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A COUNTEREXAMPLE TO A STATEMENT CONCERNING LYAPUNOV STABILITY

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Abstract. We find a class of weakly unimodal C^∞ maps of an interval with zero topological entropy such that no such map f is Lyapunov stable on the set $\mathrm{Per}(f)$ of its periodic points. This disproves a statement published in several books and papers, e.g., by V. V. Fedorenko, S. F. Kolyada, A. N. Sharkovsky, A. G. Sivak and J. Smítal.

1. Introduction and Preliminaries

In a series of papers and books (cf., e.g., [**3**], [**4**], [**7**], [**8**], [**11**]) it is stated that a function $f \in C(I, I)$ has zero topological entropy if and only if f is stable in the sense of Lyapunov on the set $\mathrm{Per}(f)$ of its periodic points. However, this statement is false, see Theorems A and B below. It seems that this false result appeared for the first time, without proof, in [**11**] and then it was only cited in the other papers.

Actually, the above quoted papers and books contain long lists of conditions for continuous maps of the interval, which are equivalent to the condition that f has zero topological entropy. However, another counterexample disproving any of these equivalences is given in [**12**].

By the period of a periodic point we mean its smallest period. If the periods of points in $\mathrm{Per}(f)$ are the powers of 2, then f is of **type** 2^∞. Recall [**9**] that a function f in the class $C(I, I)$ of continuous maps of the compact unit interval I has zero topological entropy if and only if it is of type 2^∞. A map $f \in C(I, I)$ is **unimodal** if there exists $c \in (0, 1)$ such that f is strictly increasing on $[0, c]$ and strictly decreasing on $[c, 1]$. A map f is **weakly unimodal** if there exists $c \in (0, 1)$ such that f is non-decreasing on $[0, c]$ and non-increasing on $[c, 1]$.

Let f be weakly unimodal. We shall say that $x, y \in I$ are **equivalent** (denoted by $x \sim y$) if there exists $n \geq 1$ such that f^n is constant on $[x, y]$. Clearly, $\sim$ is an equivalence relation. Let $\tilde{I} = I/\sim$ be the factor space obtained by identifying to a point each equivalence class. These classes are closed intervals (possibly

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degenerated). The natural projection $\pi\colon I \to \tilde{I}$ is continuous and non-decreasing. Since f is continuous, $x \sim y$ implies $f(x) \sim f(y)$. Therefore there exists a unique map $\tilde{f}\colon \tilde{I} \to \tilde{I}$ such that

$$f \circ \pi = \pi \circ \tilde{f}. \tag{1}$$

This $\tilde{f}$ is continuous and either monotone or unimodal.

A map $f \in C(I, I)$ is **Lyapunov stable** on a set $A \subseteq I$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $|x - y| < \delta$ for x and y in A then $|f^n(x) - f^n(y)| < \varepsilon$ for any n.

Finally, we recall the Feigenbaum map $\Phi\colon I \to I$ which we use as the main tool in our argument. It is the unique unimodal map of type 2^∞ vanishing at the endpoints of I, with a critical point c, a continuous derivative and a compact periodic interval J of period 2, containing c in its interior and such that the composition of $\Phi^2|J$ with an affine scaling is the original map Φ. The properties of Φ are well-known, cf., e.g., [**2**], or [**6**]. In particular, we have the following result:

Lemma 1. *Let a_n be the periodic point of Φ of period 2^n with the largest image under Φ, and let c be the critical point of Φ. Then*

$$a_1 < a_3 < \cdots < c < \cdots < a_4 < a_2 < a_0 < \Phi(c),$$

and

$$\Phi(a_0) < \Phi(a_1) < \Phi(a_2) < \Phi(a_3) < \cdots < \Phi(c).$$

To prove our results we use the approach from [**10**]; in particular, we use the following two lemmas.

Lemma 2. *(a) If $x \in I$ is a periodic point of f of period k then $\pi(x)$ is a periodic point of $\tilde{f}$ of period k. (b) If $y \in \tilde{I}$ is a periodic point of $\tilde{f}$ of period k then there exists a unique periodic point $x \in I$ of f for which $\pi(x) = y$. The period of x is k.*

For a map f let $J_f = \{x \in I;\ f(x) \geq f(y) \text{ for all } y \in I\}$, and let $\mathcal{F}$ be the class of all weakly unimodal maps f, for which

(2) the set J_f contains more than one point,

(3) f is of type 2^∞.

Lemma 3. *If $f \in \mathcal{F}$ then $\tilde{f}$ is unimodal and satisfies* (3).

The following result is well known. See, e.g., [**5**, Proposition 4.3 and 1.3].

Lemma 4. *If a map f is unimodal and satisfies (3), then the relative position of the critical point, its images and the periodic points are the same for Φ and f.*

2. Main Results

Theorem A. *No $f \in \mathcal{F}$ is Lyapunov stable on* $\mathrm{Per}(f)$. *On the other hand, $\mathcal{F}$ consists of mappings with zero topological entropy and contains a C^∞ map.*

Proof. Let $f \in \mathcal{F}$. By (3), f has zero topological entropy. By Lemma 3, $\tilde{f}$ is unimodal and satisfies (3). By Lemma 4, the relative position of the critical point, its images and the periodic points are the same for Φ and $\tilde{f}$. Since π is non-decreasing, by Lemma 2 it is the same also for f. Thus, by Lemma 1, we have

$$b_1 < b_3 < \cdots < d < \cdots < b_4 < b_2 < b_0 < f(d), \tag{4}$$

and

$$f(b_0) < f(b_1) < f(b_2) < f(b_3) < \cdots < f(d), \tag{5}$$

where $d \in J_f$ and b_n is the periodic point of f of period 2^n with the largest image under f. Let $\varepsilon > 0$ be the length of interval J_f, $\delta > 0$, and let $p = f(b_{2n+1})$, $q = f(b_{2n})$. Then p, q are periodic points and by (5), $0 < p - q < \delta$ for any sufficiently large n. But $f^{2^{2n+1}-1}(p) = b_{2n+1}$ and $f^{2^{2n+1}-1}(q) = b_{2n}$. Thus, by (5), $f^{2^{2n+1}-1}(p) - f^{2^{2n+1}-1}(q) > \varepsilon$. To finish the proof note that $\mathcal{F}$ contains a C^∞ map [**10**]. $\square$

For completeness, we show that the condition of zero topological entropy is necessary for Lyapunov stability on the set of periodic points. The proof is based on a standard argument.

Theorem B. *Let $f \in C(I,I)$. If f is Lyapunov stable on* $\mathrm{Per}(f)$ *then f has zero topological entropy.*

Proof. Assume, contrary to what we wish to show, that f has a positive topological entropy. Apply a known result (cf., e.g., [**1**]) that $f \in C(I,I)$ has positive topological entropy if and only if, for some positive integer n there exist compact disjoint subintervals J, K of I such that $J \cup K \subseteq f^n(J) \cap f^n(K)$. Without loss of generality assume that $n = 1$ (otherwise replace f by f^n). Let K, L be compact disjoint intervals in I such that $K \cup L \subseteq f(K) \cap f(L)$. Put $\varepsilon = \mathrm{dist}(K, L)$. Choose an arbitrary $\delta > 0$. By induction we can see that, for any positive integer m, there exist 2^m disjoint compact intervals $K_1^m, K_2^m, \dots, K_{2^m}^m$ contained in K such that

(6) $f^m(K_i^m) = K$ for $1 \le i \le 2^{m-1}$ and $f^m(K_i^m) = L$ for $2^{m-1} < i \le 2^m$.

For a sufficiently large m, there are i, j such that $1 \le i \le 2^{m-1} < j \le 2^m$ and the diameter of the set $K_i^m \cup K_j^m$ is less then δ. Since$K_i^m \cup K_j^m \subset f^{m+1}(K_i^m) \cap f^{m+1}(K_j^m)$, there exist periodic points $p \in K_i^m$ and $q \in K_j^m$ such that $f^{m+1}(p) = p$ and $f^{m+1}(q) = q$. By (6), $|f^m(p) - f^m(q)| > \varepsilon$, i.e., f is not Lyapunov stable on $\mathrm{Per}(f)$. $\square$

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