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NAMBU-LIE GROUP ACTIONS

N. CICCOLI

Abstract. The purpose of this work is to describe the action of Nambu-Lie groups on Nambu spaces and to identify Nambu quotients. In this process we introduce a notion of dual Lie group of a Nambu-Lie group and as an example we generalize dressing actions.

1. Introduction

In last years much interest has been drawn by manifold supporting an n-ary operation on their function algebra which generalizes Poisson brackets. The first example of a ternary bracket on a linear space was introduced by Y. Nambu, in 1973 ([**11**]), to discuss some properties of integrable Hamiltonian dynamical systems. Recently Takhtajan, in [**12**], proposed a general algebraic definition of Nambu-Poisson brackets of order n. A vector space is a Nambu algebra (Nambu-Takhtajan in some authors) of rank n if it is endowed with an n-linear completely antisymmetric bracket $\{\dots\}$ such that:

$$\{f_1,\dots,f_{n-1},g_1g_2\} = \{f_1,\dots,f_{n-1},g_1\}g_2 + g_1\{f_1,\dots,f_{n-1},g_2\} \tag{1.1}$$

$$\{f_1,\dots,f_{n-1},\{g_1,\dots,g_n\}\} = \sum_{i=1}^{n}\{g_1,\dots,\{f_1,\dots,f_{n-1},g_i\},\dots,g_n\} \tag{1.2}$$

When the algebra of smooth functions on a manifold can be given a Nambu algebra structure the manifold itself is called a Nambu manifold. Such manifolds share many properties with the more studied Poisson manifolds. For example they support a canonically defined (non regular) foliation, composed of 0 and n-dimensional leaves, where n is the rank of the Nambu structure. On higher dimensional leaves the Nambu tensor (i.e. the n-vector Λ such that $\{f_1,\dots,f_n\} = \Lambda(df_1,\dots,df_n)$ defines a volume form ([**10**]). Volume manifolds (i.e. manifolds with a fixed volume form) can then be thought as the analogue, in this context, of symplectic manifolds. Despite these similarities, when $n > 2$ Nambu structures appear to be much more rigid than Poisson ones. Such rigidity can be considered related to the decomposability of every Nambu n-vector of rank greater than 2 (conjectured by Takhtajan and proven in [**1**]). In such context the study of Nambu tensors on Lie

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groups which satisfies reasonable invariance conditions arises quite naturally. Not much literature is available for Nambu-Lie groups of rank greater than 2, much more so if compared with the great amount of papers on Poisson-Lie groups. A review of the main results is [**13**]. Here we would like to show how a relevant part of the theory of Poisson-homogeneous spaces admits a generalization to the Nambu setting. We will show how a suitable analogue of the concept of coisotropic subgroup can be used to reduce Nambu structures and prove that all non regular Nambu-homogeneous spaces can be obtained this way (see [**2**] for the ($n = 2$)-case). We will also briefly deal with the problem of defining a dual group of a Nambu-Lie group, as in the Poisson case, propose a definition and show how it enters in the theory of Nambu actions providing local dressing fields. We propose to pursue this study further on and more work is at present going on in this direction. The author would like to acknowledge Dr. L. Guerra for useful conversations on the subject. He is also grateful to R. Ibanez for pointing out an error in an earlier version of the paper.

2. Nambu-Lie Groups

Let us recall the definition of Nambu-Lie group ([**13**]).

Definition 2.1. A Lie group G with a rank n Nambu tensor P is said to be a Nambu-Lie group if the tensor P is G-multiplicative:

$$P(g_1g_2) = l_{g_1,*}P(g_2) + r_{g_2,*}P(g_1)\,,$$

where $l_{g_1,*}$ and $r_{g_2,*}$ respectively denote the extension to the space of n-tensors of the derivatives of the left and right translation operators.

We will denote with $\sharp_P : \Omega^{n-1}(G) \to \mathfrak{X}(G)$ the linear map defined as:

$$\sharp_P(\alpha_1 \wedge \cdots \wedge \alpha_{n-1}) \lrcorner\, \beta = P(\alpha_1 \wedge \cdots \wedge \alpha_{n-1} \wedge \beta)\,.$$

More details about general properties of Nambu manifolds can be found in [**10**]. In the context of Nambu geometry an analogue of Lie algebras is given as follows:

Definition 2.2. Let V be a vector space together with a linear antisymmetric n-bracket

$$V^n \ni (f_1, \ldots, f_n) \mapsto [f_1, \ldots, f_n] \in V\,.$$

If such bracket verifies the identity

$$[f_1, \ldots, f_{n-1}, [g_1, \ldots, g_n]] = \sum_{i=1}^{n} [g_1, \ldots, [f_1 \ldots, f_{n-1}, g_i], \ldots, g_n] \tag{2.3}$$

for every $g_1, \ldots, g_n, f_1, \ldots, f_{n-1} \in V$ then $(V, [.,.,.])$ is called a Filippov algebra.

More on the algebraic properties of n-brackets can be found in [**4**] and [**5**].

Between Filippov algebras and Nambu-Lie groups one can establish the following relevant connection.

Proposition 2.3 ([**6**]). *Let (G, P) be a Nambu-Lie group and let $\delta_P : \mathfrak{g} \to \wedge^n \mathfrak{g}$ denote the intrinsic derivative $\delta_P(X) = L_X(P)(e)$. The map $\delta_P^* \colon \wedge^n \mathfrak{g}^* \to \mathfrak{g}^*$ defines a Filippov bracket on $\mathfrak{g}^*$. If $\alpha_1, \ldots, \alpha_n$ are in $\mathfrak{g}^*$ and $\alpha_1^l, \ldots, \alpha_n^l$ denote the corresponding left invariant differential 1-forms on G then*

$$[\alpha_1, \ldots, \alpha_n] = \{\alpha_1^l, \ldots, \alpha_n^l\}(e)$$

where in the left hand side we have the n-bracket between differential 1-form defined by P (see [**13**]*).*

We will refer to a pair $(\mathfrak{g}, \delta)$ such that $\mathfrak{g}$ is a Lie algebra and δ^* is a Filippov bracket on the dual as a Filippov-Lie bialgebra.

The n-bracket between differential 1-forms can also be used to verify if a given Nambu tensor on G is multiplicative. If, in fact, G is a connected Lie group and P a Nambu tensor on G such that $P(e) = 0$ then (G, P) is a Nambu-Lie group if and only if for every $\alpha_1, \ldots, \alpha_n \in \Omega^1_{inv}(G)$ we have

$$\{\alpha_1, \ldots, \alpha_n\} \in \Omega^1_{inv}(G)\,. \tag{2.4}$$

The map δ_P appearing in proposition 2.3 belongs to the space of 1-cocycles of the Lie algebra $\mathfrak{g}$ with values in $\wedge^n \mathfrak{g}$ with respect to the usual $ad^{(n)}$ action, i.e. it verifies $ad_X^{(n)}(\delta_P(Y)) - ad_Y^{(n)}(\delta_P(X)) = \delta_P([X, Y])$. The cocycle condition can also be restated as follows: for every X, $Y \in \mathfrak{g}$ and for every $\alpha_1, \ldots, \alpha_n \in \mathfrak{g}^*$

$$\begin{aligned} \langle [\alpha_1, \ldots, \alpha_n], [X, Y] \rangle = \sum_{k=1}^{n} &\langle [\alpha_1, \ldots, ad_Y^* \alpha_k, \ldots, \alpha_n], X \rangle \\ &- \langle [\alpha_1, \ldots, ad_X^* \alpha_k, \ldots, \alpha_n], Y \rangle\,. \end{aligned} \tag{2.5}$$

There are two interesting kind of subgroups of a Nambu-Lie group.

Definition 2.4. A subgroup H of a Nambu-Lie group (G, P) is called a Nambu subgroup if the immersion of H in G is a Nambu-map, i.e. if $P\big|_H$ defines a n-Nambu vector on H. A subgroup H of a Nambu-Lie group (G, P) is said to be coisotropic if for every n functions $f_1, \ldots, f_n \in \mathcal{C}^\infty(G)$ such that $f_i\big|_H = 0$, and for every $i = 1, \ldots, n$

$$\{f_1, \ldots, f_n\}\big|_H = P(df_1, \ldots, df_n)\big|_H = 0\,. \tag{2.6}$$

As an example of Nambu-Lie subgroup we can consider the set G_0 of all those points in which the Nambu tensor is zero.

There exists a more general notion of coisotropic submanifold of a Nambu-Lie manifold introduced in [**8**]: if N is a submanifold of the Nambu manifold (M, P) and we let

$$Ann^j(T_x N) = \big\{\alpha \in \Omega^{n-1}(T_x^* M) \;\big|\; (v_1 \wedge \cdots \wedge v_j) \lrcorner\, \alpha = 0, \quad \forall v_1, \ldots, v_j \in T_x N\big\}$$

we will say that N is j-coisotropic in M if:

$$\sharp_P(Ann^j(T_x N)) \subseteq T_x N\,. \tag{2.7}$$

In particular we can say that Nambu subgroups are 1-coisotropic submanifolds and coisotropic Nambu subgroups are $(n-1)$-coisotropic submanifolds.

If the subgroup H is connected we can detect its properties at the infinitesimal level; it is a Nambu-Lie subgroup iff its Lie algebra $\mathfrak{h}$ is a Nambu-Lie subalgebra, or, in other words, its annihilator $\mathfrak{h}^0$ is a Filippov ideal in $\mathfrak{g}^*$.

Proposition 2.5. *Let (G,P) be a Nambu-Lie group of rank n and let H be a connected subgroup of G with Lie algebra $\mathfrak{h}$. The following are equivalent:*

1. *H is coisotropic;*
2. *$\alpha_1,\dots,\alpha_n \in \mathfrak{h}^0 \Rightarrow [\alpha_1,\dots,\alpha_n] \in \mathfrak{h}^0$, that is to say $\mathfrak{h}^0$ is a Filippov subalgebra of $(\mathfrak{g}^*,[.,\dots,.])$;*
3. *$\delta_P(\mathfrak{h}) \subseteq \mathfrak{h} \wedge (\bigwedge^{n-1}\mathfrak{g})$;*
4. *for every $1 \le j \le n$ and for every $x \in H$, $\sharp_P(Ann^j T_x H) \subseteq T_x H$*

Proof. Let H be coisotropic and connected; $\alpha \in \mathfrak{h}^0$ if and only if $\alpha = d_e f$ where $f \in \mathcal{C}^\infty(G)$, $f\big|_H = 0$. It is then clear that

$$\begin{aligned} \{f_1,\dots,f_n\}\big|_H = 0 \quad &\Longleftrightarrow \quad [\alpha_1,\dots,\alpha_n] \in \mathfrak{h}^0 \\ &\Longleftrightarrow \quad \langle [\alpha_1,\dots,\alpha_n], X\rangle = 0 \quad \forall X \in \mathfrak{h} \\ &\Longleftrightarrow \quad \langle \alpha_1 \wedge \dots \wedge \alpha_n, \delta_P(X)\rangle = 0 \quad \forall X \in \mathfrak{h} \\ &\Longleftrightarrow \quad \delta(X) \in (\alpha_1 \wedge \dots \wedge \alpha_n)^0 \quad \forall X \in \mathfrak{h} \end{aligned}$$

from which all claimed equivalences follow. □

Examples. 1. Let us remark that every subgroup such that $\dim \mathfrak{h}^0 < \operatorname{rank} P$ is coisotropic. In particular all subgroups of Nambu-Lie group of maximal rank are coisotropic.

2. Another relevant example of coisotropic subgroup of a given Nambu-Lie group is its **core subgroup**. If we let $\mathfrak{p}$ be equal to the **core** ideal of $\mathfrak{g}$ ([**13**]) a simple case-by-case analysis proves that $\mathfrak{p}$ verifies the third condition of Proposition 2.5.

The connections which the more general j-coisotropic subgroups, which we will not use in the following, can be easily proven.

Proposition 2.6. *Let H be a closed connected subgroup of a Nambu-Lie group (G,P). The following are equivalent:*

1. *H is j-coisotropic;*
2. *for every $\alpha_1,\dots,\alpha_{n-j+1} \in \mathfrak{h}^0$ e $\beta_1,\dots,\beta_{j-1} \in \mathfrak{g}^*$*
$$[\alpha_1,\dots,\alpha_{n-j+1},\beta_1,\dots,\beta_{j-1}] \in \mathfrak{h}^0\,.$$

Proof. Using translation operators to identify tangent spaces in different points to the identity tangent space one has:

$$Ann^j(T_x H) \simeq (\wedge^{n-j}\mathfrak{h}^0)\bigwedge(\wedge^{j-1}\mathfrak{g}^*)\,.$$

With such identification the conditions

$$\sharp_P\big((\wedge^{n-j}\mathfrak{h}^0)\bigwedge(\wedge^{j-1}\mathfrak{g}^*)\big) \subseteq \mathfrak{h}$$

and

$$\alpha_1,\dots,\alpha_{n-j+1} \in \mathfrak{h}^0 \Rightarrow [\alpha_1,\dots,\alpha_{n-j+1},\beta_1,\dots,\beta_{j-1}] \in \mathfrak{h}^0$$

for every $\beta_1,\dots,\beta_{j-1} \in \mathfrak{g}^*$, are dual of each other. □

Remark. As in the Poisson case ([**2**]) the correspondence between subgroups and homogeneous spaces breaks down due to the fact that every subgroup conjugated to a connected Nambu subgroup of a Nambu-Lie group is no longer Nambu.

3. The Dual of a Nambu-Lie Group

The purpose of this section is to construct for any given Nambu-Lie group a Lie algebra structure generalizing the usual dual Lie algebra for Poisson-Lie groups. Unfortunately the theory seems less meaningful than in the Poisson case. Nevertheless, as we will see, dual Lie algebras of a Nambu-Lie group will play a role in analyzing actions of Nambu-Lie groups.

Definition 3.1. Given a Nambu-Lie group (G, P) with associated Filippov-Lie bialgebra $(\mathfrak{g}^*, \delta^*)$ we define the following bracket in $\wedge^{n-1}\mathfrak{g}^*$:

$$\llbracket \eta_1 \wedge \dots \eta_{n-1}, \xi_1 \wedge \dots \wedge \xi_{n-1} \rrbracket = \sum_{j=1}^{n-1} \xi_1 \wedge \dots [\eta_1, \dots, \eta_{n-1}, \xi_j] \wedge \dots \xi_{n-1} \,. \tag{3.8}$$

Proposition 3.2. *With the bracket $\llbracket , \rrbracket$ just defined $\wedge^{n-1}\mathfrak{g}^*$ is a (left) Leibniz algebra.*

Proof. We will prove that

$$\begin{aligned} &\llbracket \eta_{(n-1)}, \llbracket \xi_{(n-1)}, \chi_{(n-1)} \rrbracket \rrbracket \\ &\quad - \llbracket \llbracket \eta_{(n-1)}, \xi_{(n-1)} \rrbracket, \chi_{(n-1)} \rrbracket - \llbracket \xi_{(n-1)}, \llbracket \eta_{(n-1)}, \chi_{(n-1)} \rrbracket \rrbracket = 0. \end{aligned} \tag{3.9}$$

We compute separately the three summands of (3.9), call them a, b, c in what follows.

$$\begin{aligned} a = &\sum_{j>i=1}^{n-1} \chi_1 \wedge \dots [\eta_{(n-1)}, \chi_j] \dots [\xi_{(n-1)}, \chi_i] \dots \wedge \chi_{n-1} \\ &+ \sum_{i=1}^{n-1} \chi_1 \wedge \dots [\eta_{(n-1)}, [\xi_{(n-1)}, \chi_i]] \dots \wedge \chi_{n-1} \\ &+ \sum_{i>j=1}^{n-1} \chi_1 \wedge \dots [\xi_{(n-1)}, \chi_i] \dots [\eta_{(n-1)}, \chi_j] \dots \wedge \chi_{n-1} \end{aligned}$$

$$b = \sum_{i=1}^{n-1} \chi_1 \wedge \dots [\llbracket \eta_{(n-1)}, \xi_{(n-1)} \rrbracket, \chi_j] \dots \wedge \chi_{n-1}$$

$$
\begin{aligned}
c = & \sum_{j>i=1}^{n-1} \chi_1 \wedge \ldots [\xi_{(n-1)}, \chi_j] \ldots [\eta_{(n-1)}, \chi_i] \ldots \wedge \chi_{n-1} \\
& + \sum_{i=1}^{n-1} \chi_1 \wedge \ldots [\xi_{(n-1)}, [\eta_{(n-1)}, \chi_i]] \ldots \wedge \chi_{n-1} \\
& + \sum_{i>j=1}^{n-1} \chi_1 \wedge \ldots [\eta_{(n-1)}, \chi_i] \ldots [\xi_{(n-1)}, \chi_j] \ldots \wedge \chi_{n-1}.
\end{aligned}
$$

The first and third term of a cancel respectively with the third and first term of c. For the remaining summands the fundamental identity (2.3) is enough to prove the claim. □

This bracket is not, in general, antisymmetric as the following example proves: let (G, P) be the generalized Nambu-Heisenberg group $H(1, p)$ as defined in [**13**], $p \geq 2$. Then left invariant 1-forms on the group are spanned by

$$dx_1, \ldots, dx_p, dy, dz_1 - x_1 dy, \ldots, dz_p - x_p dy$$

and the Nambu tensor is

$$P = y \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial y}.$$

A direct computation allows to prove that

$$
\begin{aligned}
&[\![dy \wedge (dz_1 - x_1 dy), dx_1 \wedge dx_2]\!] = dx_2 \wedge dy \\
&[\![dx_1 \wedge dx_2, dy \wedge (dz_1 - x_1 dy)]\!] = 0
\end{aligned}
$$

so that the bracket $[\![,]\!]$ is not antisymmetric. On the other hand we remark that $\wedge^2 Lie(H(1,1))^*$ is a Lie algebra.

The following standard construction (see [**9**]) allows to construct a Lie algebra from any Leibniz algebra: consider the subspace

$$\mathfrak{k}_n = \{[\![\xi, \xi]\!] \quad | \quad \xi \in \wedge^{n-1} \mathfrak{g}^*\}$$

then $\wedge^{n-1}\mathfrak{g}^*/\mathfrak{k}_n$ is a Lie algebra. It will be called the **Nambu-Lie dual** of $\mathfrak{g}$.

We will call $(\wedge^{n-1}\mathfrak{g}^*/\mathfrak{k}_n, [\![,]\!])$ the tangent Lie dual of the given Nambu-Lie group. The connected simply-connected group G^* integrating such Lie algebra will be called **the dual group**. Let us remark that when $n = 2$ we obtain the usual Poisson-Lie duals. This is slightly different from the construction in [**3**], where the Lie quotient considered has always trivial center and, thus, does not restrict to the Poisson dual in the $(n = 2)$-case. The universal property of $\wedge^{n-1}\mathfrak{g}^*/\mathfrak{k}_n$, or more trivially a straightforward remark from definitions, allows to prove that the Daletskij-Takhtajan Lie algebra is a quotient of the Nambu-Lie dual.

Remark. The bracket defined in (3.8) can be extended to the space of $(n-1)$-forms on the group G by letting:

$$
\begin{aligned}
&[\![f\alpha^l, \beta^l]\!] = f[\![\alpha, \beta]\!] + \sharp_P \alpha^l(f)\beta^l \\
&[\![\alpha^l, f\beta^l]\!] = f[\![\alpha, \beta]\!] - i_P(\sharp_P \alpha^l)(df \wedge \beta^l)
\end{aligned}
$$

for every $f \in \mathcal{C}^\infty(G)$ and for every left invariant $(n-1)$-forms α^l, β^l with values $\alpha, \beta \in \wedge^{n-1}\mathfrak{g}^*$ in the identity.

Remark. In terms of the Nambu-Lie dual the infinitesimal characterization of the Nambu properties of subgroups of G can be given as follows

1. H is a connected Nambu-Lie subgroup of $G \Rightarrow \wedge^{n-1}\mathfrak{h}^*$ is a (left) Leibniz ideal in $\wedge^{n-1}\mathfrak{g}^*$.
2. H is a connected coisotropic subgroup of $G \iff \wedge^{n-1}\mathfrak{h}^*$ is a Leibniz subalgebra in $\wedge^{n-1}\mathfrak{g}^*$.

From this remark it is obvious that if $codim\mathfrak{h} \leq n-2$ then $\wedge^{n-1}\mathfrak{h}^\perp = 0$ is a Leibniz left ideal and so $\mathfrak{h}$ corresponds to a Nambu-Lie subgroup; furthermore if $codim\mathfrak{h} = n-1$ then $\wedge^{n-1}\mathfrak{h}^\perp$ is a 1-dimensional subspace and, thus, a Lie subalgebra, which proves that $\mathfrak{h}$ integrates to a coisotropic subgroup.

Corollary 3.3. *Every subgroup of a maximal rank Nambu-Lie group is coisotropic.*

Whichever is the rank of P we can also define an action of $\wedge^{n-1}\mathfrak{g}^*$ on $\mathfrak{g}$ as follows

$$\langle \phi, ad^*_\xi X\rangle = -\langle [\xi, \phi], X\rangle \tag{3.10}$$

where $\xi \in \wedge^{n-1}\mathfrak{g}^*$ and $\phi \in \mathfrak{g}^*$ (so that the bracket on the right hand side should be intrepretated as $\sum c_{i_1\ldots i_{n-1}}[\xi_{i_1}, \ldots, \xi_{i_{n-1}}, \phi]$). Remark that this action shouldn't be confused with the coadjoint action of a Lie algebra on its dual (and thus on the wedge products of the dual); we will denote the coadjoint action with *coad*.

Lemma 3.4. *The map ad^* defines a Leibniz algebra representation of $\wedge^{n-1}\mathfrak{g}^*$ on $\mathfrak{g}$.*

Proof. Let ξ, η be in $\wedge^{n-1}\mathfrak{g}^*$ and prove that for every $\phi \in \mathfrak{g}^*$

$$0 = \langle \phi, ad^*_{[\![\xi,\eta]\!]}X - ad^*_\eta ad^*_\xi X + ad^*_\xi ad^*_\eta X\rangle \tag{3.11}$$

The right hand side of (3.11) equals

$$-\langle [[\![\xi,\eta]\!], \phi] - [\xi, [\eta, \phi]] + [\eta, [\xi, \phi]], X\rangle$$

which is zero due to the fundamental identity of Filippov algebras (2.3). The last statement of the lemma is trivial. □

This representation factors through the two sided ideal $\mathfrak{k}_n$ to give a Lie algebra representation of $\wedge^{n-1}\mathfrak{g}^*/\mathfrak{k}_n$ on $\mathfrak{g}$.

Corollary 3.5. *The cocycle condition* (2.5) *is satisfied if and only if*

$$\begin{aligned} &\langle \phi, ad^*_\alpha[X,Y] - [Y, ad^*_\alpha X] + [X, ad^*_\alpha Y]\rangle \\ &= \langle [ad^*_Y\alpha, \phi], X\rangle - \langle [ad^*_X\alpha, \phi], Y\rangle \end{aligned} \tag{3.12}$$

for every $\alpha \in \wedge^{n-1}\mathfrak{g}^*$, $\phi \in \mathfrak{g}^*$, $X, Y \in \mathfrak{g}$. Equivalently

$$ad^*_\alpha[X,Y] - [X, ad^*_\alpha Y] + [Y, ad^*_\alpha X] = ad^*_{ad^*_Y\alpha}X - ad^*_{ad^*_X\alpha}Y\,.$$

Proof. Just rewrite (2.5) for $\alpha = \alpha_1 \wedge \ldots \wedge \alpha_{n-1}$ and $\phi = \alpha_n$. Then:

$$\begin{aligned} \langle[\alpha,\phi],[X,Y]\rangle &= \langle[ad^*_Y\alpha,\phi]+[\alpha,ad^*_Y\phi],X\rangle \\ &\quad - \langle[ad^*_X\alpha,\phi]+[\alpha,ad^*_X\phi],Y\rangle \\ \Longleftrightarrow\ \langle\phi,ad^*_\alpha[X,Y]\rangle &= \langle[ad^*_Y\alpha,\phi],X\rangle + \langle ad^*_Y\phi, ad^*_\alpha X\rangle \\ &\quad - \langle[ad^*_X\alpha,\phi],Y\rangle - \langle ad^*_X\phi, ad^*_\alpha Y\rangle \\ \Longleftrightarrow\ 0 &= \langle\phi, ad^*_\alpha[X,Y]-[Y,ad^*_\alpha X]+[X,ad^*_\alpha Y]\rangle \\ &\quad + \langle[ad^*_Y\alpha,\phi],X\rangle - \langle[ad^*_X\alpha,\phi],Y\rangle \end{aligned}$$

This last equality, being verified for every $\phi \in \mathfrak{g}^*$ implies (3.13). □

4. Nambu Actions

Let M be a G-manifold with respect to the action $\phi\colon G\times M \to M$. For every $g \in G$ we will denote with $\phi_g\colon M \to M$ the map $\phi_g(x) = \phi(g,x)$ and for every $x \in M$ we will denote with $\phi_x\colon G \to M$ the map $\phi_x(g) = \phi(g,x)$. The usual notation of a differentiable map with a low index $*$ will be used both for the derivative and for any of its wedge products when needed. When (G,P) is a Nambu-Lie group and (M,S) is a Nambu manifold the latter will be called a Nambu (G,P)-space if

$$S_{\phi(g,x)} = \phi_{g,*}S_x + \phi_{x,*}P_g\ .$$

In such case we will say that the Nambu structure S on M is (G,P)-multiplicative and that the action σ is a Nambu action. Let us remark that the special case $P = 0$ corresponds to G-invariance of S: $S_{\phi(g,x)} = S_x$. When the action is homogeneous we will also say that (M,S) is a (G,P)-Nambu homogeneous space. Lastly, for any chosen $x_0 \in M$ we will denote with G_{x_0} its stabilizer subgroup in G.

Definition 4.1. Let $(\mathfrak{g},\delta)$ be a Filippov-Lie bialgebra. A Lie algebra antihomomorphism $\sigma\colon \mathfrak{g} \to \mathfrak{X}(M)$ with values in the Lie algebra of vector fields on a Nambu manifold (M,S) is called an infinitesimal Nambu action if

$$L_{\sigma(X)}S = \sigma(\delta_P(X))\,. \tag{4.14}$$

An infinitesimal action of a Nambu-Lie group is an infinitesimal action of its tangent Filippov-Lie bialgebra.

Remark that in the definition we used the convention of denoting with the same letter the map σ and its extension to wedge product spaces.

The link between Nambu actions and their infinitesimal counterpart is given by the following proposition.

Proposition 4.2. *Let (G,P) be a connected Nambu-Lie group and let ϕ be a G-action on a rank n Nambu manifold (M,S). Let us denote with σ the corresponding infinitesimal G-action. The following are equivalent:*

1. *ϕ is a Nambu action;*
2. *σ is an infinitesimal Nambu action;*

3. *for every* $f_1, \ldots, f_n \in \mathcal{C}^\infty(M)$ *and every* $X \in \mathfrak{g}$

$$\sigma(X)\{f_1, \ldots, f_n\} - \sum_{i=1}^{n}\{f_1, \ldots, \sigma(X)f_i, \ldots f_n\} = \langle [df_1, \ldots, df_n], X\rangle \tag{4.15}$$

where the right hand side contains the 1*-form bracket defined by the Nambu tensor* S*.*

4. *for every* $X \in \mathfrak{g}$ *and* $\omega_1, \ldots, \omega_n$ *differentiable* 1*-forms on* M*:*

$$(L_{\sigma(X)}S)(\omega_1, \ldots, \omega_n) = \langle [\xi_{\omega_1}, \ldots, \xi_{\omega_n}], X\rangle \tag{4.16}$$

where ξ_{ω_i} *are functions from* M *to* $\mathfrak{g}^*$ *defined by* $\langle \xi_{\omega_i}, X\rangle = \langle \omega_i, \sigma(X)\rangle$ *and their bracket is the pointwise bracket.*

Proof. Let us prove that (2) $\Rightarrow$ (1). Consider the multiplicative property

$$S_{\phi(g,x)} = \phi_{g,*}S_x + \phi_{x,*}P_g$$

and apply $\phi_{g^{-1},*}$ on both sides. Then

$$\phi_{g^{-1},*}S_{\phi(g,x)} = S_x + \phi_{x,*}l_{g^{-1},*}P_g$$

Let now $g = exp(tX)$, $t \in \mathbb{R}$, $X \in \mathfrak{g}$:

$$\phi_{e^{-tX},*}S_{\phi(e^{tX},x)} = S_x + \phi_{x,*}l_{e^{-tX},*}P_{e^{tX}} \tag{4.17}$$

Now differentiate (4.17) and evaluate it at $t = 0$. Then the left hand side equals $L_X S(x)$ and the right hand side equals $\phi_{x,*}\delta(X)$. Let us now prove the opposite implication. The idea is, first of all, to prove that (4.17) holds for every $t \in \mathbb{R}$. Clearly the relation holds for $t = 0$. Furthermore for generic t the derivative of the left hand side equals

$$\sigma_{e^{-tX},*}L_{\sigma(X)}S(e^{tX}x) = \sigma_X Ad_{e^{-tX}}(L_X P)(e)$$

and the derivative of the right hand side equals

$$\sigma_{x,*}\frac{d}{dt}l_{e^{-tX},*}P(e^{tX}) = \sigma_{x,*}Ad_{e^{-tX}}L_X P(e)$$

From the fact that such derivatives are equal we can conclude that (4.17) holds for every $x \in M$, at least for all g in a neighbourhood of the identity. Being G connected every neighbourhood of the identity generates G. This fact, together with P being multiplicative, is enough to prove the claim.

To prove that (2) implies (4) we simply have to apply to both sides of 4.14 to the n-form $\omega_1 \wedge \ldots \wedge \omega_n$. $\square$

We remark that when $n \geq 3$ this proposition still does not allow to conclude that every infinitesimal action of a Filippov-Lie bialgebra is integrable to a Nambu action of a Nambu-Lie group, due to the existence of non integrable Filippov-Lie bialgebras (see [**13**] for an explicit example).

Proposition 4.3. *Let* $\sigma\colon \mathfrak{g} \to \mathfrak{X}(M)$ *be an infinitesimal Nambu action of a Filippov-Lie bialgebra* $(\mathfrak{g}, \delta_P)$ *on the Nambu manifold* (M, S)*. If* $\mathfrak{h}$ *is a coisotropic subalgebra of* $\mathfrak{g}$ *the algebra of invariant functions on* M *is closed under Nambu* *brackets. If $\mathfrak{h}\backslash M$ is a manifold there exists a unique Nambu tensor on the quotient manifold such that the projection $M \to \mathfrak{h}\backslash M$ is a Nambu map.*

Proof. Let $\phi_1, \ldots, \phi_n \in \mathcal{C}^\infty(M)$ and let $X \in \mathfrak{g}$. Then:

$$\sigma(X)\{\phi_1, \ldots, \phi_n\} = \sum_{i=1}^{n} \{\phi_1, \ldots, \sigma(X)\phi_i, \ldots, \phi_n\} + \langle [\xi_{\phi_1}, \ldots, \xi_{\phi_n}], X\rangle$$

where $\xi_{\phi_i} : M \to \mathfrak{g}^*$ verifies $\langle \xi_{\phi_i}, Y\rangle = \sigma(Y)\phi_i$. If every function ϕ_i is $\mathfrak{h}$-invariant the first n summands are zero. Considering that the maps ξ_{ϕ_i} take values in $\mathfrak{h}^0$ and that such space is a Filippov subalgebra of $\mathfrak{g}^*$ the last summand is zero as well. The other claims of the proposition then follow immediatly. $\square$

Let now $\alpha \in \wedge^{(n-1)}\mathfrak{g}^*$ and denote with α^l and α^r respectively the left and right invariant differential $(n-1)$-form on G. Define the following two maps:

$$\lambda\colon \wedge^{n-1} \mathfrak{g}^* \to \mathfrak{X}(G);\ \alpha \mapsto \sharp_P(\alpha^l) \tag{4.18}$$

$$\rho\colon \wedge^{n-1} \mathfrak{g}^* \to \mathfrak{X}(G);\ \alpha \mapsto -\sharp_P(\alpha^r) \tag{4.19}$$

Remark that λ is a Leibniz algebra antihomomorphism and ρ is a Leibniz algebra homomorphism. This follows from the fact that $\sharp_P$ induces a Leibniz morphism from the algebra of $(n-1)$ forms to the Lie algebra of vector fields on G, as proven in [**8**], Proposition 3.3.

Let now $\xi \in \wedge^{n-1}\mathfrak{g}^*$ be such that $[\![\xi, \xi]\!] = 0$. Then $\xi \in Ker\sharp_P$ so that $\lambda(\xi) = \rho(\xi) = 0$. We conclude that $\mathfrak{k}_n \subseteq \operatorname{Ker}\lambda, \rho$ and that λ and ρ induce a well defined Lie algebra (anti)homomorphism from $\wedge^{n-1}\mathfrak{g}^*/\mathfrak{k}_n$ to $\mathfrak{X}(G)$. We will denote such maps respectively with λ_n and ρ_n.

Definition 4.4. The maps λ_n and ρ_n are called infinitesimal dressing actions. If they can be integrated to actions of G^* on G then the corresponding integrated action will be called dressing action of G^* on G. The Nambu-Lie group G is called complete if every dressing field is complete.

As one can straightforwardly remark from definitions the infinitesimal dressing actions are tangent to the volume leaves. In fact, a simple proof allows to show that up to connected components the canonical foliation on G is given by the orbits of such infinitesimal action.

Proposition 4.5. *For every $\alpha \in \wedge^{n-1}\mathfrak{g}^*/\mathfrak{k}_n$ and for every $g, h \in G$ we have the twisted multiplicative property*

$$\begin{aligned} \lambda_n(\alpha)(gh) &= l_{g,*}\lambda_n(\alpha)(h) + r_{h,*}\lambda_n(Ad_{h^{-1}}\alpha)(g) \\ \rho_n(\alpha)(gh) &= l_{g,*}\rho_n(Ad_g\alpha)(h) + r_{h,*}\rho_n(\alpha)(g) \end{aligned} \tag{4.20}$$

Furthermore the linearization of the dressing action is the action ad^* of $\wedge^{n-1}\mathfrak{g}^*$ on $\mathfrak{g}$ defined in 3.11. *These properties uniquely characterize the infinitesimal dressing action.*

Proof. We will give the proof only for the infinitesimal left dressing action λ_n. The identity (4.20) is proven by direct computation:

$$\begin{aligned}\lambda_n(\alpha)(gh) &= (\alpha^l \lrcorner P)(gh) = \alpha^l(gh) \lrcorner P(gh) \\ &= \alpha^l(gh) \lrcorner (l_{g,*}P(h) + r_{h,*}P(g)) \\ &= l_{g,*}(\alpha^l(h) \lrcorner P(h)) + \alpha^l(gh) \lrcorner r_{h,*}P(g) \\ &= l_{g,*}\lambda_n(\alpha)(h) + r_{h,*}Ad_{h^{-1}}\alpha^l(g) \lrcorner r_{h,*}P(g)\end{aligned}$$

To prove the second claim remark that being $\lambda_n(\alpha)(e) = 0$ we can linearize it to map from $\mathfrak{g}$ to $\mathfrak{g}$ given by $X \mapsto [\bar{X}, \lambda_n(\alpha)]_e$, where $\bar{X}$ is any vector field over G such that $\bar{X}(e) = X$. Let's choose $\bar{X}$ right invariant. Then for any $\beta \in \mathfrak{g}^*$

$$\begin{aligned}\langle[\bar{X}, \lambda_n(\alpha)_e], \beta\rangle &= \frac{d}{dt}\langle l_{e^{-tX},*}\lambda_n(\alpha)(e^{tX}), \beta\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\langle \lambda_n(\alpha)(e^{tX}), l_{e^{tX},*}\beta\rangle\Big|_{t=0} \\ &= \frac{d}{dt}\left[\alpha \wedge l_{e^{tX},*}\beta \lrcorner P^l\right]\Big|_{t=0} \\ &= \langle X, [\alpha, \beta]\rangle = \langle ad^*_\alpha X, \beta\rangle\end{aligned}$$

Lastly let $L\colon \wedge^{n-1}\mathfrak{g}^*/\mathfrak{k}_n \to \mathfrak{X}(M)$ be a map verifying (4.20) and the linearization condition. Then $L - \lambda_n$ still verifies (4.20). If $\alpha \in \mathfrak{g}^*$ and $\beta \in \wedge^{n-1}\mathfrak{g}^*$ and with the l superscript we denote the corresponding left invariant forms we can define the n-vector field P_0 on G by

$$P_0(\alpha^l, \beta^l) = \langle \alpha^l, (L - \lambda)(\beta)\rangle$$

Being $L - \lambda$ twisted multiplicative then P_0 is a multiplicative n-vector field on G; its linearization at the identity is 0 so that $P_0 = 0$ and the claim follows. □

5. Homogeneous Nambu Spaces

Proposition 5.1. *Let (M, S) be a (G, P)-Nambu homogeneous space. Then (M, S) is non regular if and only if there exists a coisotropic subgroup H of G such that $M \simeq G/H$ and S is the reduction of P on M.*

Lemma 5.2. *For every $x_0 \in M$ there exists a bijective correspondence among:*

1. *(G, P)-multiplicative n-vectors S on M;*
2. *elements $\rho \in \wedge^n T_{x_0}M$ such that*

$$\rho = \phi_{h,*}\rho + \phi_{x_0,*}P_h \qquad \forall h \in G_{x_0} \tag{5.21}$$

Proof. Let S be a multiplicative n-vector. Apply such condition to $S(x_0)$ for any $h \in G_{x_0}$ to obtain exactly (5.21).

On the other hand let ρ be an n-vector for which (5.21) holds and let

$$S(x) := \phi_{g,*}\rho + \phi_{x_0,*}P_g$$

where $g \in G$ is such that $x = gx_0$. Then S is a well defined multiplicative n-vector field on M. □

Deriving condition (5.21) in the group identity e one obtains the following infinitesimal carachterization.

Lemma 5.3. *Let G_{x_0} be connected. There exists a bijective correspondence between*

1. *(G, P)-multiplicative n-vector fields S on M;*
2. *tangent n-vectors $\rho \in \wedge^n T_{x_0} M$ such that*

$$\phi_{x_0,*}(\delta_P(X)) + L_X \rho = 0 \qquad \forall X \in \mathfrak{g}_{x_0} \tag{5.22}$$

where $\mathfrak{g}_{x_0}$ stands for the Lie algebra of G_{x_0}.

Proof. Let now $\tilde{\rho} \in \wedge^n \mathfrak{g}$ be any lift of ρ. Condition (5.22) can be rewritten as

$$\delta_P(X) + ad_X^{(n)} \tilde{\rho} \in \mathfrak{g} \wedge \ldots \wedge \mathfrak{g} \wedge \mathfrak{g}_{x_0} \qquad \forall X \in \mathfrak{g}_{x_0} \tag{5.23}$$

Then (5.23) admits $\rho = 0$ as a solution if and only if G_{x_0} is a coisotropic subgroup of the given Nambu-Lie group (see proposition 2.5).

We have then proved that in M there exists at least one point in which the homogeneous Nambu structure is zero in and only if there exists a cosiotropic subgroup H such that M is the Nambu quotient G/H; such are all non regular Nambu spaces. □

The multiplicative condtion can be dualized and expressed as follows:

$$[\alpha_1, \ldots, \alpha_n] + \sum_{i=1}^{n} (-1)^i ad^*_{\langle R\alpha_{\hat{i}} \rangle}(\alpha_i) \in \mathfrak{g}^0 \qquad \forall \alpha_1, \ldots, \alpha_n \in \mathfrak{h}^0$$

where $\alpha_{\hat{i}} = \alpha_1 \wedge \ldots \wedge \alpha_{i-1} \wedge \alpha_{i+1} \ldots \wedge \alpha_n$. The condition that guarantees that $p_*(P + R)$ is Nambu is not as easy as in the Poisson case.

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N. Ciccoli, Università degli studi di Perugia, Dipartimento di Matematica, Via Vanvitelli 1, 06123 Perugia, Italia