Acta Math. Univ. Comenianae
Vol. LXX, 2(2001), pp. 229–240

PERRON CONDITIONS AND UNIFORM EXPONENTIAL STABILITY OF LINEAR SKEW-PRODUCT SEMIFLOWS ON LOCALLY COMPACT SPACES

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Abstract. The aim of this paper is to give necessary and sufficient conditions for uniform exponential stability of linear skew-product semiflows on locally compact metric spaces with Banach fibers. Thus, there are obtained generalizations of some theorems due to Datko, Neerven, Clark, Latushkin, Montgomery-Smith, Randolph, van Minh, Räbiger and Schnaubelt.

1. Introduction

A well developed area in the field of differential equations is the theory of linear skew-product flows, which arise as solution operators for variational equations

$$\frac{d}{dt}u(t) = A(\sigma(\theta,t))u(t),$$

where σ is a flow on a locally compact metric space Θ and $A(\theta)$ an unbounded linear operator on X, for every $\theta \in \Theta$. In the last few years significant progress has been made in the study of asymptotic behaviour of linear skew-product flows in infinite dimensional spaces giving an unifield answer to an impresive list of classical problems (see [**1**]–[**5**], [**9**], [**20**]). There has been studied the dichotomy of linear skew-product semiflows defined on compact spaces (see [**2**]–[**5**]), and on locally compact spaces, respectively (see [**10**]). An answer concerning stability of linear skew-product semiflows, on locally compact spaces, has been done in [**13**], where this property is characterized in terms of Banach function spaces, generalizing some results contained in [**11**] and [**12**]. In [**10**], dichotomy of strongly continuous linear skew-product flows was expressed in terms of hyperbolicity of a family of weighted shift operators and thus it was extended the classical theorem of Perron, which connects dichotomy to the existence and uniqueness of bounded, continuous mild solutions of an inhomogeneous equation.

The purpose of this paper is to answer questions concernig uniform exponential stability of linear skew-product semiflows on locally compact metric spaces. Therefore we consider a concept of exponential stability for linear skew-product

Received November 27, 2000.
2000 *Mathematics Subject Classification.* Primary 34D05, 35B40, 93D05.
Key words and phrases. Uniform exponential stability, linear skew-product semiflow.

semiflows, which is an extension of the classical concept of exponential stability for time-dependent linear differential equations in Banach spaces (see, e.g. [**7**], [**8**], [**18**]). Thus we give theorems of characterization for uniform exponential stability of linear skew-product semiflows in terms of boundedness of a family of linear operators acting on $C_0(\mathbf{R}_+, X)$ and $L^p(\mathbf{R}_+, X)$, respectively. We obtain that the uniform exponential stability of a linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is equivalent to uniform $(C_0(\mathbf{R}_+, X), C_b(\mathbf{R}_+, X))$ — stability of a certain family of linear operators, associated to π. It is proved that the property of uniform $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$ — stability of the associated family, is a sufficient condition for the uniform exponential stability of π and it is also necessary for $p \leq q$. An example shows that this result fails for $p > q$. We obtain here theorems of Perron type, which generalise some theorems contained in [**6**], [**8**], [**15**], [**16**], [**17**].

2. Linear Skew-Product Semiflows

Let X be a fixed Banach space — the state space — let $\Theta = (\Theta, d)$ be a locally compact metric space and let $\mathcal{E} = X \times \Theta$. We shall denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators from X into itself.

Definition 2.1. A mapping $\sigma \colon \Theta \times \mathbf{R}_+ \to \Theta$ is called a **semiflow** on Θ, if it has the following properties:

(i) $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$;
(ii) $\sigma(\theta, s + t) = \sigma(\sigma(\theta, s), t)$, for all $(\theta, s, t) \in \Theta \times \mathbf{R}_+^2$;
(iii) σ is continuous.

Definition 2.2. A pair $\pi = (\Phi, \sigma)$ is called a **linear skew-product semiflow** on $\mathcal{E} = X \times \Theta$ if σ is a semiflow on Θ and $\Phi \colon \Theta \times \mathbf{R}_+ \to \mathcal{B}(X)$ satisfies the following conditions:

(i) $\Phi(\theta, 0) = I$, the identity operator on X, for all $\theta \in \Theta$;
(ii) $\Phi(\theta, t + s) = \Phi(\sigma(\theta, t), s)\Phi(\theta, t)$, for all $(\theta, t, s) \in \Theta \times \mathbf{R}_+^2$ (**the cocycle identity**);
(iii) $t \mapsto \Phi(\theta, t)x$ is continuous for all $(\theta, x) \in \Theta \times X$;
(iv) there are $M \geq 1$ and $\omega > 0$ such that

$$||\Phi(\theta, t)|| \leq Me^{\omega t}, \tag{2.1}$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$.

Remark 2.1. If $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ then for every $\beta \in \mathbf{R}$ the pair $\pi_\beta = (\Phi_\beta, \sigma)$, where $\Phi_\beta(\theta, t) = e^{-\beta t}\Phi(\theta, t)$ for all $(\theta, t) \in \Theta \times \mathbf{R}_+$, is also a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$.

Example 2.1. Let Θ be a locally compact metric space, let σ be a semiflow on Θ and let $\mathbf{T} = \{T(t)\}_{t \geq 0}$ be a C_0 – semigroup on X. Then the pair $\pi_T = (\Phi_T, \sigma)$, where

$$\Phi_T(\theta, t) = T(t),$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$, is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$, which is called **the linear skew-product semiflow generated by the C_0 – semigroup T and the semiflow σ.**

Example 2.2. Let $\Theta = \mathbf{R}_+$, $\sigma(\theta, t) = \theta + t$ and let $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ be an evolution operator on the Banach space X. We define

$$\Phi(\theta, t) = U(t + \theta, \theta),$$

for all $(\theta, t) \in \mathbf{R}_+^2$. Then $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ called **the linear skew-product semiflow generated by the evolution operator $\mathcal{U}$ and the semiflow σ.**

Example 2.3. Let Θ be a compact metric space and let $\sigma\colon \Theta \times \mathbf{R}_+ \to \Theta$ be a semiflow on Θ. Let $A\colon \Theta \to \mathcal{B}(X)$ be a continuous mapping, where X is a Banach space. Let $\Phi(\theta, t)$ be the solution of the linear differential system

$$\dot{u}(t) = A(\sigma(\theta, t))\, u(t), \quad t \geq 0.$$

Then the pair $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$.

These equations arise from the linearization of nonlinear equations (see [**20**] and the references therein).

Example 2.4. Let X be a Banach space and let $Y := C(\mathbf{R}_+, \mathbf{R})$ be the space of all continuous functions with the topology of uniform convergence on compact subsets on $\mathbf{R}_+$. This space is metrizable with the metric

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x, y)}{1 + d_n(x, y)},$$

where $d_n(x, y) = \sup\limits_{t \in [0, n]} |x(t) - y(t)|$.

On the Banach space X, we consider the nonautonomous differential equation

$$\dot{x}(t) = a(t)\, x(t), \quad t \geq 0$$

where $a\colon \mathbf{R}_+ \to \mathbf{R}_+$ is an uniformly continuous function such that there exists $\alpha := \lim\limits_{t \to \infty} a(t) < \infty$.

If we denote by $a_s(t) = a(t + s)$ and by $\Theta =$ closure $\{a_s : s \in \mathbf{R}_+\}$, then

$$\sigma\colon \Theta \times \mathbf{R}_+ \to \Theta, \quad \sigma(\theta, t)(s) := \theta(t + s),$$

is a semiflow on Θ. For

$$\Phi\colon \Theta \times \mathbf{R}_+ \to \mathcal{B}(X), \quad \Phi(\theta, t)x = \exp\left(\int_0^t \theta(\tau)\, d\tau\right) x,$$

we have that $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$.

Definition 2.3. A linear skew-product semiflow $\pi = (\Phi, \sigma)$ on $\mathcal{E} = X \times \Theta$ is said **uniformly exponentially stable** if there are $N \geq 1$ and $\nu > 0$ such that

$$||\Phi(\theta, t)|| \leq N e^{-\nu t},$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$.

Example 2.5. Let $\beta \in \mathbf{R}_+$. Consider the linear skew-product semiflow $\pi_\beta = (\Phi_\beta, \sigma)$, where
$$\Phi_\beta(\theta, t) = e^{-\beta t}\, \Phi(\theta, t),$$
and $\pi = (\Phi, \sigma)$ is the linear skew-product semiflow given in Example 2.4.

It is easy to see that for $\beta > \alpha$, π_β is uniformly exponentially stable and for $\beta \in [0, \alpha]$ and $\theta_0(\tau) = \alpha$, for all $\tau \geq 0$ we have
$$||\Phi_\beta(\theta_0, t)x|| = \begin{cases} ||x||, & \text{if } \beta = \alpha \\ e^{\alpha-\beta}\, ||x||, & \text{if } \beta < \alpha, \end{cases}$$
so π_β is not uniformly exponentially stable.

Proposition 2.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$. If there are $t_0 > 0$ and $c \in (0, 1)$ such that*
$$||\Phi(\theta, t_0)|| \leq c,$$
for all $\theta \in \Theta$, then π is uniformly exponentially stable.

Proof. Let $M \geq 1$ and $\omega > 0$ given by (2.1). Let ν be a positive number such that $c = e^{-\nu t_0}$.

Let $\theta \in \Theta$ be fixed. For $t \in \mathbf{R}_+$ there are $n \in \mathbf{N}$ and $r \in [0, t_0)$ such that $t = nt_0 + r$. Then
$$||\Phi(\theta, t)|| \leq ||\Phi(\sigma(\theta, nt_0), r)||\, ||\Phi(\theta, nt_0)|| \leq M\, e^{\omega t_0}\, e^{-n\nu t_0} \leq N\, e^{-\nu t},$$
where $N = Me^{(\omega+\nu)t_0}$. So, π is uniformly exponentially stable. $\square$

Let $C_b(\mathbf{R}_+, X)$ be the linear space of all bounded continous functions $u \colon \mathbf{R}_+ \to X$ and
$$C_0(\mathbf{R}_+, X) = \{u \in C_b(\mathbf{R}_+, X) : u(0) = \lim_{t\to\infty} u(t) = 0\}.$$
Endowed with the sup-norm:
$$|||u||| := \sup_{t\geq 0} ||u(t)||,$$
$C_0(\mathbf{R}_+, X)$ and $C_b(\mathbf{R}_+, X)$ are Banach spaces.

We denote by $\mathcal{F}$ the linear space of all Bochner measurable functions $u \colon \mathbf{R}_+ \to X$ identifying the functions which are equal almost everywhere. For every $p \in [1, \infty)$ the linear space
$$L^p(\mathbf{R}_+, X) = \{u \in \mathcal{F} : \int_0^\infty ||u(t)||^p\, dt < \infty\}$$
is a Banach space with respect to the norm:
$$||u||_p := \left(\int_0^\infty ||u(t)||^p\, dt\right)^{1/p}.$$

Throughout the paper, we shall denote by $L^1_{\text{loc}}(\mathbf{R}_+, X)$ the set of all locally integrable functions $u \colon \mathbf{R}_+ \to X$.

Definition 2.4. A subspace E of $C_b(\mathbf{R}_+, X)$ is said to be **boundedly locally dense** in $C_b(\mathbf{R}_+, X)$ if there exists $c > 0$ such that

(i) for every $T > 0$ and every $u \in C_b(\mathbf{R}_+, X)$ there exists a sequence $(u_n) \subset E$ with $u_n \to u$ almost everywhere on $[0, T]$;
(ii) $|||u_n||| \leq c\,|||u|||$, for all $n \in \mathbf{N}$.

Remark 2.2. (i) It is easy to see that $C_c(\mathbf{R}_+, X)$ – the space of all X – valued, continuous functions on $\mathbf{R}_+$ with compact support is an example of boundedly locally dense subspace of $C_b(\mathbf{R}_+, X)$.

(ii) Let $BUC(\mathbf{R}_+, X)$ be the space of all X – valued, bounded, uniformly continuous functions on $\mathbf{R}_+$ and $AP(\mathbf{R}_+, X)$ – the closure in $BUC(\mathbf{R}_+, X)$ of the linear span of the functions $\{e^{i\lambda(\cdot)}x : \lambda \in \mathbf{R}, x \in X\}$ (see [**17**]). Then $BUC(\mathbf{R}_+, X)$ and $AP(\mathbf{R}_+, X)$ are two remarkable examples of boundedly locally dense subspaces of $C_b(\mathbf{R}_+, X)$.

Definition 2.5. Let $p \in [1, \infty)$. A subspace E of $L^p(\mathbf{R}_+, X)$ is said to be **boundedly locally dense** in $L^p(\mathbf{R}_+, X)$ if there exists $c > 0$ such that

(i) for every $T > 0$ and every $u \in L^p(\mathbf{R}_+, X)$ there exists a sequence $(u_n) \subset E$ with $u_n \to u$ in $L^p([0, T], X)$;
(ii) $||u_n||_p \leq c\,||u||_p$, for all $n \in \mathbf{N}$.

Remark 2.3. $\mathcal{S}(\mathbf{R}_+, X)$ — the space of all measurable simple functions $s \mapsto \mathbf{R}_+ \to X$ and $C_c(\mathbf{R}_+, X)$ are boundedly locally dense subspaces of $L^p(\mathbf{R}_+, X)$, for every $p \in [1, \infty)$.

If $\pi = (\Phi, \sigma)$ is linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ then for every $\theta \in \Theta$ we define

$$P_\theta \colon L^1_{\mathrm{loc}}(\mathbf{R}_+, X) \to L^1_{\mathrm{loc}}(\mathbf{R}_+, X), \quad (P_\theta u)(t) := \int_0^t \Phi(\sigma(\theta, \tau), t - \tau) u(\tau)\, d\tau.$$

Definition 2.6. Let $U, Y \in \{C_0(\mathbf{R}_+, X), C_b(\mathbf{R}_+, X)\} \cup \{L^p(\mathbf{R}_+, X), p \in [1, \infty)\}$ and let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$. We say that the family $\{P_\theta\}_{\theta \in \Theta}$ is **uniformly** (U, Y)**-stable** if for every $u \in U$ and every $\theta \in \Theta$ $P_\theta u$ belongs to Y and there is $K > 0$ such that

$$||P_\theta u||_Y \leq K ||u||_U,$$

for all $(u, \theta) \in U \times \Theta$.

Proposition 2.2. *Let $\pi = (\Phi, \sigma)$ be an uniformly exponentially stable linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ and $p, q \in [1, \infty)$ with $p \leq q$. Then the family $\{P_\theta\}_{\theta \in \Theta}$ is uniformly $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$-stable.*

Proof. It follows using Hölder's inequality and the cocycle identity. □

3. The Main Results

We shall start with a generalization of a theorem of characterization of exponential stablity of evolution operators in Banach spaces (see [**5**, Theorem 2.2]) at the case of linear skew-product semiflows.

Theorem 3.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$. Then the following assertions are equivalent:*

(i) *π is uniformly exponentially stable;*
(ii) *the family $\{P_\theta\}_{\theta\in\Theta}$ is uniformly $(C_0(\mathbf{R}_+, X), C_0(\mathbf{R}_+, X))$-stable;*
(iii) *the family $\{P_\theta\}_{\theta\in\Theta}$ is uniformly $(C_0(\mathbf{R}_+, X), C_b(\mathbf{R}_+, X))$-stable.*

Proof. The implication $(i) \Rightarrow (ii)$ is a simple exercise and $(ii) \Rightarrow (iii)$ is obvious. Suppose that (iii) holds and hence there is $K > 0$ such that

$$|||P_\theta u||| \le K\,|||u|||, \tag{3.1}$$

for all $(u, \theta) \in C_0(\mathbf{R}_+, X) \times \Theta$.

Consider $M \ge 1$ and $\omega > 0$ given by (2.1).

Let $\theta \in \Theta$ and $x \in X$. If $\alpha\colon \mathbf{R}_+ \to [0, 2]$ is a continuous function with the support contained in $(0, 1)$ and with the property that

$$\int_0^1 \alpha(s)\,ds = 1,$$

then we consider the function

$$u\colon \mathbf{R}_+ \to X, \quad u(t) = \alpha(t)\Phi(\theta, t)x.$$

Hence $u \in C_0(\mathbf{R}_+, X)$ and

$$|||u||| = \sup_{t\in[0,1]} ||u(t)|| \le 2Me^{\omega}||x||.$$

For $t \ge 1$, we observe that

$$(P_\theta u)(t) = \int_0^t \alpha(s)\,\Phi(\sigma(\theta, s), t - s)\Phi(\theta, s)x\,ds = \Phi(\theta, t)x.$$

Then using (3.1) we obtain

$$||\Phi(\theta, t)x|| \le |||P_\theta u||| \le 2KMe^{\omega}||x||. \tag{3.2}$$

But, for $t \in [0, 1]$ we have

$$||\Phi(\theta, t)|| \le Me^{\omega}, \tag{3.3}$$

so, denoting by $L = (2K + 1)Me^{\omega}$ and using relations (3.2) and (3.3), it follows that

$$||\Phi(\theta, t)|| \le L,$$

for all $(\theta, t) \in \Theta \times \mathbf{R}_+$.

Consider $\nu = e/4LK$ and

$$\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+, \quad \varphi(t) = \int_0^t se^{-\nu s}\,ds.$$

The function φ is strictly increasing on $\mathbf{R}_+$ with

$$\lim_{t\to\infty} \varphi(t) = \frac{1}{\nu^2},$$

so, we can choose $\delta > 0$ such that $\varphi(\delta) > 1/2\nu^2$.

Let $\theta \in \Theta$ and $x \in X$. Define the function

$$v\colon \mathbf{R}_+ \to X, \quad v(t) = te^{-\nu t}\Phi(\theta, t)x.$$

Then $v \in C_0(\mathbf{R}_+, X)$ and

$$|||v||| \leq L||x|| \sup_{t\geq 0} te^{-\nu t} = \frac{L}{\nu e}\,||x||.$$

We observe that

$$(P_\theta v)(\delta) = \varphi(\delta)\Phi(\theta, \delta)x,$$

and hence it follows that

$$\begin{aligned}||\Phi(\theta,\delta)x|| &< 2\nu^2\varphi(\delta)||\Phi(\theta,\delta)x||\\ &\leq 2\nu^2\,|||P_\theta v|||\;\leq\;2\nu\frac{LK}{e}\,||x|| = \frac{1}{2}\,||x||.\end{aligned}$$

It results that

$$||\Phi(\theta,\delta)|| \leq \frac{1}{2},$$

for all $\theta \in \Theta$. From Proposition 2.1. we obtain that π is uniformly exponentially stable. □

Corollary 3.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ and let E be a boundedly locally dense subspace of $C_b(\mathbf{R}_+, X)$. If for every $u \in E$ and every $\theta \in \Theta$ $P_\theta u$ belongs to $C_b(\mathbf{R}_+, X)$ and there exists $L > 0$ such that*

$$|||P_\theta u||| \leq L|||u|||,$$

for all $(u, \theta) \in E \times \Theta$, then π is uniformly exponentially stable.

Proof. Let $u \in C_0(\mathbf{R}_+, X)$, $T > 0$. There is a sequence $(u_n) \subset E$ with $u_n \to u$ almost everywhere on $[0, T]$ and

$$|||u_n||| \leq c\,|||u|||,$$

for all $n \in \mathbf{N}$, where $c > 0$ is given by Definition 2.4.

Let $\theta \in \Theta$ be fixed. From Lebesgue's theorem we have that

$$(P_\theta u_n)(T) \to (P_\theta u)(T), \quad \text{as } n \to \infty.$$

Because

$$||(P_\theta u_n)(T)|| \leq |||P_\theta u_n||| \leq L|||u_n||| \leq cL|||u|||,$$

as $n \to \infty$ the relation from above gives

$$||(P_\theta u)(T)|| \leq cL|||u|||. \tag{3.4}$$

Since $T > 0$ was arbitrary chosen it follows that $P_\theta u \in C_b(\mathbf{R}_+, X)$. Moreover (3.4) holds for every $u \in C_0(\mathbf{R}_+, X)$ and every $\theta \in \Theta$, so the family $\{P_\theta\}_{\theta\in\Theta}$ is uniformly $(C_0(\mathbf{R}_+, X), C_b(\mathbf{R}_+, X))$-stable. By applying Theorem 3.1, it follows that π is uniformly exponentially stable. □

Remark 3.1. Neerven proved that a C_0 — semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ is uniformly exponentially stable if and only if convolution with $\mathbf{T}$ maps certain subspaces of $BUC(\mathbf{R}_+, X)$ into $C_b(\mathbf{R}_+, X)$. Thus, he obtained characterizations for uniform exponential stablity of C_0 — semigroups, in terms of almost periodic functions (see [**17**, p. 90-94]). So, Corollary 3.1. is a generalization of Neerven's result, for the case of linear skew-product semiflows.

In the theory of stability of evolution operators in Banach spaces a well-known result says that an evolution operator $\mathcal{U} = \{U(t,s)\}_{t\geq s\geq 0}$ is exponentially stable if and only if for every $f \in L^p(\mathbf{R}_+, X)$ the mapping P_f, where

$$P_f(t) = \int_0^t U(t,s)f(s)\,ds,$$

for all $t \geq 0$, belongs to $L^p(\mathbf{R}_+, X)$ (see e.g. [**6**, Theorem 2.5]). As a sufficient condition for exponential stability, this theorem was also treated in [**8**].

In what follows, we shall generalize this result for the case of linear skew-product semiflows on locally compact metric spaces.

Theorem 3.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ and $p, q \in [1, \infty)$. If the family $\{P_\theta\}_{\theta\in\Theta}$ is uniformly $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$-stable then π is uniformly exponentially stable.*

Proof. Let $K > 0$ given by Definition 2.6 and $M \geq 1$, $\omega > 0$ given by (2.1).

Let $\theta \in \Theta$ and $x \in X$. Let $\alpha\colon \mathbf{R}_+ \to [0,2]$ be a continuous function with the support contained in $(0,1)$ and

$$\int_0^1 \alpha(s)\,ds = 1.$$

We consider the function

$$u\colon \mathbf{R}_+ \to X, \quad u(t) = \alpha(t)\Phi(\theta,t)x.$$

Then $u \in L^p(\mathbf{R}_+, X)$ and

$$||u||_p = \left(\int_0^1 \alpha(s)^p ||\Phi(\theta,s)x||^p\,ds\right)^{\frac{1}{p}} \leq 2Me^{\omega}||x||.$$

Moreover we obtain

$$P_\theta u(t) = \Phi(\theta,t)x, \tag{3.5}$$

for all $t \geq 1$.

Since, for every $\theta \in \Theta$, $x \in X$ and $t \geq 1$ we have

$$||\Phi(\theta,t)x|| \leq Me^{\omega}\left(\int_{t-1}^t ||\Phi(\theta,\tau)x||^q\,d\tau\right)^{\frac{1}{q}}, \tag{3.6}$$

from (3.5) and (3.6), we deduce that

$$\begin{aligned}||\Phi(\theta,t)x|| &\leq Me^{\omega}\left(\int_{t-1}^{t}||(P_\theta u)(\tau)||^q\,d\tau\right)^{\frac{1}{q}}\\ &\leq Me^{\omega}||P_\theta u||_q \leq MKe^{\omega}||u||_p \leq 2M^2Ke^{2\omega}||x||,\end{aligned}$$

for every $t \geq 2$. Because for $t \in [0,2]$ we have

$$||\Phi(\theta,t)x|| \leq Me^{2\omega}||x||,$$

denoting by $L = Me^{2\omega}(2MK+1)$, we finally conclude that

$$||\Phi(\theta,t)|| \leq L, \tag{3.7}$$

for all $(\theta,t) \in \Theta \times \mathbf{R}_+$.

Let

$$\varphi\colon \mathbf{R}_+ \to \mathbf{R}_+, \quad \varphi(t) = \int_0^t se^{-s}\,ds.$$

Then, φ is a strictly increasing function, with $\lim\limits_{t\to\infty}\varphi(t) = 1$. Let $c > 0$ such that

$$\varphi(t) > \frac{1}{2}, \tag{3.8}$$

for all $t \geq c$.

Let $\theta \in \Theta$ and $x \in X$. We consider the function

$$v\colon \mathbf{R}_+ \to X, \quad v(t) = te^{-t}\Phi(\theta,t)x.$$

Then $v \in L^p(\mathbf{R}_+, X)$ and

$$||v||_p = \left(\int_0^\infty s^p e^{-sp}||\Phi(\theta,s)x||^p\,ds\right)^{\frac{1}{p}} \leq L_1||x||,$$

where $L_1 = L(\int_0^\infty s^p e^{-sp}\,ds)^{1/p}$. But

$$(P_\theta v)(t) = \varphi(t)\,\Phi(\theta,t)x,$$

for all $t \geq 0$. For $t > c$ and $\tau \in [c,t]$ using (3.7) and (3.8) we obtain that

$$\frac{1}{2}||\Phi(\theta,t)x|| \leq L\varphi(\tau)||\Phi(\theta,\tau)x||.$$

Hence, we deduce that

$$\begin{aligned}\frac{(t-c)^{1/q}}{2}||\Phi(\theta,t)x|| &\leq L(\int_c^t ||(P_\theta v)(\tau)||^q\,d\tau)^{\frac{1}{q}}\\ &\leq\; L\,||P_\theta v||_q \;\leq\; KL\,||v||_p \leq\; KLL_1\,||x||.\end{aligned}$$

Let $t_0 > 0$ with $(t_0 - c)^{1/p} > 4KLL_1$. Then

$$||\Phi(\theta,t_0)|| \leq \frac{1}{2},$$

for all $\theta \in \Theta$. From Proposition 2.1. we conclude that π is uniformly exponentially stable. □

In certain situations, the sufficient condition for uniform exponential stability of a linear skew-product semiflow, given by Theorem 3.2, becomes necessary, too, as shows

Corollary 3.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ and $p, q \in [1, \infty)$ with $p \leq q$. Then π is uniformly exponentially stable if and only if the family $\{P_\theta\}_{\theta \in \Theta}$ is uniformly $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$-stable.*

Proof. It follows from Proposition 2.2 and Theorem 3.2. $\square$

Remark 3.2. Generally, if $\pi = (\Phi, \sigma)$ is an uniformly exponentially stable linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ and $p, q \in [1, \infty)$, with $p > q$, it does not result that the family $\{P_\theta\}_{\theta \in \Theta}$ is uniformly $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$-stable. This fact is illustrated by the following example.

Example 3.1. Let $X = \Theta = \mathbf{R}$ and $\sigma(\theta, t) = \theta + t$. If

$$\Phi(\theta, t)x = e^{-t}x,$$

for all $t \geq 0$, $x, \theta \in \mathbf{R}$, then $\pi = (\Phi, \sigma)$ is a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$ which is uniformly exponentially stable.

If $p, q \in [1, \infty)$, with $p > q$, let $\delta \in (q, p)$. We consider the function

$$u \colon \mathbf{R}_+ \to \mathbf{R}, \quad u(t) = \frac{1}{(t+1)^{1/\delta}}.$$

We have that $u \in L^p(\mathbf{R}_+, \mathbf{R}) \setminus L^q(\mathbf{R}_+, \mathbf{R})$.

Let $\theta \in \Theta$. We observe that

$$(P_\theta u)(t) = e^{-t} \int_0^t e^s u(s)\, ds,$$

for all $t \geq 0$. Because

$$\lim_{t \to \infty} \frac{(P_\theta u)(t)}{u(t)} = \lim_{t \to \infty} \frac{e^t u(t)}{e^t u(t) - \frac{1}{\delta(t+1)} e^t u(t)} = 1$$

and $u \notin L^q(\mathbf{R}_+, \mathbf{R})$, we obtain that $P_\theta u \notin L^q(\mathbf{R}_+, \mathbf{R})$ and hence the family $\{P_\theta\}_{\theta \in \Theta}$ is not uniformly $(L^p(\mathbf{R}_+, \mathbf{R}), L^q(\mathbf{R}_+, \mathbf{R}))$-stable.

Corollary 3.3. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product semiflow on $\mathcal{E} = X \times \Theta$, $p, q \in [1, \infty)$ and let E be a boundedly locally dense subspace of $L^p(\mathbf{R}_+, X)$. If for every $u \in E$ and every $\theta \in \Theta$, $P_\theta u$ belongs to $L^q(\mathbf{R}_+, X)$ and there exists $L > 0$ such that*

$$||P_\theta u||_q \leq L||u||_p,$$

for all $(u, \theta) \in E \times \Theta$, then π is uniformly exponentially stable.

Proof. Let $M \geq 1$ and $\omega > 0$ given by (2.1). Let $\theta \in \Theta, u \in L^p(\mathbf{R}_+, X)$ and $T > 0$. Then there exist $c > 0$ and a sequence $(u_n) \subset E$ such that $u_n \to u$ in $L^p([0, T], X)$ and

$$||u_n||_p \leq c||u||_p,$$

for all $n \in \mathbf{N}$.

For $t \in [0,T]$ we have that

$$\begin{aligned}||(P_\theta u_n)(t) - (P_\theta u)(t)|| &\le Me^{\omega T} \int_0^T ||u_n(s) - u(s)||\, ds \\ &\le Me^{\omega T} \delta \Big(\int_0^T ||u_n(s) - u(s)||^p\, ds\Big)^{\frac{1}{p}},\end{aligned}$$

where

$$\delta = \begin{cases} 1, & p = 1 \\ T^{1/q}, & p \in (1,\infty) \text{ and } q = \frac{p}{p-1} \end{cases},$$

so,

$$(P_\theta u_n)(t) \to (P_\theta u)(t), \qquad \text{as } n \to \infty.$$

But

$$\begin{aligned}||(P_\theta u_n)(t)|| &\le Me^{\omega T} \int_0^T ||u_n(s)||\, ds \le Me^{\omega T} \delta ||u_n||_p \\ &\le Me^{\omega T} \delta\, c\, ||u||_p,\end{aligned}$$

for all $t \in [0,T]$, $n \in \mathbf{N}$. From Lebesgue's theorem, we obtain that

$$\int_0^T ||P_\theta u_n(t)||^q\, dt \to \int_0^T ||(P_\theta u)(t)||^q\, dt \qquad \text{as } n \to \infty. \tag{3.9}$$

Moreover, for every $n \in \mathbf{N}$

$$\int_0^T ||(P_\theta u_n)(t)||^q\, dt \le ||P_\theta u_n||_q^q \le L^q ||u_n||_p^q \le c^q L^q ||u||_p^q. \tag{3.10}$$

For $n \to \infty$ in (3.10) and using (3.9) we deduce that

$$\int_0^T ||(P_\theta u)(t)||^q\, dt \le c^q L^q ||u||_p^q.$$

Since $T > 0$ was arbitrary chosen, it follows that $P_\theta u \in L^q(\mathbf{R}_+, X)$ and

$$||P_\theta u||_q \le cL||u||_p,$$

for all $(u,\theta) \in L^p(\mathbf{R}_+, X) \times \Theta$. It follows that the family $\{P_\theta\}_{\theta \in \Theta}$ is uniformly $(L^p(\mathbf{R}_+, X), L^q(\mathbf{R}_+, X))$-stable, so from Theorem 3.2 we conclude that π is uniformly exponentially stable. $\square$

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