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ON ALMOST SURE CONVERGENCE WITHOUT THE RADON-NIKODYM PROPERTY

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Abstract. In this paper we obtain almost sure convergence theorems for vector-valued uniform amarts and C-sequences without assuming the Radon-Nikodym Property. Specifically, it is shown that if a limit exists in a weak sense for these martingale generalizations, then a.s. scalar and strong convergence follow. These results lead to some new versions of the Ito-Nisio theorem. Convergence results for random sequences taking values in a weakly compact space are also presented.

1. Introduction

Convergence theorems for various classes of martingale generalizations taking values in a Banach space E are obtained in general under the assumption that E possesses the Radon-Nikodym Property (RNP). Without assuming the latter property, Marraffa (1988) showed that a.s. scalar convergence of an E-valued strong amart $(X_n, n \in N)$ to an (E-valued) random variable X holds if there exists a total subset D of E^*, dual of E, such that for any $x' \in D$, $x' \circ X_n$ converges a.s. to $x' \circ X$. Davis et al. (1990) established strong a.s. convergence for martingales under the same assumptions. The purpose of this paper is to extend the results of these authors to the uniform weak amarts of Schmidt (Gut and Schmidt (1983)), the uniform amarts of Bellow (1978), and to weak and strong C-sequences (Tuyẽn (1981), Bouzar (1991)). As a consequence, versions of the Ito-Nisio theorem (Ito-Nisio (1968)) for uniform amarts and strong C-sequences are derived. Some related convergence results for random sequences taking values in a weakly compact space are also obtained. The paper is organized as follows. In the remainder of the section we recall a few definitions and results. In Section 2, convergence results for uniform amarts and uniform weak amarts are given. C-sequences are the object of Section 3. In Section 4, we discuss the case of random sequences taking values in a weakly compact space.

Throughout the paper, let $(E, \|.\|)$ be a Banach space and E^* its dual. A subset D of E^* is said to be a total set over E if $x'(x) = 0$ for each $x' \in D$ implies $x = 0$.

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A subset I of the unit ball of E^* is said to be norming for E if for each $x \in E$, $\|x\| = \sup\{|x'(x)| : x' \in I\}$.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(\mathcal{F}_n, n \in N)$ an increasing sequence of sub-σ-algebras of $\mathcal{F}$. We denote by $\mathcal{F}_\infty$ the σ-algebra generated by $\bigcup_{n \in N} \mathcal{F}_n$ and by T the set of bounded stopping times for $(\mathcal{F}_n, n \in N)$. Let $T(n) = \{\tau \in T : \tau \geq n\}$, $n \in N$. For each $\tau \in T$, we associate the σ-algebra $\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty : A \cap [\tau = n] \in \mathcal{F}_n, \forall n \in N\}$.

A random variable (*rv*) is any mapping X from Ω into E that is strongly $\mathcal{F}$-measurable. Unless specified otherwise, all the *rv*'s in the sequel are E-valued. A sequence $(X_n, n \in N)$ of *rv*'s is said to converge scalarly a.s. to the *rv* X if for any $x' \in E^*$, $\lim_n x' \circ X_n = x' \circ X$ a.s. It is said to converge weakly a.s. to X if it converges scalarly to X outside a null set independent of the functionals x'. Note that if E^* is separable, then scalar convergence and weak convergence are equivalent.

An adapted sequence $(X_n, \mathcal{F}_n, n \in N)$ of Bochner integrable *rv*'s is said to be

i) a **strong** (**resp. weak**) **amart** if the net $(E(X_\tau), \tau \in T)$ converges strongly (**resp.** weakly) in E;

ii) a **uniform amart** if $\limsup_{\tau \in T\, \sigma \geq \tau} E(\|E(X_\sigma|\mathcal{F}_\tau) - X_\tau\|) = 0$;

iii) a **uniform weak amart** if it is a weak amart such that for any $A \in \bigcup_{n \in N} \mathcal{F}_n$, $E(X_n I_A)$ converges weakly in E;

iv) a **weak sequential amart** if $(E(X_{\tau_n}), n \in N)$ converges weakly for each increasing sequence $(\tau_n, n \in N)$ in T;

v) a **strong (resp. weak) C-sequence** if its predictable compensator $(\widetilde{X}_n, n \in N)$ (of the Doob decomposition of $(X_n, n \in N)$) converges strongly (resp. weakly) a.s. to a *rv* $\widetilde{X}$.

Finally, an adapted sequence $(X_n, \mathcal{F}_n, n \in N)$ of Bochner integrable *rv*'s is said to be of class (B) if $\sup_{\tau \in T} E(\|X_\tau\|) < \infty$. It is said to satisfy condition (I) if for every stopping time σ, $\int_{[\sigma < \infty]} \|X_\sigma\|\, dP < \infty$.

2. Convergence of Uniform Amarts

We start out with the case of uniform weak amarts.

Proposition 2.1. *Let $(X_n, n \in N)$ be a uniform weak amart of class (B). Assume that there exist a rv X and a total subset D of E^* such that for each $x' \in D$, $(x' \circ X_n, n \in N)$ converges a.s. to $x' \circ X$. Then $(X_n, n \in N)$ converges scalarly to X.*

Proof. Since $(X_n, n \in N)$ and X are a.s. separably valued, we may and do assume that E is separable. It can be easily seen that $(X_{\sigma \wedge n}, \mathcal{F}_{\sigma \wedge n}, n \in N)$ is a uniform weak amart. Therefore, by the maximal inequality of Chacon and Sucheston (1975) and a classical stopping time argument, we can assume that $E(\sup_n \|X_n\|) < \infty$. This implies that the (finitely additive) set function

$$\mu(A) = \lim_{n \to \infty} E(X_n I_A), \quad A \in \bigcup_{n \in N} \mathcal{F}_n$$

is absolutely continuous and is of bounded variation. Its extension to $\mathcal{F}_\infty$, which we also denote by μ, satisfies

$$\mu(A) = \lim_{n\to\infty} E(X_n I_A), \quad A \in \mathcal{F}_\infty.$$

Letting $A_k = [\|X\| \leq k]$, $k \in N$, and noting that XI_{A_k} is Bochner integrable, it follows that for any $x' \in D$

$$x' \circ \mu(A_k) = \lim_{n\to\infty} E((x' \circ X_n)I_{A_k}) = E((x' \circ X)I_{A_k}) = x'(E(XI_{A_k})).$$

Since D is total, we have

$$\mu(A_k) = E(XI_{A_k}), \quad k \in N$$

which implies that

$$E(\|X\|I_{A_k}) = \|\mu\|(A_k) \leq E(\sup_n \|X_n\|)$$

which in turn implies that X is Bochner integrable. Repeating the same argument as above, we also have

$$\mu(A) = E(XI_A), \quad A \in \mathcal{F}_\infty.$$

Now for each $x' \in E^*$, $(x' \circ X_n, n \in N)$ is an L^1-dominated, real-valued amart, therefore $x' \circ X_n$ converges a.s. and for $A \in \mathcal{F}_\infty$

$$E((x' \circ X)I_A) = x'(E(XI_A)) = \lim_{n\to\infty} E((x' \circ X_n)I_A) = E(\lim_{n\to\infty}(x' \circ X_n)I_A)$$

which implies that $x' \circ X = \lim_{n\to\infty} x' \circ X_n$. □

Since weak sequential amarts and strong amarts are themselves uniform weak amarts, we have

Corollary 2.2. *Let $(X_n, n \in N)$ be a weak sequential amart (***resp.*** a strong amart) of class (B). Assume that there exist a rv X and a total subset D of E^* such that for each $x' \in D$, $(x' \circ X_n, n \in N)$ converges a.s. to $x' \circ X$. Then $(X_n, n \in N)$ converges scalarly a.s. to X.*

For uniform amarts the conclusions will be shown to be stronger. We begin with the case of uniform amarts taking values in the dual of a normed space.

Proposition 2.3. *Let F be a normed space and let $(X_n, n \in N)$ be an L^1-bounded uniform amart with values in F^*. Assume that there exists an F^*-valued rv X such that $x \circ X_n$ converges to $x \circ X$ a.s. for each $x \in F$. Then $(X_n, n \in N)$ converges strongly a.s. to X.*

Proof. Since $(X_n, n \in N)$ and X are a.s. separably valued, we may assume (by possibly passing to subspaces) that F^* and hence F are separable. There exists therefore a countable dense subset I of $\{x \in F : \|x\| = 1\}$ that norms F^*. Since L^1-boundedeness of $(X_n, n \in N)$ and the class (B) property are equivalent (see Bellow (1978)), the maximal inequality and a stopping time argument allow us again to reduce the proof to the case where $E(\sup_n \|X_n\|) < \infty$. The conclusion follows then immediately from Proposition 1 of Bellow (1978). □

It must be noted that Davis et al. (1990) obtained Proposition 2.3 for martingales by using the submartingale lemma of Neveu (1975) and a renorming theorem of Davis and Johnson (1973). In the case of uniform amarts we needed a different proof as Neveu's result is unapplicable.

Proposition 2.4. *Let $(X_n, n \in N)$ be an L^1-bounded uniform amart. Let X be a rv. Then the set $Y = \{x' \in E^* : \lim_n x' \circ X_n = x' \circ X \text{ a.s.}\}$ is a weak*-closed linear subspace of E^*.*

Proof. As before, we will assume without loss of generality that E is separable. It can be easily seen that Y is a linear subspace of E^*. By the Krein-Smulyan theorem, we only need to prove that the unit ball of Y, $B = \{x' \in Y : \|x'\| \le 1\}$, is weak*-closed in E^*. Let x' be in the weak*-closure of B and let $(x'_n, n \in N)$ be a sequence in B that weak*-converges to x'. Such a sequence exists since E is separable. Denote by F the linear subspace (of E^*) generated by $(x'_n, n \in N)$ and let S be the canonical map from E to F^*. From the inequality $\|S(x)\|_{F^*} \le \|x\|_E$ it can be deduced that $\|S \circ X\|_{L^1(F^*)} \le \|X\|_{L^1(E)}$ for a Bochner integrable *rv* X. Likewise, for an E-valued (finitely additive) measure of bounded variation ν on an algebra, we have $\|S \circ \nu\|(\cdot) \le \|\nu\|(\cdot)$. For $\tau \in T$ and $A \in \mathcal{F}_\tau$, let $\mu_\tau(A) = E(X_\tau I_A)$ and let μ be the limiting measure of $(\mu_\tau, \tau \in T)$ defined on $\bigcup_n \mathcal{F}_n$ (Bellow (1978)). Then for each $\tau \in T$,

$$\|S \circ \mu_\tau - ((S \circ \mu)|\mathcal{F}_\tau)\| = \|S \circ (\mu_\tau - (\mu|\mathcal{F}_\tau))\| \le \|\mu_\tau - (\mu|\mathcal{F}_\tau)\|$$

which implies that $(S \circ X_n, n \in N)$ is an L^1-bounded uniform amart with limiting measure $S \circ \mu$. Now, for each $k \in N$, $x'_k \circ X_n$ converges a.s. to $x' \circ X$. Applying Proposition 2.3 to $(S \circ X_n, n \in N)$, we deduce that $\lim_n \|S \circ X_n - S \circ X\|_{F^*} = 0$ which implies that $\lim_n x'_k \circ (X_n - X) = 0$ uniformly in k. This in turn implies that $\lim_n x' \circ X_n = x' \circ X$ and hence $x' \in B$. $\square$

The main convergence result for uniform amarts follows next.

Proposition 2.5. *Let $(X_n, n \in N)$ be an L^1-bounded uniform amart. Assume that there exist a rv X and a total subset D of E^* such that for each $x' \in D$, $(x' \circ X_n, n \in N)$ converges a.s. to $x' \circ X$. Then $(X_n, n \in N)$ converges strongly a.s. to X.*

Proof. We have $D \subseteq H \subseteq \{x' \in E^* : \lim_n x' \circ X_n = x' \circ X \text{ a.s.}\} \subseteq E^*$, where H is the smallest linear subspace in E^* that contains D. Since D is total in E^*, so is H, and hence H is weak*-dense. From Proposition 2.4 it follows that $\lim_n x' \circ X_n = x' \circ X$ a.s. for every $x' \in E^*$. The conclusion is then obtained from Proposition 2.3. $\square$

Doob's local convergence theorem is shown to extend easily to uniform amarts.

Corollary 2.6. *Let $(X_n, n \in N)$ be a uniform amart such that $E(\sup_n \|X_{n+1} - X_n\|) < \infty$. Assume that there exist a rv X and a total subset D of E^* such that for each $x' \in D$, $(x' \circ X_n, n \in N)$ converges a.s. to $x' \circ X$. Then $(X_n, n \in N)$ converges strongly a.s. to X in the set $\{\omega \in \Omega : \sup_n \|X_n(\omega)\| < \infty\}$.*

Proof. For $a > 0$, we define the stopping time

$$\sigma_a(\omega) = \begin{cases} \inf\{n : \|X_n(\omega)\| > a\}, & \text{if } \sup_n \|X_n(\omega)\| > a, \\ +\infty, & \text{otherwise.} \end{cases}$$

We have $E(\|X_{\sigma_a \wedge n}\|) \leq a + E(\sup_n(\|X_{n+1} - X_n\|)$. By the optional stopping theorem of Bellow (1978), the uniform amart $(X_{\sigma_a \wedge n}, n \in N)$ is L^1-bounded. Moreover, it can be easily seen that for each $x' \in D$, $\lim_n x' \circ X_{\sigma_a \wedge n} = x' \circ Y$ for some *rv* Y. Therefore, $(X_{\sigma_a \wedge n}, n \in N)$ converge a.s. by Proposition 2.5. The conclusion follows by noting that $\bigcup_{a>0}[\sigma_a = +\infty] = [\sup_n \|X_{n+1} - X_n\| < \infty]$. □

We conclude the section by proving the uniform amart version of the Ito-Nisio theorem. A lemma that may be of independent interest is obtained first.

Lemma 2.7. *Let H be a linear subspace of E^* and let $(X_n, n \in N)$ be a sequence of rv's of class (B). Let S be the canonical injection from E to H^*. Further, assume that for any $x' \in H$, the sequence $(x' \circ X_n, n \in N)$ converges a.s. Then there exists a weak*-measurable, H^*-valued rv ϕ such that for each $x' \in H$, $x' \circ X_n$ converges a.s. to $\phi(x')$.*

Proof. By the maximal inequality of Chacon and Sucheston (1975), $\sup_n \|X_n\| < \infty$ a.s. and hence $\sup_n \|S(X_n)\| < \infty$ a.s. The weak*-compactness of the unit ball in H^* implies that for almost every $\omega \in \Omega$, $S(X_n(\omega))$ has a weak*-limit point $\phi(\omega)$ in H^* which is weak*-measurable. The conclusion follows by noting that $S(X_n)(x') = x' \circ X_n$. □

Proposition 2.8. *Assume E separable and let H be a total subspace of E^*. Let $(X_n, n \in N)$ be an L^1-bounded uniform amart. The following assertions are equivalent:*

(1) *X_n converges a.s.*
(2) *X_n converges in distribution.*
(3) *For almost all $\omega \in \Omega$, $(X_n(\omega), n \in N)$ has a cluster point in the topology $\sigma(E, H)$.*
(4) *There is a distribution μ on E such that for each $x' \in H$, $x' \circ X_n$ converges in distribution to $x' \circ \mu$.*

Proof. We proceed as in Davis et al. (1990). The implications (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (4), and (2) $\Rightarrow$ (3) are true in general, and the details are omitted. Next we prove that (3) $\Rightarrow$ (1). Let S be the canonical injection from E to H^* and note that for each x' in H, $(x' \circ X_n, n \in N)$ is a real valued, L^1-bounded amart, and hence converges a.s. By (3) and Lemma 2.7 (recall again that for uniform amarts L^1-boundedness and the class (B) property are equivalent), $S(X) = \phi$ a.s. where ϕ is H^*-valued and weak*-measurable. Since E is separable, so is $S(E)$. Therefore, ϕ is almost separably valued and by a well-known theorem of Pettis, it is H^*-strongly measurable. Since S^{-1} is Borel measurable, a theorem of Lusin implies that $X = S^{-1} \circ \phi$ is E-strongly measurable. Since $\phi(x') = x' \circ X$, (1) follows then from Lemma 2.7 and Proposition 2.5. It remains to show that (4) $\Rightarrow$ (1).

By (4) the distribution of ϕ of Lemma 2.7 is equal to $S \circ \mu$ which is tight. This implies that ϕ is almost surely $S(E)$-valued and hence H^*-strongly measurable. Letting $X = S^{-1} \circ \phi$, the conclusion follows then along the same lines as that of (3) $\Rightarrow$ (1). □

Remark. 1. Proposition 2.5 can also be derived from Proposition 2.1 as follows. We assume without loss of generality that E is separable and $E(\sup_n \|X_n\|) < \infty$. A uniform amart is necessarily a weak uniform amart. Hence by Proposition 2.1, X_n converges scalarly to X. Since E is separable, there exists a countable norming subset D (subset of the unit ball of E^*) for E. The conclusion follows again from Proposition 1 in Bellow (1978).

2. Lemma 2.7 states that under some mild assumptions, any random sequence of class (B) has a limit, in a weak*-sense, in the second dual.

3. Convergence of C-Sequences

Proposition 3.1. *Let $(X_n, n \in N)$ be a strong (**resp.** weak) C-sequence that satisfies condition (I). Assume further that there exist a rv X and a total subset D of E^* such that for each $x' \in D$, $(x' \circ X_n, n \in N)$ converges a.s. to $x' \circ X$. Then $(X_n, n \in N)$ converges strongly (**resp.** scalarly) to X.*

Proof. Let $(\widetilde{X}_n, n \in N)$ be the predictable compensator of $(X_n, n \in N)$. It is enough to prove that the martingale $M_n = X_n - \widetilde{X}_n$ converges scalarly a.s. For $a > 0$, we define the stopping time

$$\sigma_a(\omega) = \begin{cases} \inf\{n : \|\widetilde{X}_{n+1}(\omega)\| > a\}, & \text{if } \sup_n \|\widetilde{X}_{n+1}(\omega)\| > a, \\ +\infty, & \text{otherwise.} \end{cases}$$

Then for each $n \in N$, $\|\widetilde{X}_{\sigma_a \wedge n}\| \le a$, which implies that for any stopping time τ

$$\int_{[\tau<\infty]} \|M_{\sigma_a \wedge \tau}\| \, dP \le a + \int_{[\tau<\infty]} \|X_{\sigma_a \wedge \tau}\| \, dP < \infty.$$

This implies (by a result of Schmidt (1979)) that the martingale $(M_{\sigma_a \wedge n}, n \in N)$ is L^1-bounded. It is easy to deduce from the assumptions that $(\widetilde{X}_{\sigma_a \wedge n}, n \in N)$ converges scalarly (in fact strongly in the case of a strong C-sequence). Hence there exists a *rv* Y such that for each $x' \in D$, $\lim_n x' \circ M_{\sigma_a \wedge n} = x' \circ Y$. Applying Proposition 2.5 to $(M_{\sigma_a \wedge n}, n \in N)$, we have that M_n converges strongly a.s. on $[\sigma_a = +\infty]$. This in turn implies that M_n converges strongly on $\bigcup_{a>0} [\sigma_a = +\infty] = [\sup_n \|\widetilde{X}_n\| < \infty]$. The conclusion follows since the latter set has probability 1. □

The Ito-Nisio theorem as stated in Proposition 2.8 extends to C-sequences of class (B) with essentially the same proof. The details are omitted. We simply note that Lemma 2.7 does apply since if $(X_n, n \in N)$ is a C-sequence, then $(x' \circ X_n, n \in N)$ is a real-valued C-sequence of class (B) and hence converges a.s. (Bouzar (1991)).

Remark. 1. Proposition 3.1 remains valid for C-sequences of class (B) since the latter condition implies (I). Condition (I), the class (B) property, and

L^1-boundedness have been shown to be equivalent for martingales, and uniform amarts (Dubins and Freedman (1966), Schmidt (1979), Gut and Schmidt (1983)). They are, however, not equivalent for C-sequences (Bouzar (1991)).

2. Tomkins (1984) showed that L^1-bounded (even uniformly integrable) C-sequences need not converge a.s.

3. Doob's local convergence theorem, as stated in Corollary 2.6, extends to strong strict C-sequences. The proof is a combination of the arguments used in the proofs of Proposition 3.2 and Corollary 2.7, with the following stopping time

$$\sigma_a(\omega) = \inf\{n : \|X_n(\omega)\| > a \text{ or } \|\widetilde{X}_{n+1}(\omega)\| > a\}.$$

4. Random Sequences Taking Values in a Weakly Compact Set

Chatterji (1973) showed that martingales taking values in a weakly compact set of a Banach space converge strongly a.s. (see also Brunel and Sucheston (1976) for a related result on vector-valued amarts.) In this section we obtain a general convergence result for random sequences taking values in a weakly compact set. We then specialize our result to several martingale generalizations.

Proposition 4.1. *Let $(X_n, n \in N)$ be a sequence of rv's taking values in a weakly compact space K of the Banach space E. Suppose moreover that for each $x' \in E^*$, there exists $N_{x'} \in \mathcal{F}$, $P(N_{x'}) = 0$, such that $x' \circ X_n(\omega)$ converges for every $\omega \in \Omega \backslash N_{x'}$. Then X_n converges weakly a.s.*

Proof. We may assume without loss of generality that E is separable. Since K is weakly compact, for each $\omega \in \Omega$, there exists a weak limit point $X(\omega) \in K$ of the sequence $(X_n(\omega), n \in N)$. It follows from the assumptions that for every $x' \in E^*$, $\lim_n x' \circ X_n(\omega) = x' \circ X(\omega)$ outside $N_{x'}$. Since K is separable and E is separable, the weak topology in K is metrizable, and the metric d is determined by a sequence $(x'_j, j \in N)$ in E^*. Moreover, the class of Borel sets $\mathcal{B}(K)$ in (K, d) is the same as in $(K, \|.\|)$ (see, for example, Bellow and Egghe (1982).) Letting $N = \bigcup_j N_{x'_j}$, we have $P(N) = 0$. For every $\omega \in \Omega \backslash N$ and $j \in N$, $\lim_n x'_j \circ X_n(\omega) = x'_j \circ X(\omega)$. This implies that $X_n : (\Omega, \mathcal{F}) \to (K, \mathcal{B}(K))$ converges a.s. in (K, d). We define $X^* : \Omega \to K$ by

$$X^*(\omega) = \begin{cases} \lim\limits_n X_n(\omega) & \text{if the limit exists in } (K,d) \\ a \in K & \text{otherwise.} \end{cases}$$

Since the set $\{\omega : \lim_n X_n(\omega) \text{ exists in } (K, d)\}$ is in $\mathcal{F}$, X^* is measurable as a function from $(\Omega, \mathcal{F})$ to $(K, \mathcal{B}(K))$. Furthermore,

$$\Omega \backslash N \subset \{\omega : \lim_n X_n(\omega) \text{ exists in } (K, d)\}.$$

Therefore X_n converges weakly a.s. to X^*. $\square$

Next, we derive several corollaries.

Corollary 4.2. *Let $(X_n, n \in N)$ be a uniform amart (**resp.** a weak uniform amart) taking values in a weakly compact space K of E. Then $(X_n, n \in N)$ converges strongly (**resp.** weakly) a.s.*

Proof. Since K is weakly compact, K is bounded. There exists therefore $M > 0$ such that for any $x \in K$, $\|x\| \leq M$. It is easy to see that for both strong and weak amarts $(x' \circ X_n, n \in N)$, $x' \in E^*$, is a real-valued amart that is bounded by $\|x\|.M$ and hence converges a.s. (see for example Austin et al. (1974).) The conclusion follows from Proposition 4.1 for uniform weak amarts and from Propositions 4.1 and 2.5 for uniform amarts. ☐

Corollary 4.3. *Let $(X_n, n \in N)$ be a strong (***resp.*** weak) C-sequence taking values in a weakly compact space K of E. Then $(X_n, n \in N)$ converges strongly (***resp.*** weakly) a.s.*

Proof. Let $(\widetilde{X}_n, n \in N)$ be the predictable compensator of $(X_n, n \in N)$. Then for each $x' \in E^*$, $x' \circ \widetilde{X}_n$ is the predictable compensator of $x' \circ X_n$. This implies that $(x' \circ X_n, n \in N)$ is a real-valued C-sequence. Since $x' \circ X_n$ is dominated by a constant, it is necessarily of class (B) and hence converges a.s. The conclusion follows then from Propositions 4.1 and 3.2. ☐

Remark. The results of the previous two sections extend accordingly to quasimartingales (Fisk (1965)) as these are uniform amarts, and to eventual martingales (Tomkins (1975)) as these are C-sequences.

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