\rho\sinh\tau}\,d\tau. \tag{4.3}$$

See [**32**, (4), p. 176].

4.3 Definition. For a function $g \in L^2(\Sigma^{n-1})$ we define the tempered distribution μ_g by

$$\mu_g(f) \;=\; \int_{\Sigma^{n-1}} f(\xi')\, g(\xi')\, d\sigma(\xi').$$

4.4 Theorem. ([**20**, Theorem 3.10, p. 158]) *For a spherical harmonic P of degree k let $f(x) = P(x) f_0(|x|)$. Then*

$$\widehat{f}(\xi) = (2\pi)^{n/2}\, i^{-k} |\xi|^{-\nu(k)} P(\xi) \int_0^\infty f_0(r) J_{\nu(k)}(r|\xi|) r^{n/2+k}\, dr, \quad \nu(k) = \frac{n}{2} + k - 1.$$

4.5 Corollary. *For a spherical harmonic P of degree k μ_P has Fourier transform $\widehat{\mu_P}$ given by*

$$\widehat{\mu_P}(\xi) \;=\; (2\pi)^{n/2}\, i^{-k}\, |\xi|^{-\nu(k)}\, P(\xi)\, J_{\nu(k)}(|\xi|).$$

4.6 Theorem. (Guo [**8**, Lemma 3.2, p. 1333]) *Let $p > 4$. Then*

$$\sup_{k \in \mathbf{Z}} \int_0^\infty |J_k(\rho)|^p\, \rho\, d\rho \;<\; \infty.$$

Remark on the proof. To prove this theorem one may apply the two-dimensional restriction theorem for the Fourier transform (cf. e.g. [**19**, Corollary 1, p. 414]) to μ_P. $\square$

4.7 Lemma. *If $1 < b < 2$ then*

$$\sup_{\lambda \geq 0} \int_0^\infty J_\lambda(\rho)^2\, \rho^{1-b}\, d\rho \;<\; \infty. \tag{4.4}$$

Remarks on the proof. We could appeal to **the critical case of the Weber-Schafheitlin integral** ([**32**, (2), p. 403]) to show the lemma. Instead we will indicate the proof using a different method based among other things on Theorem 4.6.

The following results can be found in [**8**, Lemma 3.4 and (21) in Theorem 3.5, p. 1334]:

$$\sup_{\lambda \geq 2\rho, \lambda \geq 1} |\lambda\, J_\lambda(\rho)| \;<\; \infty, \tag{4.5a}$$

$$\sup_{\lambda \geq 0} \int_0^\infty |J_\lambda(\rho)|^p\; \rho\, d\rho \;<\; \infty, \quad p > 4. \tag{4.5b}$$

(4.5a) follows from van der Corput's lemma ([**19**, Proposition 2, p. 332]) applied to Schläfli's generalisation of Bessel's integral (4.3). (4.5b) follows from Theorem 4.6, Schläfli's generalisation of Bessel's integral (4.3) and well-known recursion formulae for Bessel functions ([**32**, (2), (4), p. 45]). For proofs of (4.5a) and (4.5b) we refer to [**8**].

To prove the bound (4.4) we consider two cases of λ.

4.7.1 *Estimate when* $\lambda \geq 1$. Make the splitting

$$\int_0^\infty J_\lambda(\rho)^2\,\rho^{1-b}\,d\rho = \int_0^{\lambda/2} + \int_{\lambda/2}^\infty . \tag{4.6}$$

According to (4.5a) there exists a number C independent of λ such that

$$\int_0^{\lambda/2} J_\lambda(\rho)^2\,\rho^{1-b}\,d\rho \;\leq\; \frac{C}{\lambda^2}\int_0^{\lambda/2} \rho^{1-b}\,d\rho.$$

Hence

$$\sup_{\lambda\geq 1}\int_0^{\lambda/2} J_\lambda(\rho)^2\,\rho^{1-b}\,d\rho \;<\infty.$$

For the second part in the splitting (4.6) choose $p > 4$ such that

$$\frac{1}{2} < \frac{p-2}{p} < \frac{b}{2} \tag{4.7}$$

and apply Hölder's inequality with exponents $(p/2)^* = p/(p-2)$ and $p/2$. We get

$$\begin{aligned}\int_{\lambda/2}^\infty & J_\lambda(\rho)^2\,\rho^{1-b}\,d\rho \\ &\leq \left(\int_{\lambda/2}^\infty \left(\rho^{-b}\right)^{p/(p-2)}\,\rho\,d\rho\right)^{(p-2)/p} \left(\int_0^\infty J_\lambda(\rho)^p\,\rho\,d\rho\right)^{2/p}.\end{aligned} \tag{4.8}$$

The first factor is $C\,\lambda^{(-pb+2p-4)/p}$, where C is independent of λ and where the exponent $(-pb+2p-4)/p$ is less than zero by (4.7). The second factor is bounded with respect to λ by (4.5b).

4.7.2 *Estimate when* $\lambda \leq 1$. Make the splitting

$$\int_0^\infty J_\lambda(\rho)^2\,\rho^{1-b}\,d\rho = \int_0^1 + \int_1^\infty . \tag{4.9}$$

We use Poisson's representation (4.2) to make the estimate

$$\begin{aligned}\int_0^1 & J_\lambda(\rho)^2\,\rho^{1-b}\,d\rho \\ &\leq \sup_{0\leq\lambda\leq 1}\frac{1}{\Gamma(\lambda+1/2)^2}\left(\int_0^1 \rho^{1-b}\,d\rho\right)\left(\int_{-1}^1 \left(1-r^2\right)^{-1/2}\,dr\right)^2.\end{aligned}$$

For the second part in the splitting (4.9) we use (4.8) with $\lambda = 2$ in the lower limit of the integrals. □

4.8 Remark. In Stempak [**21**] there is a proof of Lemma 4.7 and (4.5b) using the following intrinsic properties of the Bessel function J_λ: **There are positive numbers C and D independent of r and λ such that**

$$|J_\lambda(\rho)| \le C \times \begin{cases} \exp(-D\lambda), & 0 < \rho \le \lambda/2, \\ \lambda^{-1/4}\left(|\rho - \lambda| + \lambda^{1/3}\right)^{-1/4}, & \lambda/2 < \rho \le 2\lambda, \\ \rho^{-1/2}, & 2\lambda < \rho. \end{cases} \tag{4.10}$$

See [**21**, (4), p. 2944].

4.9 Theorem. (Cf. S. L. Wang [**31**, Theorem 2, p. 88]) *If $1 < b < n$, then there exists a number C independent of f such that*

$$\int_{\mathbf{R}^n} |\widehat{\mu_g}(x)|^2 \frac{dx}{|x|^b} \le C\, \|g\|^2_{L^2(\Sigma^{n-1})}.$$

Remark on the proof. By orthogonality of spherical harmonics one may conclude this theorem from the uniformity property of Lemma 4.7 together with the Poisson representation (4.2). For the details cf. [**31**, pp. 90–91]. □

4.10 Theorem. (Strichartz [**24**, p. 63]. Cf. also [**29**, Corollary 3.9, p. 387].) *The mean value*

$$\frac{1}{\rho}\int_0^\rho J_\lambda(r)^2\, r\, dr$$

is bounded from above by a number independent of $\rho > 0$ and $\lambda > 0$.

Proof. We shall prove this estimate using the estimate (4.10). Our proof is very similar to the work in [**21**].

4.10.1 *Estimate when* $0 < \rho \le \lambda/2$. There exist numbers C and D independent of ρ and λ such that

$$\frac{1}{\rho}\int_0^\rho J_\lambda(r)^2\, r\, dr \le \frac{C\, e^{-2D\lambda}\,\lambda}{2\rho}\int_0^\rho dr \le \frac{C}{4De}.$$

4.10.2 *Estimate when* $\lambda/2 < \rho \le 2\lambda$. The given mean value is majorized by

$$\frac{2}{\lambda}\int_0^{2\lambda} J_\lambda(r)^2\, r\, dr = \frac{2}{\lambda}\int_0^{\lambda/2} J_\lambda(r)^2\, r\, dr + \frac{2}{\lambda}\int_{\lambda/2}^{2\lambda} J_\lambda(r)^2\, r\, dr.$$

According to the previous paragraph the first part is majorized by C/e for some number C independent of ρ and λ. The second part is majorized by

$$\frac{C}{\lambda^{1/2}}\left\{\int_{\lambda/2}^{\lambda}\left(\lambda - r + \lambda^{1/3}\right)^{-1/2} dr + \int_{\lambda}^{2\lambda}\left(r - \lambda + \lambda^{1/3}\right)^{-1/2} dr\right\}$$

for some number C independent of ρ and λ. Inside the brackets the first as well as the second integral is by an explicit computation majorized by $C\lambda^{1/2}$ for some number C independent of λ.

4.10.3 *Estimate when* $2\lambda < \rho$. The given mean value is majorized by

$$\frac{1}{2\lambda}\int_0^{2\lambda} J_\lambda(r)^2\, r\, dr \;+\; \frac{1}{\rho}\int_{2\lambda}^{\rho} J_\lambda(r)^2\, r\, dr.$$

According to the previous paragraph the first part is majorized by a number C independent of λ. The second part is majorized by

$$\frac{C}{\rho}\int_{2\lambda}^{\rho} r^{-1}\, dr \;\le\; \frac{C\,(\rho-2\lambda)}{2\rho\lambda}$$

for some number C independent of ρ and λ. □

4.11 Remark. In [**29**, Corollary 3.9, p. 387] we proved Theorem 4.10 by applying the estimate

$$\int_{|x|\le\rho} \left|\widehat{\mu_g}(x)\right|^2\, dx \;\le\; C\,\rho\, \|g\|^2_{L^2(\Sigma^{n-1})} \tag{4.11}$$

to $g = P$ where P is a spherical harmonic and where C is a number independent of f and ρ. The estimate (4.11) is a special case of the estimate in Hörmander [**10**, Theorem 7.1.26, p. 173]. Cf. also Agmon, Hörmander [**1**]. Estimates reminiscent of (4.11) was studied by Hartman and Wilcox. See [**9**].

4.12 Theorem (Asymptotics of Bessel functions)**.** ([**20**, Lemma 3.11, p. 158]) *If* $\lambda > -1/2$*, then there exists a number* C *depending on* λ *but independent of* r *such that*

$$\left| J_\lambda(\rho) - \left(\frac{2}{\pi\rho}\right)^{1/2} \cos\left(\rho - \frac{\lambda\pi}{2} - \frac{\pi}{4}\right)\right| \;\le\; C\,\rho^{-3/2}, \quad \rho \ge 1.$$

5. Proofs

5.1 Definition. For a bounded and measurable function m we define

$$\left(\widetilde{S^a_m} f\right)[x](t) = \frac{1}{|t|^\gamma |x|^{b/p}} \int_{\mathbf{R}^n} m(x,|\xi|) e^{i(x\xi + t|\xi|^a)} |\xi|^{-s} f(\xi)\, d\xi, \quad t \in \mathbf{R}. \tag{5.1}$$

The main difference between $\widetilde{S^a_m}$ and S^a_m of §2.2 is that we have included the ξ-weight of the Sobolev spaces $\dot{H}^s\left(\mathbf{R}^n\right)$ under and the x- and t-weights before the integral sign. If $m = 1$ we will write $\widetilde{S^a}$ instead of $\widetilde{S^a_m}$.

5.2 Definition. For $\rho > 0$ we define $\widetilde{f^{a,s}}$

$$\widetilde{f^{a,s}}(\rho)[x] \;=\; \frac{\rho^{(n-a)/a - s/a}}{a\,|x|^{b/p}} \int_{\Sigma^{n-1}} e^{i\rho^{1/a} x\xi'}\, f(\rho^{1/a}\xi')\, d\sigma(\xi'), \quad \rho > 0$$

and for $\rho \le 0$ by $\widetilde{f}(\rho) = 0$. Note that $\widetilde{f^{a,s}}(\rho)\,\rho^\alpha = \widetilde{f^{a,s-a\alpha}}(\rho)$.

5.3 Proof of Theorem 2.6**.** Our theorem follows if we can show that for $s = (b-a)/2 + a(\gamma + 1/2 - 1/q_1) = a(\gamma - 1/q_1) + b/2$ there exists a number C independent of f such that

$$\left\| \widetilde{S^a_m} f \right\|_{L^2(\mathbf{R}^n, L^{q_1}(\mathbf{R}))} \leq C \, \|f\|_{L^2(\mathbf{R}^n)} \,. \tag{5.2}$$

The formula

$$\left(\widetilde{S^a_m} f\right)[x](t) \;=\; \frac{1}{|t|^\gamma} \int_0^\infty e^{it\rho}\, m(x, \rho^{1/a})\, \widetilde{f^{a,s}}[x](\rho)\, d\rho$$

follows by polar coordinates and change of variables in (5.1).

We would like to apply Theorem 3.2 with $p = 2$ and $q = q_1$. Then $\alpha = \\ = \gamma + 1/2 - 1/q_1$. We get that there exists a number C independent of f and x such that

$$\left\| \left(\widetilde{S^a_m} f\right)[x] \right\|_{L^{q_1}(\mathbf{R})} \leq C \, \|m\|_{L^\infty(\mathbf{R}^n \times \mathbf{R}_+)} \left\| \widetilde{f^{a,s'}}[x] \right\|_{L^2(\mathbf{R})}, \quad s' = \frac{b-a}{2}.$$

Hence, to prove (5.2) (where C may be chosen to be independent of f) it is sufficient to prove that

$$\left\| \widetilde{f^{a,s'}} \right\|_{L^2(\mathbf{R}^{n+1})} \leq C \, \|f\|_{L^2(\mathbf{R}^n)}$$

where C may be chosen to be independent of f.

Let us write $f_\rho(\xi') = f(\rho^{1/a}\xi')$. According to Theorem 4.9 there exists a number C independent of f and ρ such that

$$\begin{aligned} \left\| \widetilde{f^{a,s'}}(\rho) \right\|^2_{L^2(\mathbf{R}^n)} &= \frac{1}{a^2} \int_{\mathbf{R}^n} \left| \int_{\Sigma^{n-1}} e^{ix\xi'} \, f_\rho(\xi')\, d\sigma(\xi') \right|^2 \frac{dx}{|x|^b}\, \rho^{n/a-1} \leq \\ &\leq C \, \|f_\rho\|^2_{L^2(\Sigma^{n-1})} \, \rho^{n/a-1}. \end{aligned}$$

Integrating with respect to ρ completes the proof. □

5.4 Proof of Theorem 2.8 (a)**.** According to Theorem 3.3 and 2.6 the conclusion of the theorem holds for $\theta \in \{0, 1\}$. We would now like to combine Theorem 3.7 and 3.8 to get the conclusion also for $0 < \theta < 1$.

5.4.1 *Inner interpolation in the left hand side.* Choose $A_0 = A_1 = \mathbf{R}$, $q_0 = 2 + 2a/n$, $q_1 \geq 2$, $v_0(t) = 1$ and $v_1(t) = |t|^{-\gamma q_1}$ in Theorem 3.7 to obtain

$$\left(L^{q_0}_{v_0}(\mathbf{R}), L^{q_1}_{v_1}(\mathbf{R})\right)_{[\theta]} \;=\; L^{q_\theta}_{v_\theta}(\mathbf{R})$$

with

$$\frac{1}{q_\theta} \;=\; \frac{n(1-\theta)}{2(n+2)} + \frac{\theta}{q_1} \quad \text{and} \quad v_\theta(t) = |t|^{-\gamma q_\theta \theta}.$$

5.4.2 *Outer interpolation in the left hand side.* Now choose $A_j = L^{q_j}_{v_j}(\mathbf{R})$, $j \in \{0,1\}$, $q_0 = 2+2a/n$, $q_1 \geq 2$, $v_0(t) = 1$ and $v_1(t) = |t|^{-\gamma q_1}$. Furthermore choose $p_0 = q_0$, $p_1 = 2$, $w_0(x) = 1$ and $w_1(x) = |x|^{-b}$. For these choices of A_j, p_j and w_j apply Theorem 3.6 again to obtain

$$\big(L^{p_0}_{w_0}(L^{q_0}_{v_0}(\mathbf{R})), L^{p_1}_{w_1}(L^{q_1}_{v_1}(\mathbf{R}))\big)_{[\theta]} \;=\; L^{p_\theta}_{w_\theta}(L^{q_\theta}_{v_\theta}(\mathbf{R}))$$

with

$$\frac{1}{p_\theta} \;=\; \frac{n(1-\theta)}{2(n+2)} + \frac{\theta}{2} \quad \text{and} \quad w_\theta(x) \;=\; |x|^{-bp_\theta\theta/2}.$$

5.4.3 *Conclusion.* Finally, combine §5.4.1 and 5.4.2 with Theorem 3.8 to obtain the conclusion of our theorem. □

5.5 Lemma. ([**29**, Lemma 3.10, p. 388]) *Let $b > 1$. Then there exists a number C independent of $\rho \geq 1$ and k such that*

$$\int_\rho^\infty J_{\nu(k)}(r)^2\, r^{1-b}\, dr \;\leq\; C\, \rho^{1-b}.$$

5.6 Sketch of proof of Theorem 2.14(a). In a way very similar to the proof of [**29**, Theorem 2.1(a) and 2.2(a), p. 385] our theorem can be reduced to the following $L^1(\mathbf{R}_+)$-estimate: **There exists a number C independent of $f_0 \geq 0$ such that**

$$\int_0^\infty \int_0^\infty J_{\nu(k)}(r)^2\, \psi(\rho)\, f_0(\rho)\, \rho^{b-a-2s}\, r\, \frac{dr\, d\rho}{(\rho+r)^b} \leq C \int_0^\infty \psi(\rho)\, f_0(\rho)\, d\rho.$$

(Here $b > 1$ and $s = (1-a)/2$. The class of testfunctions for f_0 may be chosen to be e.g. $\mathcal{C}^\infty_0(\mathbf{R}_+)$.) We have

$$\int_0^\infty \int_0^\infty J_{\nu(k)}(r)^2\, \psi(\rho)\, f_0(\rho)\, \rho^{b-a-2s}\, r\, \frac{dr\, d\rho}{(\rho+r)^b} \;\leq$$
$$\leq \int_0^\infty \psi(\rho)\, f_0(\rho)\, \rho^{b-1} \left(\int_0^\infty J_{\nu(k)}(r)^2\, r\, \frac{dr}{(\rho+r)^b} \right) d\rho.$$

Here we use ρ to split the integration with respect to r into two pieces and use Theorem 4.10 for $0 \leq r \leq \rho$ and Lemma 5.5 for $r \geq \rho$ to conclude that the inner integral in the right hand side is bounded from above by a number C independent of k. □

5.7 Proof of Theorem 2.14(b)**.** By applying the assumption to a radial function f it follows that there exists a number C independent of $f_0 \geq 0$ such that

$$\int_0^\infty \int_0^\infty J_{\nu(0)}(r)^2\, \psi(\rho)\, f_0(\rho)\, \rho^{b-a-2s}\, r\, \frac{dr\, d\rho}{(\rho+r)^b} \;\leq C \int_0^\infty \psi(\rho)\, f_0(\rho)\, d\rho. \tag{5.3}$$

The assumption $\operatorname{supp}\widehat{f} \subseteq 4B^n \setminus \overline{B^n}$ on the testfunctions translates into $\operatorname{supp} f_0 \subseteq$ $\subseteq (1,4)$. We may impose further restrictions on f_0, e.g. that f_0 is non-negative and $f_0([2,3]) = \{1\}$. Under this assumption it follows that the integral

$$\int_2^3 \rho^{b-a-2s} \left(\int_0^\infty J_{\nu(0)}(r)^2 \, r \, \frac{dr}{(\rho+r)^b} \right) d\rho$$

is convergent. Hence the integral

$$\int_0^\infty J_{\nu(0)}(r)^2 \, r \, \frac{dr}{(\rho+r)^b}$$

is convergent for every $\rho \in [2,3]$. In view of the asymptotics of Bessel functions in Theorem 4.12 this implies that $b > 1$. □

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