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AVERAGING AT ANY LEVEL

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ABSTRACT. In this paper we study the behaviour of two functions which can be defined in normed spaces: they measure, for finite sets on the unit sphere, "average distances" from points in smaller spheres. We also study these functions when only pairs of opposite points are considered. We generalize some results indicated in recent papers, concerning the values of these functions when "average distances" are measured from points in the unit sphere of the space.

0. Introduction and Notation

In this paper, we consider — in a normed space — functions defined by considering finite subsets of the unit sphere, or only antipodal pairs. The results indicated here generalize some of those contained in [**1**], [**2**], [**3**].

Let $(X, \|.\|)$ be a normed space, of dimension at least 2, over the real field $\mathbb{R}$.

We shall use the following notations:

$$S_{\varepsilon,X} = \{x \in X; \|x\| = \varepsilon\}, \quad \varepsilon \geq 0;$$
$$S_X = S_{1,X} = \{x \in X; \|x\| = 1\};$$

we shall simply write S_ε and S instead of $S_{\varepsilon,X}$ and S_X when no confusion can arise. We shall denote by X^* the dual of X.

Let

$$\mathcal{F}(X) = \{F \subset X; F \text{ is finite and nonempty}\}$$

and

$$\mathcal{F}(S) = \{F \subset S; F \text{ is finite and nonempty}\}.$$

If $F = \{x_1, x_2, \dots, x_n\} \subset S$ and $x \in X$, we set

$$\mu(F, x) = \frac{1}{n} \sum_{i=1}^{n} \|x_i - x\|,$$

and, for $\varepsilon \geq 0$:

$$\mu_1^\varepsilon(F) = \inf\{\mu(F, x); \|x\| = \varepsilon\}; \tag{0.1}$$

$$\mu_2^\varepsilon(F) = \sup\{\mu(F, x); \|x\| = \varepsilon\}. \tag{0.1$'$}$$

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For $F \in \mathcal{F}(S)$, we also set

$$\mu(F, S_\varepsilon) = \{\alpha \geq 0; \text{ there exists } x \in S_\varepsilon \text{ such that } \mu(F,x) = \alpha\}$$

and

$$\mu(F,S) = \mu(F,S_1);$$

moreover

$$\mu_1(F) = \mu_1^1(F) = \inf(\mu(F,S)); \tag{0.2}$$

$$\mu_2(F) = \mu_2^1(F) = \sup(\mu(F,S)). \tag{0.2'}$$

Given $F \in \mathcal{F}(S)$, since S_ε is connected $(0 \leq \varepsilon \leq 1)$, $\mu(F, S_\varepsilon)$ is an interval; so

$$\overline{\mu(F,S_\varepsilon)} = [\mu_1^\varepsilon(F), \mu_2^\varepsilon(F)];$$

$$\overline{\mu(F,S)} = [\mu_1(F), \mu_2(F)].$$

Of course, if $\dim(X) < \infty$, then for every F there exist $x_F \in S_X, y_F \in S_X$, such that $\mu(F, x_F) = \mu_1(F)$; $\mu(F, y_F) = \mu_2(F)$, and a similar fact is true for $\mu_1^\varepsilon(F)$, $\mu_2^\varepsilon(F)$, $0 < \varepsilon < 1$. We shall denote by $\mathbb{R}_1^2$, $\mathbb{R}_2^2$, $\mathbb{R}_\infty^2$ the plane endowed with the usual sum, euclidean, max norm.

1. A Few Simple General Facts

Now set, for $\leq \varepsilon \leq 1$:

$$\mu_1^\varepsilon(X) = \sup\{\mu_1^\varepsilon(F); F \in \mathcal{F}(S)\}; \tag{1.1}$$

$$\mu_2^\varepsilon(X) = \inf\{\mu_2^\varepsilon((F); F \in \mathcal{F}(S)\}; \tag{1.1'}$$

$$\mu_1(X) = \mu_1^1(X); \tag{1.2}$$

$$\mu_2(X) = \mu_2^1(X); \tag{1.2'}$$

in other terms, we have

$$\mu_1(X) = \sup\{\mu_1(F); F \in \mathcal{F}(S)\} = \sup_{F \subset S} \inf_{x \in S} \frac{1}{n} \sum_{i=1}^n \|x_i - x\| \quad \text{(F finite)}; \tag{1.3}$$

$$\mu_2(X) = \inf\{\mu_2(F); F \in \mathcal{F}(S)\} = \inf_{F \subset S} \sup_{x \in S} \frac{1}{n} \sum_{i=1}^n \|x_i - x\| \quad \text{(F finite)}. \tag{1.3'}$$

Clearly, given X, for any $F \in \mathcal{F}(S)$ and any $\varepsilon \in [0,1]$ we have:

$$\max\{1, \mu_1^\varepsilon(F)\} \leq \min\{\mu_1^\varepsilon(X), \mu_2^\varepsilon(F)\} \leq 1 + \varepsilon; \tag{1.4}$$

$$1 \leq \mu_2^\varepsilon(X) \leq \mu_2^\varepsilon(F) \leq 1 + \varepsilon; \tag{1.4'}$$

(1.4″) $\mu_1^\varepsilon(F)$ and $\mu_2^\varepsilon(F)$ are 1-lipschitz functions of $\varepsilon \in [0,1]$,

so:

(1.4‴) $\mu_1^\varepsilon(X)$ and $\mu_2^\varepsilon(X)$ are 1-lipschitz functions of $\varepsilon \in [0,1]$.

The following result was proved in [**15**, p. 332]:

Proposition 1.1. *For any Banach space X we have*

$$\mu_1(X) \leq \mu_2(X). \tag{1.5}$$

According to Proposition 1.1, we can consider

$$[\mu_1(X), \mu_2(X)] = \bigcap\{\overline{\mu(F,S)}; F \in \mathcal{F}(S)\};$$

when $\mu_1(X) = \mu_2(X)$, we shall simply denote such value by $\mu(X)$.

We indicate a few general results on the above numbers: for the finite dimensional case, see e.g. [**5**]; for the general case, see also [**1**], [**2**], [**9**], [**13**] and [**14**].

A famous result of Gross is the following:

Proposition 1.2. *If* $\dim(X) < \infty$, *then* $\mu_1(X) = \mu_2(X) = \mu(X)$*; moreover (clearly) for every* $F \in \mathcal{F}(S)$, *there exists* x_F *such that* $\mu(F, x_F) = \mu(X)$.

Of course, if $\dim(X) < \infty$, then for every $\varepsilon \in [0,1]$ both values $\mu_i^\varepsilon(X)$, $i = 1, 2$, belong to $\mu(F, S_\varepsilon)$, for any $F \in \mathcal{F}(S)$.

If $\dim(X) = \infty$, also when $\mu_1(X) = \mu_2(X) = \mu(X)$, it is not true in general that for every $F \in \mathcal{F}(S)$ there exists x_F such that $\mu(F, x_F) = \mu(X)$: this failure may happen also in case the space is reflexive, and/or $\mu(X) \in \{1, 2\}$ (see [**1**], [**9**]).

With relation to Proposition 1.1, the following simple (2-dimensional) example shows that in general we do not have $\mu_1^\varepsilon(X) \leq \mu_2^\varepsilon(X)$.

Example. (see also Proposition 2.2 later) Let X be the space $\mathbb{R}^2_\infty$; we will show that $\mu_2^\varepsilon(X) < \mu_1^\varepsilon(X)$ for all $\varepsilon \in (0,1)$. Let $F = \{(0,1), (0,-1)\}$; then, for $\varepsilon \leq 1/2$, we have $\mu(F, x) = 1$ for every $x \in S_\varepsilon$, thus $\mu_2^\varepsilon(F) = 1$ and then also $\mu_2^\varepsilon(X) = 1$.

If $1/2 < \varepsilon \leq 1$, then for $y = (\varepsilon, \varepsilon)$ we have $\mu(F, y) = \varepsilon + 1/2 = \mu_2^\varepsilon(F)$, so $\mu_2^\varepsilon(X) \leq \varepsilon + 1/2$. Let $G = \{(1,1), (1,-1), (-1,1), (-1,-1)\}$ and $0 < \varepsilon \leq 1$; then $\mu(G, x) = 1 + \varepsilon/2$ for every $x \in S_\varepsilon$, so $\mu_1^\varepsilon(G) = 1 + \varepsilon/2$, and then, for $0 < \varepsilon < 1$, $\mu_1^\varepsilon(X) \geq 1 + \varepsilon/2 > \max\{1, \varepsilon + 1/2\} \geq \mu_2^\varepsilon(X)$.

By next proposition we indicate a few simple properties; some of them are concerned with sets, that we shall call **symmetric**, satisfying the following property: $x \in F$ implies $-x \in F$ (of course, we write $-F = \{-x; x \in F\}$). Note that $-F \cup F$ is always a symmetric set.

Proposition 1.3. *For any set* $F \in \mathcal{F}(S)$ *and any* $\varepsilon \in [0,1]$ *we have:*

(i) $\mu(F, S_\varepsilon) = \mu(-F, S_\varepsilon)$,

so

(i′) $\mu_i(F, S_\varepsilon) = \mu_i(-F, S_\varepsilon), i = 1, 2.$

(ii) *if also* $G \in \mathcal{F}(S)$, *then we have*

$$\min\{\mu_1^\varepsilon(F), \mu_1^\varepsilon(G)\} \leq \mu_1^\varepsilon(F \cap G) \leq \max\{\mu_1^\varepsilon(F), \mu_2^\varepsilon(G)\};$$

$$\min\{\mu_1^\varepsilon(F), \mu_2^\varepsilon(G)\} \leq \mu_2^\varepsilon(F \cup G) \leq \max\{\mu_2^\varepsilon(F), \mu_2^\varepsilon(G)\}.$$

(iii) *If* $\#(F) = \#(G)$, *then*

$$\frac{1}{2}\left(\mu_1^\varepsilon(F) + \mu_1^\varepsilon(G)\right) \leq \mu_1^\varepsilon(F \cup G) \leq \mu_2^\varepsilon(F \cup G) \leq \frac{1}{2}\left(\mu_2^\varepsilon(F) + \mu_2^\varepsilon(G)\right).$$

(iv) $\mu_1^\varepsilon(-F \cup F) \geq 1.$

(v) $\mu_1^\varepsilon(F) \le \mu_1^\varepsilon(-F \cup F) \le \mu_2^\varepsilon(-F \cup F) \le \mu_2^\varepsilon(F)$.

Proof. Let $F = \{x_1, x_2, \ldots, x_n\}$.

(i) (and (i$'$)): note that $\mu(F,x) = \mu(-F,-x)$, so we have (i), and then

$$\inf_{\|x\|=\varepsilon} \mu(F,x) = \inf_{\|-x\|=\varepsilon} \mu(-F,-x);$$

the same is true if we take sup instead of inf, so we have (i$'$).

(ii): almost trivial.

(iii): Let $F = \{x_1, x_2, \ldots, x_n\}$; $G = \{y_1, y_2, \ldots, y_n\}$ ($\#(F) = \#(G)$); then for any $x \in S_\varepsilon$ we have:

$$\sum_{i=1}^n \|x_i - x\| + \sum_{i=1}^n \|y_i - x\| = n(\mu(F,x) + \mu(G,x)) \ge n(\mu_1^\varepsilon(F) + \mu_1^\varepsilon(G)),$$

so

$$\frac{1}{2n}\left(\sum_{i=1}^n \|x_i - x\| + \sum_{i=1}^n \|y_i - x\|\right) \ge \frac{1}{2}(\mu_1^\varepsilon(F) + \mu_1^\varepsilon(G)),$$

which implies the left part of (iii).

The proof for the right part of (iii) is similar.

(iv): trivial, since $\|x_i - x\| + \|x_i + x\| \ge 2$ for $i = 1, \ldots, n$, any $x \in S_\varepsilon, \varepsilon \in [0,1]$.

(v): is a consequence of (i) and (ii). □

Remark. If $F, G \in \mathcal{F}(S)$ and $F \subset G$ ($\varepsilon \in [0,1]$), then it is simple to see that concerning the relation between $\mu_i^\varepsilon(F)$ and $\mu_i^\varepsilon(G), i = 1,2$, all cases $<, =, >$ are possible.

Given F, $\mu(F,x)$ is a convex function of $x \in X$; moreover, if $F \in \mathcal{F}(S)$ is symmetric, then $\mu(F,x)$ is even; i.e.:

$$\mu(F,x) = \mu(F,-x). \tag{1.6}$$

As a consequence, we have:

Proposition 1.4. *Let $F \in \mathcal{F}(S)$; then, if $\mathcal{E}$ denotes the set of extreme points of the unit sphere of X, we have:*

$$\sup_{\|x\|=\varepsilon} \mu(F,x) = \sup_{\|x\|\le\varepsilon} \mu(F,x) = \sup_{x\in\mathcal{E}} \mu(F,\varepsilon x). \tag{1.7}$$

Moreover, if F is symmetric, then $\mu_1^\varepsilon(F)$ and $\mu_2^\varepsilon(F)$ are non decreasing functions of $\varepsilon \in \mathbb{R}$.

Proof. The first assertion follows from the convexity in x of the functions $\mu(F,x)$, for any given F.

For the second part, if F is symmetric, then according to (1.6), for $\varepsilon < \sigma$ we have:

$$\mu_1^\varepsilon(F) = \inf\{\mu(F,x); \|x\| \ge \varepsilon\} \le \inf\{\mu(F,x); \|x\| \ge \sigma\} = \mu_1^\sigma(F); \tag{1.8}$$

$$\mu_2^\varepsilon(F) = \sup\{\mu(F,x); \|x\| \le \varepsilon\} \le \sup\{\mu(F,x); \|x\| \le \sigma\} = \mu_2^\sigma(F); \tag{1.8$'$}$$

this concludes the proof. □

Corollary 1.5. *To compute* $\mu_i^\varepsilon(X)$, $i = 1, 2$ $(0 \le \varepsilon \le 1)$, *it is enough to use symmetric sets; moreover:*

$$\mu_1^\varepsilon(X) = \sup_{F \subset S} \inf_{\|x\| \ge \varepsilon} \mu(F, x), \tag{1.9}$$

$$\mu_2^\varepsilon(X) = \inf_{F \subset S} \sup_{\|x\| \le \varepsilon} \mu(F, x); \tag{1.9$'$}$$

also, the functions $\mu_1^\varepsilon(X)$ *and* $\mu_2^\varepsilon(X)$ *are non decreasing, and we have* $(0 \le \varepsilon \le 1)$*:*

$$1 = \mu_1^0(X) \le \mu_1^\varepsilon(X) \le \mu_1(X) \le 2; \tag{1.10}$$

$$1 = \mu_2^0(X) \le \mu_2^\varepsilon(X) \le \mu_2(X) \le 2; \tag{1.10$'$}$$

Proof. The first assertion is a consequence of Proposition 1.3(v); moreover (use (1.8)):

$$\begin{aligned}\mu_1^\varepsilon(X) &= \sup\Big\{\inf_{\|x\|=\varepsilon} \mu(F, x); F \subset S; F \text{ symmetric}\Big\}\\ &= \sup\Big\{\inf_{\|x\|\ge\varepsilon} \mu(F, x); F \subset S; F \text{ symmetric}\Big\}\\ &\le \sup\Big\{\inf_{\|x\|\ge\varepsilon} \mu(F, x); F \subset S\Big\} \le \mu_1^\varepsilon(X),\end{aligned}$$

which is (1.9); (1.9′) can be obtained in a similar way.

Inequalities (1.10), (1.10′) are almost trivial and follow from (1.9), (1.9′). □

Proposition 1.6. *The functions* $\mu_1^\varepsilon(X), \mu_2^\varepsilon(X)$ *satisfy the following inequalities* $(0 \le \varepsilon \le 1)$*:*

$$\mu_1^\varepsilon(X) \le 1 - \varepsilon + \varepsilon\mu_1(X); \tag{1.11}$$

$$\mu_2^\varepsilon(X) \le 1 - \varepsilon + \varepsilon\mu_2(X). \tag{1.11$'$}$$

Proof. Let $F = \{x_1, \dots, x_n\} \in \mathcal{F}(S)$; for any $y \in S$ set $(0 \le \varepsilon \le 1)$:

(1.12)
$$f_{x_1,\dots,x_n,y}(\varepsilon) = \frac{1}{n}\sum_{i=1}^n \|x_i - \varepsilon y\| \quad (\text{the function } f_{x_1,\dots,x_n,y}(\varepsilon) \text{ is convex in } \varepsilon).$$

Since $\mu_1(F) \le \mu_1(X)$, given $\sigma > 0$ we can take $y \in S$ such that

$$\frac{1}{n}\sum_{i=1}^n \|x_i - y\| < \mu_1(X) + \sigma.$$

Therefore:

$$\begin{aligned}f_{x_1,\dots,x_n,y}(\varepsilon) &= f_{x_1,\dots,x_n,y}((1-\varepsilon)\cdot 0 + \varepsilon \cdot 1)\\ &\le (1-\varepsilon) f_{x_1,\dots,x_n,y}(0) + \varepsilon f_{x_1,\dots,x_n,y}(1)\\ &\le 1 - \varepsilon + \varepsilon(\mu_1(X) + \sigma).\end{aligned}$$

Since this can be done for any $\sigma \ge 0$ and any $\varepsilon \in [0, 1]$, this shows that $\mu_1^\varepsilon(F) \le 1 - \varepsilon + \varepsilon\mu_1(X)$, and then ($F$ being arbitrary) we have (1.11).

Given $\sigma > 0$, let now $F_\sigma = \{x_1, \dots, x_n\} \in \mathcal{F}(S)$ be such that $\mu_2(F_\sigma) < \mu_2(X) + \sigma$, so

$$\frac{1}{n}\sum_{i=1}^{n} \|x_i - y\| < \mu_2(X) + \sigma \quad \text{for all } y \in S.$$

This implies

$$f_{x_1,\dots,x_n,y}(\varepsilon) \le (1-\varepsilon) f_{x_1,\dots,x_n,y}(0) + \varepsilon f_{x_1,\dots,x_n,y}(1) \le 1 - \varepsilon + \varepsilon(\mu_2(X) + \sigma)$$

for all $y \in S$, so $\mu_2^\varepsilon(F_\sigma) \le 1 - \varepsilon + \varepsilon(\mu_2(X) + \sigma)$.

Since $\sigma > 0$ is arbitrary, this implies (1.11′), which completes the proof. □

Remark. Note (see (1.12)) that $\mu_2^\varepsilon(F)$ is a convex function of ε.

2. Some Other Numbers

Now set, for $y \in S$ and $0 \le \varepsilon \le 1$:

$$a(y,\varepsilon) = \sup_{\|x\| = \varepsilon} \frac{\|y+x\| + \|y-x\|}{2}.$$

For any $y \in S$, the function on the right hand side is convex (it is also even in x); thus we have:

$$a(y,\varepsilon) = \sup_{\|x\| \le \varepsilon} \frac{\|y+x\| + \|y-x\|}{2},$$

so $a(y,\varepsilon)$ is a non decreasing function of $\varepsilon \in [0,1]$ $(1 = a(y,0) \le a(y,\varepsilon) \le 1 + \varepsilon \le 2)$.

Now set, for $0 \le \varepsilon \le 1$:

$$A_1^\varepsilon(X) = \inf_{y \in S} a(y,\varepsilon). \tag{2.1}$$

$$A_2^\varepsilon(X) = \sup_{y \in S} a(y,\varepsilon); \tag{2.2}$$

Note that, according to (1.7):

$$A_1^\varepsilon(X) = \inf_{y\in S} \sup_{\|x\| \le \varepsilon} \frac{\|y+x\| + \|y-x\|}{2} = \inf_{y\in S} \sup_{x \in \mathcal{E}} \frac{\|y+\varepsilon x\| + \|y-\varepsilon x\|}{2}; \tag{2.3}$$

$$A_2^\varepsilon(X) = \sup_{y\in S} \sup_{\|x\| \le \varepsilon} \frac{\|y+x\| + \|y-x\|}{2} = \sup_{y\in S} \sup_{x \in \mathcal{E}} \frac{\|y+\varepsilon x\| + \|y-\varepsilon x\|}{2}. \tag{2.4}$$

The constants $A_1^1(X) = A_1(X)$ and $A_2^1(X) = A_2(X)$ have been studied in [**3**]. We summarize in a proposition simple properties of these constants.

Proposition 2.1. *The following facts are true (for $\varepsilon \in [0,1]$):*

a) $A_1^\varepsilon(X), A_2^\varepsilon(X)$ *are non decreasing functions of* ε*;*
b) *the function* $A_2^\varepsilon(X)$ *is convex;*
c) *the functions* $A_1^\varepsilon(X)$ *and* $A_2^\varepsilon(X)$ *are* 1*-lipschitz;*
d) $1 \le \mu_2^\varepsilon(X) \le A_1^\varepsilon(X) \le A_2^\varepsilon(X) \le 1 + \varepsilon$ *always;*
e) *if* Y *is a dense subspace of* X*, then* $A_i^\varepsilon(Y) = A_i^\varepsilon(X)$*,* $i = 1, 2$.
f) *if* $Y \subset X$*, then* $A_2^\varepsilon(Y) \le A_2^\varepsilon(X)$*;*

g) $A_1^\varepsilon(X)$ *and* $A_2^\varepsilon(X)$ *are continuous in* X *with respect to the Banach-Mazur distance, in the class of isomorphic spaces.*
h) $A_2^\varepsilon(X) = \sup\{A_2^\varepsilon(Y); Y$ *is a two dimensional subspace of* $X\}$*;*
k) $A_1^\varepsilon(X) \geq \inf\{A_1^\varepsilon(Y); Y$ *is a two dimensional subspace of* $X\}$.

Remark. Among previous statements, e), f) and g) remain true if we substitute to $A_1^\varepsilon(X)$ and $A_2^\varepsilon(X)$ the functions $\mu_1^\varepsilon(X)$ and $\mu_2^\varepsilon(X)$ respectively (and the same, as already indicated, for a), b), c)). On the other hand, (1.7), Corollary 1.5 and Proposition 1.6 are also true for $A_1^\varepsilon(X)$, $A_2^\varepsilon(X)$; in particular:

$$A_1^\varepsilon(X) \leq 1 - \varepsilon + \varepsilon A_1(X); \tag{2.5}$$

$$A_2^\varepsilon(X) \leq 1 - \varepsilon + \varepsilon A_2(X). \tag{2.5'}$$

Note that in any space, for $\varepsilon \in [0,1]$, we always have:

$$\mu_2^\varepsilon(X) \leq A_1^\varepsilon(X) \leq A_2^\varepsilon(X). \tag{2.6}$$

According to (2.6), by the Example in Section 1 we obtain:

Proposition 2.2. *Let* $X = R_\infty^2$*; then we have:*

$$\mu_2^\varepsilon(R_\infty^2) = A_1^\varepsilon(R_\infty^2) = \max\{1, \varepsilon + 1/2\};$$
$$\mu_1^\varepsilon(R_\infty^2) = 1 + \varepsilon/2.$$

Proof. We have shown (see the Example in Section 1) that $A_1^\varepsilon = 1$, thus also $\mu_2^\varepsilon = 1$, for $\varepsilon \leq 1/2$; the same Example shows that $\mu_2^\varepsilon \leq A_1^\varepsilon \leq \varepsilon + 1/2$ for $1/2 \leq \varepsilon \leq 1$; since $\mu_2^1 = 3/2$ (see [**1**]) and μ_2^ε is 1-lipschitz, we obtain the first part of the thesis.

Concerning μ_1^ε, the quoted example shows that $1 + \varepsilon/2 \leq \mu_1^\varepsilon$; since $\mu_1 = 3/2$, (1.11) implies $\mu_1^\varepsilon \leq 1 + \varepsilon/2$, which concludes the proof. □

Remark. According to the previous proposition, in R_∞^2 we also have

$$A_1^\varepsilon(X) < \mu_1^\varepsilon(X) \quad \text{for } \varepsilon \in (0,1). \tag{2.7}$$

Also: $A_2(R_\infty^2) = 2$ (see [**3**]) implies $A_2^\varepsilon(R_\infty^2) = 1 + \varepsilon$ for $0 \leq \varepsilon \leq 1$.

Note that $A_2^\varepsilon(X) - 1$ is nothing else than Lindenstrauss' modulus of smoothness, defined as:

$$\rho_X(\varepsilon) = \sup\left\{\frac{\|x+y\| + \|x-y\| - 2}{2};\ \|y\| = 1;\ \|x\| \leq \varepsilon\right\}; \quad \varepsilon \in [0,1].$$

Therefore (see e.g. [**12**, p. 63]), it is related to the modulus of convexity $\delta_X(\sigma)$ of X by the formulas:

$$\rho_X(\varepsilon) = \sup\left\{\frac{\varepsilon\delta}{2} - \delta_{X^*}(\sigma);\ 0 \leq \sigma \leq 2\right\};$$
$$\rho_{X^*}(\varepsilon) = \sup\left\{\frac{\varepsilon\sigma}{2} - \delta_X(\sigma); 0 \leq \sigma \leq 2\right\}.$$

According to this equivalence, since $\rho_X(\varepsilon) = \rho_{X^{**}}(\varepsilon)$ always, we deduce that $A_2^\varepsilon(X) = A_2^\varepsilon(X^*)$ always (cf. [**3**, Proposition 2.2]).

Also: $\rho_X(\varepsilon) \geq \sqrt{1+\varepsilon^2} - 1$ always, while equality for every $\varepsilon \in [0,1]$ characterizes inner product spaces; therefore, in any space X, we have:

$$A_2^\varepsilon(X) \geq \sqrt{1+\varepsilon^2}. \tag{2.8}$$

For more general characterizations of inner product spaces with these numbers, see [**12**, p. 82].

3. General Estimates Concerning $A_1^\varepsilon(X)$

We have seen (Proposition 2.2) that it is possible to have $\mu_2^\varepsilon(X) = A_1^\varepsilon(X) = 1$ for $\varepsilon \leq 1/2$. But we shall prove that if ε is not too small ($\varepsilon > 2/3$), then $A_1^\varepsilon(X) > 1$.

Set

$$f(\varepsilon) = \frac{1 + 2\varepsilon + \sqrt{1 + 16\varepsilon + 4\varepsilon^2}}{6}.$$

$f(\varepsilon)$ is an increasing function of ε and we have $f(2/3) = 1$; $f(1) = \frac{3+\sqrt{21}}{6}$ (which is ≈ 1.264, so less than $4/3$). Therefore next proposition gives a non trivial estimate for $\varepsilon \in (2/3, 1]$.

Proposition 3.1. *Let* $2/3 < \varepsilon \leq 1$*; then*

$$A_1^\varepsilon(X) > \frac{1 + 2\varepsilon + \sqrt{1 + 16\varepsilon + 4\varepsilon^2}}{6}. \tag{3.1}$$

Proof. It is enough to prove the statement for any 2-dimensional space X; also, for $\varepsilon = 1$ the result was proved in [**3**], so we assume $\varepsilon \in (2/3, 1)$.

Let $A_1^\varepsilon(X) < 2\varepsilon$ (otherwise the assumption $\varepsilon \geq 2/3$ already implies the thesis). Take $h \in (A_1^\varepsilon(X), 2\varepsilon)$, then choose $x \in S$ such that $\|x - v\| + \|x + v\| \leq 2h$ for every $v \in S_\varepsilon$; now take $y \in S_\varepsilon$ so that $\|x - y\| = \|x + y\|$, say $= k$ ($1 \leq k \leq h$). Set $v = \frac{\varepsilon(x+y)}{k}$ ($v \in S_\varepsilon$); we have: $\|x + v\| = \left\|x\left(1 + \frac{\varepsilon}{k}\right) + \frac{\varepsilon y}{k}\right\| = \left(1 + \frac{\varepsilon}{k}\right) \cdot \left\|x + \frac{\varepsilon}{k+\varepsilon} y\right\|$; it is not a restriction to assume that $\|x + ty\| \geq 1$ for all $t > 0$ (otherwise we exchange y with $-y$), so

$$\begin{aligned} &\|x + v\| \geq 1 + \frac{\varepsilon}{k}, \\ &\|x - v\| = \left\|x\left(1 - \frac{\varepsilon}{k}\right) - \frac{\varepsilon y}{k}\right\| = \left(1 - \frac{\varepsilon}{k}\right) \cdot \left\|x - \frac{\varepsilon}{k - \varepsilon} y\right\|. \end{aligned}$$

Now we want estimate $\left\|x - \frac{\varepsilon}{k-\varepsilon} y\right\|$. From $\varepsilon < 1 < k \leq h < 2\varepsilon$ we obtain $\frac{\varepsilon}{k-\varepsilon} > 1$, so

$$\begin{aligned} \left\|x + \frac{\varepsilon}{k - \varepsilon} y\right\| &\leq \|x + y\| + \left\|-y + \frac{\varepsilon}{k - \varepsilon} y\right\| = k + \varepsilon \cdot \left|\frac{\varepsilon}{k - \varepsilon} - 1\right| \\ &= k + \varepsilon \cdot \frac{2\varepsilon - k}{k - \varepsilon} = \frac{k^2 + 2\varepsilon^2 - 2\varepsilon k}{k - \varepsilon}; \end{aligned}$$

therefore, since $\left\|x - \frac{\varepsilon}{k-\varepsilon} y\right\| + \left\|x + \frac{\varepsilon}{k-\varepsilon} y\right\| \geq \frac{2\varepsilon^2}{k-\varepsilon}$, we obtain

$$\left\|x - \frac{\varepsilon y}{k - \varepsilon}\right\| \geq \frac{2\varepsilon^2}{k - \varepsilon} - \frac{k^2 + 2\varepsilon^2 - 2\varepsilon k}{k - \varepsilon},$$

which implies

$$\|x-v\| \geq \Big(1-\frac{\varepsilon}{k}\Big)\Big(\frac{2\varepsilon^2}{k-\varepsilon}-\frac{k^2+2\varepsilon^2-2\varepsilon k}{k-\varepsilon}\Big)=\frac{2\varepsilon k-k^2}{k}=2\varepsilon-k.$$

Thus we have, for every $k<h$: $2h \geq \|x+v\|+\|x-v\| \geq 1+\frac{\varepsilon}{k}+2\varepsilon-k$, i.e.: $2h \geq 1+\frac{\varepsilon}{h}+2\varepsilon-h$, or $3h^2-2\varepsilon h-h-\varepsilon \geq 0$, which implies (3.1). □

Remark. By using Proposition 3.1, we can see that if a space satisfies $\mu_2^\varepsilon(X)=1+\alpha$, with α small, then at least for some $\varepsilon>2/3$, "big" sets are necessary to approach such value; in fact, given $\sigma>0$, let $F=\{x_1,x_2,\ldots,x_n\}$ and $\sum_{i=1}^n \frac{\|x-x_i\|}{n}<1+\alpha+\sigma$ for every $x\in S_\varepsilon$; it is the same for $F'=\{\pm x_1,\pm x_2,\ldots,\pm x_n\}$ (see Proposition 1.3(v)). But $\sup_{\|x\|=\varepsilon}\frac{\|x+x_1\|+\|x-x_1\|}{2} \geq A_1^\varepsilon(X)>1$ (see the previous proposition); so we obtain, for some $x\in S_\varepsilon$:

$$2n(1+\alpha+\sigma)>\sum_{i=1}^n\|x-x_i\|+\sum_{i=1}^n\|x+x_i\| \geq 2(n-1)+2(A_1^\varepsilon(X)-\sigma).$$

This means that $A_1^\varepsilon(X)<1+\sigma+n(\alpha+\sigma)$, so $n>\frac{A_1^\varepsilon-1-\sigma}{\alpha+\sigma}$; also, if we have $\alpha=0$ and we take $\varepsilon=1$, according to (3.1) we obtain $n>\frac{\sqrt{21}-3}{6\sigma}-1$.

Next proposition gives, for 2-dimensional spaces, a non trivial upper bound concerning $A_1^\varepsilon(X)$, for every $\varepsilon>0$.

Proposition 3.2. *If* $\dim(X)=2$, *then we always have*

$$A_1^\varepsilon(X) \leq 1-\varepsilon+\frac{\varepsilon}{4}(1+\sqrt{33}). \tag{3.2}$$

Proof. It was proved in [**3**, Proposition 2.8], that we always have

$$A_1(X) \leq \frac{1}{4}(1+\sqrt{33}). \tag{3.2$'$}$$

By using (2.5) we obtain (3.2). □

Remark. More precisely (see [**3**, Proposition 2.8]), we can obtain the above estimates, with $4p-1$ instead of 33, p being the perimeter of the unit sphere of X: see [**11**] for the definition of perimeter and Section 4 there for some general results on it ($6 \leq p \leq 8$ in any space). Also, since $A_2^\varepsilon(X)$ is convex, the inequality $A_2(X) \leq p/4$ (see [**3**, Proposition 2.6]) implies for $\varepsilon\in[0,1]$ (see (2.5′)):

$$A_2^\varepsilon(X) \leq 1-\varepsilon+\varepsilon\frac{p}{4}. \tag{3.3}$$

4. New and Old Constants

Note that in any space X, for $0 \leq \varepsilon \leq 1$:

$$\begin{aligned} 1=\sup_{y\in S}\inf_{x\in S}\frac{\|y+\varepsilon x\|+\|y-\varepsilon x\|}{2} &\leq \mu_1^\varepsilon(X) \leq \mu_1(X) \leq \mu_2(X) \\ \leq \inf_{y\in S}\sup_{x\in S}\frac{\|y+x\|+\|y-x\|}{2} &= A_1(X) \leq A_2(X). \end{aligned}$$

Recall that in the euclidean plane (see [**13**]), $\mu_1(X) = \mu_2(X) < A_1(X) = A_2(X) = \sqrt{2}$ and that $1 < \mu_1(X) = \mu_2(X) < 2$ in any finite dimensional space. Moreover, $\mu_2(X) < 2$ if X is reflexive. Also, Proposition 2.2 shows that for $\varepsilon \in [0,1]$, $\mu_1^\varepsilon(X) \le A_1^\varepsilon(X)$ is not true in general (so according to d) of Proposition 2.1, neither $\mu_1^\varepsilon(X) \le \mu_2^\varepsilon(X)$ is true, as said in the Example in Section 1).

Now we want to show another general inequality.

Proposition 4.1. *We always have:*

$$\mu_1^\varepsilon(X) \le A_2^\varepsilon(X). \tag{4.1}$$

Proof. Let $\delta > 0$; according to Corollary 1.5, we can find a symmetric finite set $F \subset S$ such that

$$\inf_{x \in S_\varepsilon} \frac{1}{2n}\left(\sum_{i=1}^{n} \|x_i - x\| + \sum_{i=1}^{n} \|x_i + x\|\right) > \mu_1^\varepsilon(X) - \delta.$$

But also, for every $x \in S_\varepsilon$ and $i = 1, \ldots, n$, we have:

$$\frac{\|x_i - x\| + \|x_i + x\|}{2} \le A_2^\varepsilon(X),$$

so – for every $x \in S_\varepsilon$ – $\mu_1^\varepsilon(X) - \delta < \frac{1}{2n}\left(\sum_{i=1}^{n} \|x_i - x\| + \sum_{i=1}^{n} \|x_i + x\|\right) \le \frac{1}{2n}(2n \cdot A_2^\varepsilon(X))$; since $\delta > 0$ is arbitrary, we obtain (4.1). $\square$

Now we want to indicate some relations among the constants A_1, A_2, studied in [**3**], and the following two "Gao constants", studied in [**4**] and [**7**]:

$$g'(X) = \inf_{y \in S} \sup_{x \in S} \inf(\|x - y\|, \|x + y\|); \tag{4.2}$$

$$G'(X) = \sup_{y \in S} \sup_{x \in S} \inf(\|x - y\|, \|x + y\|). \tag{4.2'}$$

We could also study the "evolution" of the last numbers when we use, in the above definitions, points $x \in S_\varepsilon$, but we do not enter into these details here.

Of course, we always have

$$1 \le g'(X) \le G'(X) \le 2. \tag{4.3}$$

Recall the definition of the **modulus of convexity** of a space:

$$\delta(\varepsilon) = 1 - \sup\left\{\frac{\|x + y\|}{2}; x, y \in S; \|x - y\| \ge \varepsilon\right\}; \quad 0 \le \varepsilon \le 2. \tag{4.4}$$

A space is said to be **uniformly non square** if $\lim_{\varepsilon \to 2^-} \delta(\varepsilon) > 0$; uniformly non square spaces are reflexive, but the converse is not true (see e.g. [**8**] for these results).

The following other facts are known:

$$G'(X) = \sup\left\{\varepsilon; \delta(\varepsilon) \le 1 - \frac{\varepsilon}{2}\right\} \quad \text{(see [4, Proposition 2.1]);} \tag{4.5}$$

in particular:

(4.6) $\sqrt{2} \leq G'(X) \leq 2$ in any space (see [**4**]);

(4.7) $G'(X) = 2$ if and only if X is uniformly non square (see [**4**]);

(4.8) if $\dim(X) = 2$, then $g'(X) \leq \sqrt{2}$ (see [**6**]);

also, if X is two-dimensional, then $g'(X) = \sqrt{2}$ if the unit ball is a circle or a regular octagon; $g'(X) = 4/3$ in case the unit ball is the affine regular hexagon (see [**6**]); $g'(X) = 1$ if the unit sphere is a parallelogram. Also, $g'(X) = 1 \Rightarrow G'(X) = 2$ (see e.g. [**10**, Section 3]); thus $g'(X) = 1 \Rightarrow X$ is not uniformly non square, but not conversely.

It is clear that the following inequalities hold in any space:

$$g'(X) \leq A_1(X); \tag{4.9}$$

$$G'(X) \leq A_2(X); \tag{4.10}$$

it is not difficult to see that no general relation exists between $A_1(X)$ and $G'(X)$. Also, according to (4.7) and to [**3**, (2.7)], we have:

$$G'(X) = 2 \Leftrightarrow A_2(X) = 2 \Leftrightarrow X \text{ is not uniformly non square.} \tag{4.11}$$

Concerning the other extreme value for these constants, we indicated in [**3**] the conjecture that $A_2(X) = \sqrt{2}$ imply that X is an i.p.s. if $\dim(X) \geq 3$; there are 2-dimensional examples of non inner product spaces where $A_2(X) = \sqrt{2}$, and so also where $G'(X) = \sqrt{2}$: note that $G'(X) = \sqrt{2} \Leftrightarrow \delta(\sqrt{2}) = 1 - \sqrt{2}/2$ (as in inner product spaces). So, according to [**12**, p. 70], we can ask the following:

let $\dim(X) \geq 3$; do we have $G'(X) = \sqrt{2} \Leftrightarrow A_2(X) = \sqrt{2} \Leftrightarrow X$ is an i.p.s.?

Concerning the relations between $g'(X)$ and $A_1(X)$, it is known that $g'(X)$ can be 1 while $A_1(X)$ is always larger than 1 (see [**3**, Proposition 2.5]). But it is almost immediate to see that $A_1(X) = 2$ ($\Leftrightarrow A_1^{\varepsilon}(X) = 1 + \varepsilon$ for $\varepsilon \in [0, 1]$) implies $g'(X) = 2$, so (according to (4.9)):

$$g'(X) = 2 \Leftrightarrow A_1(X) = 2. \tag{4.14}$$

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