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# LYAPUNOV EXPONENTS FOR THE PARABOLIC ANDERSON MODEL

M. CRANSTON, T. S. MOUNTFORD AND T. SHIGA

ABSTRACT. We consider the asymptotic almost sure behavior of the solution of the equation

$$u(t,x) \quad = \quad u_0(x) + \kappa \int_0^t \Delta u(s,x)ds + \int_0^t u(s,x)\partial B_x(s)$$
$$u(0,x) = u_0(x)$$

where $\{B_x : x \in \mathbf{Z}^d\}$ is a field of independent Brownian motions.

## 1. INTRODUCTION

We make an asymptotic study of the Lyapunov exponent of the parabolic Anderson model with white noise potential. Start with a collection of independent, one-dimensional, Brownian motions $\{B_x(t) : x \in \mathbf{Z^d}, t \geq 0\}$. This provides a random environment defined on a probability space $(\Omega, \mathcal{F}, Q)$. Fixing a $\kappa > 0$ denote by $(X(t), t \geq 0, \mathcal{F}_t, P_x)$ the symmetric random walk on $\mathbf{Z^d}$ with jump rate $\kappa$. We assume this process is independent of the field $\{B_x(t) : x \in \mathbf{Z^d}, t \geq 0\}$. Under $P_x$, $\{X(t), t \geq 0\}$ is the pure jump Markov process on $\mathbf{Z^d}$ started at $x$ with generator $\kappa\Delta$ where $\Delta$ is the discrete Laplacian defined by $\Delta f(x) = \frac{1}{2d}\sum_{|y-x|=1}(f(y) - -f(x))$. Then we consider the solution of the stochastic equation

$$u(t,x) \quad = \quad u_0(x) + \kappa \int_0^t \Delta u(s,x)ds + \int_0^t u(s,x)\partial B_x(s) \tag{1.1}$$
$$u(0,x) = u_0(x) \tag{1.2}$$

where $\partial$ denotes the Stratonovich differential. A result of Carmona and Molchanov [**4**] is that provided $u_0$ is bounded, the solution is given by means of the Feynman-Kac formula

$$u(t,x) = E_x[u_0(X(t))e^{\int_0^t dB_{X(s)}(t-s)}] \,.$$

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For important background material on the significance of the model we refer the reader to the memoir [**4**] and the references therein. The equation (1.1) and its variants arise in a variety of circumstances. Equation (1.1) can be written in differential form as
$$\frac{\partial u}{\partial t} = \kappa \Delta u + V u$$
with a potential $V = \partial B$. The elliptic version of this equation originated in the work of Anderson on entrapment of electrons in crystals with impurities. The parabolic equation can be viewed as a scalar version of magnetic fields in the presence of turbulence as described in Molcahnov and Ruzmaikin [**10**] and also has an interpretation as a population model. Additional references on the subject is Zeldovitch, Molchanov, Ruzmaikin, and Sokoloff [**14**] and Shiga [**12**].

When $\kappa = 0$, the solution is simply $u(t,x) = u_0(x)e^{-B_x(t)}$. In this case, $\lim_{t\to\infty} \frac{1}{t}\log u(t,x) = 0$. However, when $\kappa > 0$ the situation is quite different. A result of [**4**] is that when $u_0$ has compact support, then
$$\lim_{t\to\infty} \frac{1}{t}\log u(t,x) = \lambda(\kappa),\ Q \text{ a.s.}$$
The positive constant $\lambda(\kappa)$ is called the Lyapunov exponent. This was proven by a subadditivity argument which did not extend to the noncompact support case. Moreover, $\lambda(\kappa)$ is independent of $u_0$ and a principal result of [**4**] was the existence of constants $c_1, c_2$ such that for all small $\kappa > 0$,
$$c_1 \big/ \log\frac{1}{\kappa} \le \lambda(\kappa) \le c_2 \log\log\frac{1}{\kappa} \big/ \log\frac{1}{\kappa}\ .$$
This was improved in their later work [**5**] to
$$c_1 \big/ \log\frac{1}{\kappa} \le \lambda(\kappa) \le c_2 \big/ \log\frac{1}{\kappa}\,,$$
again for small $\kappa$. That is, $\lambda$ is continuous at 0 but increases dramatically with $\kappa$. In the present paper we remove the restriction that $u_0$ has compact support by showing[1]
$$\lim_{t\to\infty} \frac{1}{t}\log u(t,x) = \lambda(\kappa)$$
when $u_0$ is a bounded nonnegative, not identically 0 function and that again $\lambda(\kappa)$ is independent of $u_0$. This will follow from the case where $u_0$ has compact support and a recurrence property of oriented percolation. We also show
$$\lim_{\kappa\downarrow 0} \lambda(\kappa)\log\frac{1}{\kappa} = c$$
where we identify $c$ by means of a subadditive ergodic theory argument. We would like to remark that subadditivity plays a major role in the theory of stochastic flows. It is the main ingredient in the proof of Oseledets Theorem. We refer the reader to the books [**2**] and [**3**] for further information on the subject. In the final

[1]After this work was submitted, L. Koralov kindly pointed out the reference *Asymptotics for the almost sure Lyapunov exponent for the solution of the parabolic Anderson problem.* Random Oper. Stochastic Equations, **9** (2001), no.1, 77–86, by R. Carmona, L. Koralov, S. Molchanov, where this limit was also established.

section we use results of [**7**] to show that $\lambda(\kappa)$ is the derivative at $p = 0$ of the moment Lyapunov exponent, $M_p(t) = E_Q[u^p(t,x)]$.

Our results differ in approach from earlier work on the subject in their use of the subadditive ergodic theorem and percolation theory. This approach was developed in Mountford [**11**] in dealing with survival probabilities of the random walk $X(t)$ in a disastrous environment. In the present context this relies on introducing the space of deterministic paths on which the measure $P_x$ is concentrated when the paths of $X$ are restricted to have a specific number of jumps. For an interval $[a,b] = I \subset [0,\infty)$ and $\gamma : I \to \mathbf{Z}$ let $N(\gamma, I)$ denote the number of jumps of $\gamma$ in the interval $I$. We shall only deal with paths $\gamma$ which are right continuous with left limits. Then for $x \in \mathbf{Z^d}$ define

$$\Gamma^x_{I,j} = \{\gamma : I \to \mathbf{Z^d}, \gamma(a) = x, |\gamma(s) - \gamma(s-)| \leq 1, \text{ for all } s \in I, N(\gamma, I) = j\}.$$

We also define $\Gamma^{x,\epsilon}_{I,j}$ to be the intersection of $\Gamma^x_{I,j}$ with those paths with jump times separated from one another by $\epsilon$ and such that there are no jump times within $\frac{\epsilon}{2}$ of the boundary of $I$. Now we consider the functional

$$A^x_{I,j} = \sup_{\gamma \in \Gamma^x_{I,j}} \int_a^b dB_{\gamma(s)}(b-s)\ .$$

The corresponding functional when restricting jumps to be $\epsilon$ apart will be

$$A^{x,\epsilon}_{I,j} = \sup_{\gamma \in \Gamma^{x,\epsilon}_{I,j}} \int_a^b dB_{\gamma(s)}(b-s)\ .$$

Obviously, $A^x_{I,j} \geq A^{x,\epsilon}_{I,j}$. The functional $A^{x,\epsilon}_{I,j}$ will be important in establishing the lower bound in Theorem 2.14. We simplify the notation a little by first setting $A_{0,n} = A^0_{[0,n],n}$. The fact that time is running in opposite directions in the functional $A$ makes it a little clumsy to work with. Instead we define a functional with the same distribution. This will suffice to give all the conclusions we need about $A$. Thus, we set

$$\vec{A}^x_{I,j} = \sup_{\gamma \in \Gamma^x_{I,j}} \int_a^b dB_{\gamma(s)}(s).$$

Make the simplified notation $\vec{A}^0_{[0,n],n} = \vec{A}_{0,n}$. Similarly, set

$$\vec{A}^{x,\epsilon}_{I,j} = \sup_{\gamma \in \Gamma^{x,\epsilon}_{I,j}} \int_a^b dB_{\gamma(s)}(s)$$

and use the simplified notation $\vec{A}^\epsilon_{0,n} = \vec{A}^{0,\epsilon}_{[0,n],n}$. One thing to notice here is that time in the $dB$ and $\gamma$ terms are running in the same direction for the functionals denoted by $\vec{A}$ . A very important fact is a scaling relation for the functionals $A_{[0,n],j}$ and $\vec{A}_{[0,n],j}$ inherited from the scaling properties of the Brownian field $\{B_x(t) : t \geq 0, x \in \mathbf{Z}^d\}$. Namely, for $j, n \in \mathbf{Z}$, (using the notation $\stackrel{\mathcal{L}}{=}$ for equality in law)

$$\begin{aligned}
A^0_{[0,n],jn} &= \sup_{\gamma\in\Gamma^0_{[0,n],jn}} \int_0^n dB_{\gamma(s)}(n-s) = \sup_{\gamma\in\Gamma^0_{[0,n],jn}} \int_0^{jn} dB_{\gamma(\frac{s}{j})}(n-\frac{s}{j}) \\
&= \sup_{\gamma\in\Gamma^0_{[0,jn],jn}} \int_0^{jn} dB_{\gamma(s)}(\frac{1}{j}(jn-s)) \stackrel{\mathcal{L}}{=} \frac{1}{\sqrt{j}} \sup_{\gamma\in\Gamma^0_{[0,jn],jn}} \int_0^{jn} dB_{\gamma(s)}(jn-s) \\
&= \frac{1}{\sqrt{j}} A^0_{[0,jn],jn}.
\end{aligned}$$

A similar computation gives the same scaling relation for $\vec{A}_{[0,n],jn}$.

Another observation is that $\vec{A}_{0,n}$ is the supremum of the centered Gaussian field $\{\int_a^b dB_{\gamma(s)}(s) : \gamma \in \Gamma^0_{[0,n],n}\}$. Thus, results such as Borell's inequality are available for our analysis. For the reader's convenience, we state the results from the theory of Gaussian processes which we shall use. Both results may be found in Adler [**1**].

**Theorem 1.1. (C. Borell)** *Let $T$ be a Polish space and $\{X_t\}_{t\in T}$ be a centered, separable, Gaussian field with $\sup_{t\in T} X_t < \infty$ a.s.. Then $E\left(\sup_{t\in T} X_t\right) < \infty$ and for all $\lambda > 0$*

$$P\left(|\sup_{t\in T} X_t - E\left(\sup_{t\in T} X_t\right)| > \lambda\right) \le 2e^{-\lambda^2/2\sigma_T^2},$$

*where $\sigma_T^2 = \sup_{t\in T} E(X_t^2)$.*

Let $(X_t)_{t\in T}$ be a centered separable Gaussian field with the pseudo-metric $\rho$ on $T$;

$$\rho(t,s) = \sqrt{E(X_t - X_s)^2}.$$

**Theorem 1.2. (Fernique-Talagrand)** *There exists a universal constant $K > 0$ such that*

$$E(\sup_{t\in T} X_t) \le K \int_0^\infty \sqrt{\log N(\epsilon)}\, d\epsilon$$

*where $N(\epsilon)$ is the least number of $\rho$-balls of radius $\epsilon$ required to cover $T$.*

Next, given $m, n \in \mathbf{Z}^+$ determine $x \in \mathbf{Z^d}$ as follows, let $x$ be the smallest value of $\gamma(m)$ under any well ordering of $\mathbf{Z^d}$ taken among paths $\gamma$ which achieve the maximum for $\int_0^m dB_{\gamma(s)}(s)$. Then set $\vec{A}_{m,m+n} = \vec{A}^x_{[m,m+n],n}$. Since $\vec{A}_{0,m+n}$ is a supremum over paths which have $m+n$ jumps on the interval $[0, m+n]$ which doesn't impose any restriction on how many jumps occur in $[0, m]$ or $[m, m+n]$ it follows that

$$\vec{A}_{0,m+n} \ge \vec{A}_{0,m} + \vec{A}_{m,m+n}.$$

Then by Liggett's subadditive ergodic theorem (applied to $-\vec{A}_{0,n}$) we have the following

**Theorem 1.3.** *There is a positive $\alpha$ such that*

$$\lim_{n\to\infty} \frac{\vec{A}_{0,n}}{n} = \alpha\,.$$

*For $\epsilon > 0$ there is an $\alpha(\epsilon)$ such that*

$$\lim_{n\to\infty} \frac{\vec{A}^{\epsilon}_{0,n}}{n} = \alpha(\epsilon)\ .$$

*Moreover,* $\lim_{\epsilon\downarrow 0} \alpha(\epsilon) = \alpha$.

*Proof.* The conditions of Liggett's subadditive ergodic theorem [**9**] apply. The positivity of $\alpha$ is already apparent if one considers the supremum of $\int_0^t dB_{\gamma(s)}(s)$ over paths which are constrained to visit only two adjacent sites in $\mathbf{Z^d}$. The finiteness of $\alpha$ follows from the entropy bound in **Theorem 1.2**. Namely, for some universal constant $K$,

$$E_Q[\vec{A}_{0,n}] \le K\int_0^\infty \sqrt{\log N(\epsilon)}d\epsilon,$$

where $N(\epsilon)$ is the smallest number of $\epsilon$ balls required to cover $\Gamma^0_{[0,n],n}$ using balls with respect to the canonical metric $\rho$. By scaling, $E_Q[\vec{A}_{0,n}] = \sqrt{n}E_Q[\vec{A}_{[0,1],n}]$. Thus we will apply the entropy bound to $E_Q[\vec{A}_{[0,1],n}]$. Let $W_n$ be the set of $n$-step random walk paths on $\mathbb{Z}^d$ and let

$$T_n = \{t = (t_1, \cdots, t_n) : 0 < t_1 \le \cdots \le t_n \le 1\}.$$

Identify $\Gamma_{[0,1],n}$ with $W_n \times T_n$, where $W_n$ represents the sequence of jumps and $T_n$ gives the jump times. Then it holds that

$$\begin{aligned}\rho((w,t),(\tilde{w},s))^2 &= E_Q[\sum_{k=0}^{n}(B_{w_k}(t_{k+1}) - B_{w_k}(t_k)) - \sum_{l=0}^{n}(B_{\tilde{w}_l}(s_{l+1}) - B_{\tilde{w}_l}(s_l))]^2\\ &= 2(1 - \sum_{k=1}^{n}\sum_{l=1}^{n}\delta_{w_k\tilde{w}_l}|(t_k,t_{k+1})\cap(s_l,s_{l+1})|) \le 2,\end{aligned}$$

with $t_{n+1} = s_{n+1} = 1$. As is easily seen, the $\rho$ balls of radius $\sqrt{2\epsilon n}$ centered at the points

$$(w,(t_1,\cdots,t_n)),$$

which have $k_i$ evenly spaced jumps in $((i-1)\epsilon, i\epsilon], 1 \le i \le \frac{1}{\epsilon}$, and such that $\sum_{i=1}^{\frac{1}{\epsilon}} k_i = n$ cover $W_n \times T_n$. And since the number of vectors $(k_1, k_2, \cdots, k_{\frac{1}{\epsilon}})$ which have nonnegative integer components and with $\sum_{i=1}^{\frac{1}{\epsilon}} k_i = n$ is by elementary combinatorics, $\binom{[\frac{1}{\epsilon}]+n}{n}$, we have

$$N(\sqrt{2\epsilon n}) \le (2d)^n \binom{[\frac{1}{\epsilon}]+n}{n}.$$

Thus,

$$N(\epsilon) \leq (2d)^n \binom{[\frac{2n}{\epsilon^2}] + n}{n}.$$

But by Stirling's formula,

$$\binom{[\frac{2n}{\epsilon^2}] + n}{n} \simeq c \frac{((1+\frac{2}{\epsilon^2})n)^{(1+\frac{2}{\epsilon^2})n+\frac{1}{2}}}{n^{n+\frac{1}{2}}(\frac{2}{\epsilon^2}n)^{\frac{2}{\epsilon^2}n+\frac{1}{2}}} \leq cC^n\epsilon^{-2n}.$$

Hence

$$N(\epsilon) \leq c(2dC)^n\epsilon^{-2n}.$$

Thus,

$$\int_0^\infty \sqrt{\log N(\epsilon)}d\epsilon \leq \sqrt{n}\int_0^2 \sqrt{\log \frac{M}{\epsilon^2}}d\varepsilon \leq C\sqrt{n}.$$

Thus we obtain

$$E_Q[\vec{A}_{[0,n],n}] \leq Cn,$$

which implies

$$\alpha = \lim_{n\to\infty} \frac{1}{n} E_Q[\vec{A}_{[0,n],n}] = \sup_n \frac{1}{n} E_Q[\vec{A}_{[0,n],n}] \leq C.$$

The proof for the second claim is entirely analogous. The proof that $\lim_{\epsilon\downarrow 0}\alpha(\epsilon) = \alpha$ is straightforward and is omitted. □

From the scaling property we have immediately

**Corollary 1.4.** *For any $j \in \mathbf{Z}$*

$$\lim_{n\to\infty} \frac{1}{n}\vec{A}^0_{[0,n],jn} = \sqrt{j}\alpha,\ Q\,a.\,s..$$

The Corollary quantifies the improvement in the functional $\vec{A}^0_{[0,n],jn}$ as a function of $j$: with more jumps, a greater value is achieved. Now when considering which paths of the Markov process $X$ contribute the principal term in $u(t,x)$, paths with $jt$ jumps give a potentially greater value for $\int_0^t dB_{X(s)}(t-s)$ but the probability of making $jt$ jumps in $[0,t]$ decreases with $j$ for $j$ large. Our work shows that $j^*t$ jumps, with

$$j^* = \frac{\alpha^2}{4\log^2\frac{1}{\kappa}}$$

is the optimal number of jumps during $[0,t]$ in terms of balancing payoff $\int_0^t dB_{X(s)}(t-s)$ versus probability. Thus, the principal contribution in the Feynman-Kac representation for $u(t,x)$ arises from paths with $j^*t$ jumps.

## 2. Existence of a limit for $\frac{\log u(t,x)}{t}$

We aim to establish the existence, for arbitrary bounded, nonnegative $u_0$, of the

$$\lim_{t\to\infty} \frac{1}{t} \log u(t,x),\ Q\,a.\,s..$$

In fact, we shall assume without loss of generality that $u_0(0) > 0$. Recall, this was established by Carmona and Molchanov [**4**] in the case of compactly supported $u_0$. We first establish an existence result for a functional with the same distribution as $u(t,x)$ when $u_0$ is taken to be identically 1. In this paper we treat explicitly the case where dimension $d = 1$. This is simply to minimize notation. It will be clear that the arguments easily extend to higher dimensions. Also we make use of some large deviations results, Theorem 1.1 and Theorem 2.11.

**Theorem 2.1.** *There is a $\lambda_1(\kappa)$ such that for any $x \in \mathbf{Z^d}$*

$$\lambda_1(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log E_x[e^{\int_0^t dB_{X(s)}(s)}],\ Q\,a.\,s.$$

*Proof.* By translation invariance we may take $x$ equal to 0. Set

$$v(t,y) = E_0[e^{\int_0^t dB_{X(s)}(s)} 1_y(X(t))]$$
$$Z(t) = \max_{y\in\mathbf{Z}} v(t,y)\,.$$

The functional $\log Z(t)$ is superadditive and Liggett's subadditive ergodic Theorem (applied to $-\log Z(t)$) gives the $Q\,a.\,s.$ existence of

$$\lim_{t\to\infty} \frac{\log Z(t)}{t} = \lambda_1(\kappa)$$

for some constant $\lambda_1(\kappa)$. From its definition we have that

$$Z(t) \ \geq\ E_0[e^{\int_0^t dB_{X(s)}(s)} 1_0(X(t))]$$

and so (recalling the results of Carmona and Molchanov [**4**] mentioned in the Introduction) we have that $\lambda_1(\kappa) \geq \lambda(\kappa)\ >\ 0$.

However,

$$\log E_0[e^{\int_0^t dB_{X(s)}(s)}] \geq \log Z(t)$$

so

$$\liminf_{t\to\infty} \frac{1}{t} \log E_0[e^{\int_0^t dB_{X(s)}(s)}] \geq \lambda_1(\kappa),\ Q\,a.s.$$

On the other hand,

$$\begin{aligned}\frac{1}{t}\log E_0[e^{\int_0^t dB_{X(s)}(s)}] &= \frac{1}{t}\log\big(E_0[\Sigma_{|x|\leq t^2} e^{\int_0^t dB_{X(s)}(s)} 1_x(X(t))] + \\ &\quad + E_0[e^{\int_0^t dB_{X(s)}(s)} 1_{(t^2,\infty)}(|X(t)|)]\big) \\ &\leq \frac{1}{t}\big(\log(ct^2 e^{Z(t)} + E_0[e^{\int_0^t dB_{X(s)}(s)} 1_{(t^2,\infty)}(|X(t)|)])\big).\end{aligned}$$

But,

$$E_Q[E_0[e^{\int_0^t dB_{X(s)}(s)}1_{(t^2/2,\infty)}(|X(t)|)]] = E_0[E_Q[e^{\int_0^t dB_{X(s)}(s)}1_{(t^2/2,\infty)}(|X(t)|)]]$$
$$= e^{\frac{t}{2}}P_0(|X(t)| \geq t^2/2) \leq e^{-ct^2},$$

for some positive $c$. Chebychev's inequality then implies

$$Q(E_0[e^{\int_0^t dB_{X(s)}(s)}1_{(t^2/2,\infty)}(|X(t)|)] > \frac{1}{K}) \leq e^{-ct^2}$$

for $K$ very large but fixed. So by Borel-Cantelli,

$$Q(E_0[e^{\int_0^n dB_{X(s)}(s)}1_{(n^2/2,\infty)}(|X(n)|)] > \frac{1}{K},\ i.o.) = 0.$$

Now define a stopping time $T$ with respect to $\mathcal{F}_t = \sigma\{B_x(s) : x \in \mathbf{Z}, 0 \leq s \leq t\}$ by $T = \inf\{t \geq N_0 : E_0[e^{\int_0^t dB_{X(s)}(s)}1_{(t^2,\infty)}(|X(t)|)] \geq 1\}$.

A simple two moment argument then gives that on the event $\{T < \infty\}$, for $T \in (n-1, n]$, one has $Q(E_0[e^{\int_0^n dB_{X(s)}(s)}1_{(n^2/2,\infty)}(|X(n)|)] < \frac{1}{K}|\mathcal{F}_T)$ tends to 0 as $N_0$ tends to $\infty$. Letting $N_0$ tend to infinity one obtains that $Q$ a.s. there exists $t_0$ so that

$$E_0[e^{\int_0^t dB_{X(s)}(s)}1_{(t^2,\infty)}(|X(t)|)] \ \leq\ 1, \text{ for } t \geq t_0.$$

Thus, for $t$ large,

$$\begin{aligned}\frac{1}{t}\log E_0[e^{\int_0^t dB_{X(s)}(s)}] &\leq \frac{1}{t}\log(ct^2 e^{Z(t)} + 1)\\ &\leq \frac{Z(t)}{t} + \frac{O(\log t)}{t}.\end{aligned}$$

Consequently, $Q\,a.s.$,

$$\limsup_{t\to\infty}\frac{1}{t}\log E_0[e^{\int_0^t dB_{X(s)}(s)}] \leq \lambda_1(\kappa)$$

and therefore,

$$\lim_{t\to\infty}\frac{1}{t}\log E_0[e^{\int_0^t dB_{X(s)}(s)}] = \lim_{t\to\infty}\frac{1}{t}\log Z(t)$$

and the Theorem is proved. □

Our goal of proving the existence of $\lim_{t\to\infty}\frac{1}{t}\log u(t,x), Q\,a.\,s.$, is enabled by the introduction of a few auxiliary functionals.

Set

$$Z(t,t^2) = \max_{x\in\mathbf{Z}^d, |x|\leq t^2} v(t,x).$$

Notice that $v(t,x) \stackrel{\mathcal{L}}{=} E_0[e^{\int_0^t dB_{X(s)}(t-s)}1_x(X(t))]$ and we will use information about $v$ to get the result on $u$. We also remark that by translation invariance, $v(t,x-y) \stackrel{\mathcal{L}}{=}$ $\stackrel{\mathcal{L}}{=} E_y[e^{\int_0^t dB_{X(s)}(s)}1_x(X(t))]$.

**Theorem 2.2.** *For any* $x \in \mathbf{Z^d}$

$$\lambda(\kappa) = \lim_{t\to\infty}\frac{1}{t}\log E_x[e^{\int_0^t dB_{X(s)}(s)}],\ Q\,a.\,s..$$

The proof will be broken up into a sequence of lemmas.

**Lemma 2.3.** *For $M$ sufficiently large, if $a < 2$, then*

$$Q(E_0[e^{\int_0^t dB_{X(s)}(s)} 1_{[Mt,\infty)}(N(X,t))] \le e^{at}) \ge 1 - e^{-(2+a)t}.$$

*Proof.* By Fubini and Cauchy-Schwarz,

$$\begin{aligned} E_Q[E_0[e^{\int_0^t dB_{X(s)}(s)} 1_{[Mt,\infty)}(N(X,t))]] &= E_0[E_Q[e^{\int_0^t dB_{X(s)(s)}}] 1_{[Mt,\infty)}(N(X,t))] \\ &= e^{t/2} P_0(N(X,t) \in [Mt,\infty)) \\ &\le e^{(1/2+e\kappa-\kappa-M)t}. \end{aligned}$$

Then, provided $M$ is large enough, Chebychev's inequality completes the proof. □

Now fix an $\epsilon > 0$. In all that follows we shall assume, without loss of generality, that $Mt \in \mathbf{Z}$.

**Lemma 2.4.** *For $M$ and $t$ sufficiently large,*

$$Q(E_0[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{100})t}) \ge 1 - \epsilon^{100}.$$

*Proof.* Using Lemma 2.3, with $a = -1$, and Theorem 2.1, we have for large $t$, with $Q$ probability at least $1 - \epsilon^{100}$, that

$$\begin{aligned} E_0[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t))] &= E_0[e^{\int_0^t dB_{X(s)}(s)}] \\ &- E_0[e^{\int_0^t dB_{X(s)}(s)} 1_{[Mt,\infty)}(N(X,t))] \\ &\ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{50})t} - e^{-t} \end{aligned}$$

and the lemma follows by a little algebra. □

An immediate consequence is the following,

**Corollary 2.5.** *For $t$ and $M$ sufficiently large, given any $x \in \mathbf{Z}$, there exists an $x^*(t,x) \in \sigma(B_y(s) : y \in \mathbf{Z}, |y-x| \le Mt, s \le t)$ such that*

$$Q(E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x^*(t,x)}(X(t))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99})t}) \ge 1 - \epsilon^{100}.$$

A more refined consequence is

**Corollary 2.6.** *Given any $x \in \mathbf{Z}$ there exist $x_+^* = x_+^*(t,x) > x$ and $x_-^* = x_-^*(t,x) < x$, with $x_\pm^*(t,x) \in \sigma(B_y(s) : y \in \mathbf{Z}, |y-x| \le Mt, s \le t)$ such that*

$$Q(E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x_+^*}(X(t))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99})t}) \ge 1 - \epsilon^{50}$$

*and*

$$Q(E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x_-^*}(X(t))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99})t}) \ge 1 - \epsilon^{50}.$$

*Proof.* This is an easy use of the FKG inequalities [**8**] and the observation that the events

$$\{E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x^*}(X(t))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99})t}, \text{for some } x^* > x\}$$

and

$$\{E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x^*}(X(t))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99})t}, \text{for some } x^* < x\}$$

are positively correlated, equally likely and the probability of their union is at least $1-\epsilon^{100}$ by Corollary 2.5. $\square$

We can move the points $x^*_\pm$ a little further away from $x$,

**Lemma 2.7.** *For $M$ and $t$ sufficiently large there is a $c_1 = c_1(\kappa, M)$ such that for all $x \in \mathbf{Z}$ there exist $x^*_+ = x^*_+(t,x) \in [x + \frac{\epsilon}{2}Mt, x + (1-\frac{\epsilon}{2})Mt]$ and $x^*_- = x^*_-(t,x) \in [x - (1-\frac{\epsilon}{2})Mt, x - \frac{\epsilon}{2}Mt]$, $x^*_\pm(t,x) \in \sigma(B_y(s) : y \in \mathbf{Z}, |y-x| \le Mt, s \le t)$ such that*

$$Q(E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x^*_+}(X(t))] \ge e^{(\lambda_1(\kappa)-c_1\epsilon)t}) \ge 1-\epsilon^{49}$$

*and*

$$Q(E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x^*_-}(X(t))] \ge e^{(\lambda_1(\kappa)-c_1\epsilon)t}) \ge 1-\epsilon^{49}.$$

*Proof.* By symmetry, we only need to argue the first case. By Corollary 2.6, with probability at least $1-\epsilon^{50}$, there is an $x^*_+(t(1-\epsilon)) \in [x, x+(1-\epsilon)Mt]$ such that

$$E_x[e^{\int_0^{t(1-\epsilon)} dB_{X(s)}(s)} 1_{[0,Mt(1-\epsilon)]}(N(X,(t(1-\epsilon)))) 1_{x^*_+(t(1-\epsilon))}(X(t(1-\epsilon)))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99})(t-\epsilon)}.$$

If $x^*_+(t(1-\epsilon)) \in [x+\frac{\epsilon}{2}Mt, x+(1-\frac{\epsilon}{2})Mt]$, we consider the constant path identically equal to $x^*_+(t(1-\epsilon))$ over the interval $[t(1-\epsilon), t]$. Then $P_{x^*_+(t(1-\epsilon))}(N(X,\epsilon t) = 0) = e^{-\kappa\epsilon t}$ and $Q(e^{\int_{t(1-\epsilon)}^t dB_{x^*_+(t(1-\epsilon))}(s)} \ge e^{-\epsilon\frac{1}{2}t}) \ge 1-e^{-\epsilon ct}$. Thus, in the case, we can paste the piece of path from 0 to $x^*_+(t(1-\epsilon)) > x+\frac{\epsilon Mt}{2}$ on $[0, t(1-\epsilon)]$ to a constant path on $[t(1-\epsilon), t]$ and by setting $x^*_+(t) = x^*_+(t(1-\epsilon)) \in [x+\frac{\epsilon}{2}Mt, x+(1-\frac{\epsilon}{2})Mt]$, we get

$$Q(E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x^*_+(t)}(X(t))] \ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99}-\epsilon\frac{1}{2}t)} \ge 1-\epsilon^{50}-e^{-\epsilon ct}.$$

This gives the result if $x^*_+(t(1-\epsilon)) \in [x+\frac{\epsilon}{2}Mt, x+(1-\frac{\epsilon}{2})Mt]$. On the other hand, if first $x < x^*_+(t(1-\epsilon)) < x+\frac{\epsilon}{2}Mt$, we consider paths in $\Gamma^{x^*_+(t(1-\epsilon))}_{[t(1-\epsilon),t],\epsilon Mt}$ for which all jumps are to the right. Call this set $\Gamma_{right}$. Then, for some $c > 0$ it follows that for all $\gamma \in \Gamma_{right}$

$$Q(e^{\int_{t(1-\epsilon)}^t dB_{\gamma(s)}(s)} \ge e^{-\epsilon\frac{1}{2}t}) \ge 1-e^{-\epsilon ct}.$$

Also, for $K_1 = 2M\log\frac{2M}{e\kappa}$, $P_{x^*_+(t(1-\epsilon))}(X(\cdot - t(1-\epsilon)) \in \Gamma_{right}) \ge e^{-\epsilon K_1 t}$. Thus, using the Markov property and the above estimates, we have

$$\begin{aligned} Q(E_x[e^{\int_0^t dB_{X(s)}(s)} 1_{[0,Mt]}(N(X,t)) 1_{x^*_+(t(1-\epsilon))+\epsilon Mt}(X(t))] &\ge e^{(\lambda_1(\kappa)-\frac{\epsilon}{99}-\epsilon(\frac{1}{2}+K_1))t}) \\ &\ge 1-\epsilon^{50}-e^{-\epsilon ct} \end{aligned}$$

which implies the existence of $x^*_+ \in [x+\frac{\epsilon}{2}Mt, x+(1-\frac{\epsilon}{2})Mt]$ and $c_1$ satisfying the lemma. When $x^*_+(t(1-\epsilon)) > x+(1-\frac{\epsilon}{2})Mt$, we use an analogous argument with $\Gamma_{left}$. The measurability claim is an easy consequence of Corollary 2.6 and the use of paths in $\Gamma_{right}$. $\square$

We now establish a block argument, a la percolation theory, starting with

**Lemma 2.8.** *Let $x^*_\pm(\epsilon,t,x)$ be the points given by Lemma 2.7 when using the Brownian fields $\{B_y(s), y\in \mathbf{Z},\ \epsilon^2 t\le s\le t\}$. Then,*

$$x^*_+(\epsilon,t,x)\in[x+\frac{\epsilon}{2}Mt(1-\epsilon^2), x+(1-\frac{\epsilon}{2})Mt(1-\epsilon^2)]\subset[x+\frac{\epsilon}{4}Mt, x+(1-\frac{\epsilon}{4})Mt]$$

*and*

$$x^*_-(\epsilon,t,x)\in[x-(1-\frac{\epsilon}{2})Mt(1-\epsilon^2), x-\frac{\epsilon}{2}Mt(1-\epsilon^2)]\subset[x-(1-\frac{\epsilon}{4})Mt, x-\frac{\epsilon}{4}Mt].$$

*and*

$$\begin{gathered}Q(\forall x\in\{-Mt,(-M+\epsilon^2)t,(-M+2\epsilon^2)t,\dots,(M-\epsilon^2)t,Mt\},\\ E_x[e^{\int_{\epsilon^2 t}^t dB_{X(s-\epsilon^2 t)}(s)}1_{[0,Mt(1-\epsilon^2)]}(N(X,(1-\epsilon^2)t)1_{x^*_+(\epsilon,t,x)}(X(t))]\ge\\ \ge e^{(\lambda_1(\kappa)-c_1\epsilon)t(1-\epsilon^2)})\ge 1-\epsilon^{46}\end{gathered}$$

*and*

$$\begin{gathered}Q(\forall x\in\{-Mt,(-M+\epsilon^2)t,(-M+2\epsilon^2)t,\dots,(M-\epsilon^2)t,Mt\},\\ E_x[e^{\int_{\epsilon^2 t}^t dB_{X(s-\epsilon^2 t)}(s)}1_{[0,Mt(1-\epsilon^2)]}(N(X,(1-\epsilon^2)t)))1_{x^*_-(\epsilon,t,x)}(X(t))]\ge\\ \ge e^{(\lambda_1(\kappa)-c_1\epsilon)t(1-\epsilon^2)})\ge 1-\epsilon^{46}\end{gathered}$$

*provided $\epsilon$ is sufficiently small.*

*Proof.* Since there are $\frac{2M}{\epsilon^2}$ points $x$ in question, this follows easily from Lemma 2.7, if $\epsilon$ is sufficently small. □

As the next step we establish the previous result for all $x\in\{-Mt,-Mt+1,\dots$ $\dots\ Mt-1, Mt\}$.

**Lemma 2.9.** *For each $x\in[-Mt,Mt]\cap\mathbf{Z}$, there are points $x^*_\pm(t,x)$ lying in $[x+\frac{\epsilon}{4}Mt, x+(1-\frac{\epsilon}{4})Mt], [x-(1-\frac{\epsilon}{4})Mt, x-\frac{\epsilon}{4}Mt]$, respectively, with $x^*_\pm(t,x)$ measurable with respect to $\sigma(B_y(s): y\in\mathbf{Z}, |y-x|\le Mt, s\le t)$ such that for some $c=c(\kappa,M)$*

$$\begin{aligned}&Q(\forall x\in\{-Mt,-Mt+1,\dots\ Mt-1,Mt\},\\ &E_x[e^{\int_0^t dB_{X(s)}(s)}1_{[0,Mt]}(N(X,t))1_{x^*_+(t,x)}(X(t))]\ge e^{(\lambda_1(\kappa)-c\epsilon)t})\ge 1-\epsilon^{46},\end{aligned}$$

*with the analogous statement holding for $x^*_-(t,x)$. Thus the probability of the intersection of the events for $x^*_+(t,x)$ and $x^*_-(t,x)$ is at least $1-\epsilon^{45}$.*

*Proof.* Start by observing that to each $x\in\{-Mt,-Mt+1,\dots\ Mt-1,Mt\}$, we can associate a $y\in\{-Mt,(-M+\epsilon^2)t,(-M+2\epsilon^2)t,\dots,(M-\epsilon^2)t,Mt\}$ such that $|x-y|\le\frac{\epsilon^2}{2}t$ and $P_x(N(X,\epsilon^2 t)=|x-y|, X(\epsilon^2 t)=y)\ge e^{-K_3\epsilon^2 t}$, with $K_3$ independent of $\epsilon$ and $t$. Also, for some $K_4, K_5$ independent of $\epsilon$ and $t$ we have by Theorem 1.1

$$Q(\inf_{\gamma\in\Gamma^x_{[0,\epsilon^2 t],|x-y|:\gamma(\epsilon^2 t)=x}}\int_0^{\epsilon^2 t} dB_{\gamma(s)}(s)\ge -K_4\epsilon^2 t)\ge 1-e^{-K_5\epsilon^2 t}.$$

Put $x^*_+(t,x)=x^*_+(\epsilon,t,y)$ and $x^*_-(t,x)=x^*_-(\epsilon,t,y)$ where the latter are given by Lemma 2.8. Then, piecing together paths on $[0,\epsilon^2 t]$ which connect $x$ to $y$ and on

$[\epsilon^2 t, Mt]$ linking $y$ to $x^*_\pm(\epsilon, t, y)$, and using the fact that there are $2Mt+1$ points in $\{-Mt, -Mt+1, \dots\ Mt-1, Mt\}$, we have the result. $\square$

We next iterate so we can set up a percolation scheme.

**Lemma 2.10.** *For $i \in \{0, 1, \dots, (\frac{8}{\epsilon}-1)\}$, and $x \in [-Mt + \frac{i\epsilon}{4}Mt, Mt + \frac{i\epsilon}{4}Mt]$, $x' \in [-Mt - \frac{i\epsilon}{4}Mt, Mt - \frac{i\epsilon}{4}Mt]$ there exist $x^*_+(i,x) \in [-Mt + \frac{(i+1)\epsilon}{4}Mt, Mt + \frac{(i+1)\epsilon}{4}Mt]$, $x^*_-(i,x') \in [-Mt - \frac{(i+1)\epsilon}{4}Mt, Mt - \frac{(i+1)\epsilon}{4}Mt]$, measurable with respect to the Brownian fields $\{B_y(s+it) : 0 \le s \le t\}$, such that the events*

$$A(i) = \{\forall x \in [-Mt + \frac{i\epsilon t}{4}Mt, Mt + \frac{i\epsilon t}{4}Mt],$$
$$E_x[e^{\int_0^t dB_{X(s)}(it+s)} 1_{x^*_+(i,x)}(X(t)) 1_{[0,Mt]}(N(X,t))] \ge e^{(\lambda_1(\kappa)-\epsilon)t}\}$$

*and*

$$B(i) = \{\forall x \in [-Mt - \frac{i\epsilon t}{4}Mt, Mt - \frac{i\epsilon t}{4}Mt],$$
$$E_x[e^{\int_0^t dB_{X(s)}(it+s)} 1_{x^*_-(i,x)}(X(t)) 1_{[0,Mt]}(N(X,t))] \ge e^{(\lambda_1(\kappa)-c\epsilon)t}\}$$

*satisfy, for $\epsilon$ sufficiently small (depending only on $M$)*

$$Q(\cap_{i=0}^{\frac{8}{\epsilon}-1} A(i)) \ge 1 - \epsilon^{44}$$

*and*

$$Q(\cap_{i=0}^{\frac{8}{\epsilon}-1} B(i)) \ge 1 - \epsilon^{44}.$$

*Proof.* We can apply Lemma 2.9. If $x \in [-Mt + \frac{i\epsilon}{4}Mt, Mt + \frac{i\epsilon}{4}Mt]$, we consider the two cases $x \in [-Mt + \frac{i\epsilon}{4}Mt, \frac{\epsilon}{4}Mt + \frac{i\epsilon}{4}Mt]$ and $x \in (\frac{\epsilon}{4}Mt + \frac{i\epsilon}{4}Mt, Mt + \frac{i\epsilon}{4}Mt]$. In the first case, take $x^*_+(i,x) = x^*_+(t,x)$ from Lemma 2.9, using the Brownian fields $\{B_y(s+ti) : 0 \le s \le t, |y-x| \le Mt\}$. Then we will have

$$\begin{aligned} x^*_+(i,x) \in & [-Mt + \frac{(i+1)\epsilon}{4}Mt, \frac{\epsilon}{4}Mt + \frac{i\epsilon}{4}Mt + (1 - \frac{\epsilon}{4})Mt] \subset \\ & \subset [-Mt + \frac{(i+1)\epsilon}{4}Mt, Mt + \frac{(i+1)\epsilon}{4}Mt]. \end{aligned}$$

In the second case, take $x^*_+(i,x) = x^*_-(t,x)$ from Lemma 2.9, using the Brownian fields $\{B_y(s+ti) : 0 \le s \le t, |y-x| \le Mt\}$. Then we will have

$$x^*_+(i,x) \in [x - (1 - \frac{\epsilon}{4})Mt, x - \frac{\epsilon}{4}Mt] \subset [-Mt + \frac{(i+1)\epsilon}{4}Mt, Mt + \frac{(i+1)\epsilon}{4}Mt].$$

An analogous argument handles the selection of the points $x^*_-(i,x')$. Notice that $1 - \frac{8M}{\epsilon}\epsilon^{46} \ge 1 - \epsilon^{44}$ for $\epsilon$ sufficiently small (depending only on $M$) to get the probability estimates. $\square$

*Proof.* We now prove Theorem 2.2, namely that

$$\lambda(\kappa) = \lim_{t\to\infty} \frac{1}{t} \log E_x[e^{\int_0^t dB_{X(s)}(s)}].$$

It is clear that we need only do this for $x = 0$. Since

$$\lambda(\kappa) \;=\; \lim_{t\to\infty}\frac{1}{t}\log v(t,0) \le \lim_{t\to\infty}\frac{1}{t}\log E_0[e^{\int_0^t dB_{X(s)}(s)}]$$

and both limits exist $Q\,a.s.$ we only need show that

$$Q(\lim_{t\to\infty}\frac{1}{t}\log v(t,0) \ge \lim_{t\to\infty}\frac{1}{t}\log E_0[e^{\int_0^t dB_{X(s)}(s)}]) \ge \frac{1}{4}.$$

Using Lemma 2.10, we can conclude that with $Q$-probability at least $1-\epsilon^{43}$, corresponding to the starting point 0 there exist sequences, $\{x_+^*(i,0)\}_{i=0}^{\frac{8}{\epsilon}}, \{x_-^*(i,0)\}_{i=0}^{\frac{8}{\epsilon}}$, such that $x_\pm(0,0) = 0$ with $x_+^*(i,0) \in [-Mt + \frac{i\epsilon}{4}Mt, Mt + \frac{i\epsilon}{4}Mt]$, $x_-^*(i,0) \in$ $\in [-Mt - \frac{i\epsilon}{4}Mt, Mt - \frac{i\epsilon}{4}Mt]$, and for $0 \le i \le \frac{8}{\epsilon} - 1$,

$$E_{x_+^*(i,0)}[e^{\int_0^t dB_{X(s)}(s+it)}1_{x_+^*(i+1,0)}(X(t))1_{[0,Mt]}(N(X,t))] \ge e^{(\lambda_1(\kappa)-c\epsilon)t}$$

and

$$E_{x_-^*(i,0)}[e^{\int_0^t dB_{X(s)}(s+it)}1_{x_-^*(i+1,0)}(X(t))1_{[0,Mt]}(N(X,t))] \ge e^{(\lambda_1(\kappa)-c\epsilon)t}.$$

Thus, by the Markov property

$$E_0[e^{\int_0^{\frac{8}{\epsilon}t} dB_{X(s)}(s)}1_{x_\pm(\frac{8}{\epsilon},0)}(X(\frac{8}{\epsilon}t))1_{[0,M\frac{8}{\epsilon}t]}(N(X,\frac{8}{\epsilon}t))] \ge e^{(\lambda_1(\kappa)-c\epsilon)\frac{8t}{\epsilon}}.$$

This induces the following percolation system:

$$\{\psi(l,n), n \ge 0, l \in \mathbf{Z}, l+n \equiv 0 \mod 2\}.$$

We say that $(l,n) \to (l+1,n+1)$ if $\forall x \in [(2l-1)Mt,(2l+1)Mt], \exists x_0,\ldots,x_{\frac{8}{\epsilon}-1}$, with $x_0 = x$, such that $x_i \in [(2l-1)Mt + \frac{i\epsilon}{4}Mt, (2l+1)Mt + \frac{i\epsilon}{4}Mt]$, and

$$E_{x_i}[e^{\int_0^t dB_{X(s)}(\frac{8nt}{\epsilon}+it+s)}1_{[0,Mt]}(N(X(t)))1_{x_{i+1}}(X(t))] \ge e^{(\lambda_1(\kappa)-c\epsilon)t}.$$

And we make an analogous definition for $(l,n) \to (l-1,n+1)$. By the measurability claim in Lemma 2.10, this is a 2-dependent percolation scheme with $Q$-probability of an open bond at least $1-\epsilon^{43}$. Thus, by Durrett [**6**] we have for any $n$, with $Q$-probability at least $\frac{1}{2}$ that there is a path $(0,0) \to (l_1,1) \to \cdots \to (0,2n)$. Thus, for some $x^* \in [-Mt, Mt]$,

$$E_0[e^{\int_0^{\frac{8}{\epsilon}t2n} dB_{X(s)}(s)}1_{x^*}(X(\frac{8}{\epsilon}t2n))] \ge e^{(\lambda_1(\kappa)-c\epsilon)\frac{8}{\epsilon}t2n}.$$

But, for some $h > 0$, independent of $t, n, \epsilon, x^*$ with $Q$-probability at least $\frac{1}{2}$,

$$E_{x^*}[e^{\int_0^{\frac{16t}{\epsilon}} dB_{X(s)}(s+\frac{8}{\epsilon}t2n)}1_0(X(\frac{16}{\epsilon}t))] \ge e^{-h\frac{16t}{\epsilon}}.$$

Thus, with $Q$-probability at least $\frac{1}{4}$ we have

$$E_0[e^{\int_0^{\frac{8}{\epsilon}t2(n+1)} dB_{X(s)}(s)}1_0(X(\frac{8}{\epsilon}t2(n+1)))] \ge e^{(\lambda_1(\kappa)-2c\epsilon)\frac{8}{\epsilon}t2(n+1)}.$$

T his proves the theorem. □

The next result will lead to the existence of $\lim_{t\to\infty}\frac{1}{t}\log u(t,x)$.

**Theorem 2.11.** *Given $\epsilon > 0$ there is a positive constant $c$ such that*

$$Q(E_0[e^{\int_0^t dB_{X(s)}(s)}] \geq e^{(\lambda_1(\kappa)+\epsilon)t}) \leq e^{-ct}.$$

Before proving this Theorem we derive the following consequence,

**Corollary 2.12.**

$$\limsup_{t\to\infty} \frac{1}{t} \log E_0[e^{\int_0^t dB_{X(s)}(t-s)}] \leq \lambda_1(\kappa).$$

*Proof.* Since the distribution of $E_0[e^{\int_0^t dB_{X(s)}(t-s)}]$ and $E_0[e^{\int_0^t dB_{X(s)}(s)}]$ are identical, we can apply Theorem 2.11 and Borel-Cantelli to get the result. □

**Corollary 2.13.** *For $u_0$ positive, strictly positive somewhere and bounded,*

$$\lim_{t\to\infty} \frac{1}{t} \log u(t,x) \;=\; \lambda(\kappa).$$

*Proof.* Without loss of generality we take $u_0$ bounded by 1 and strictly positive at the origin. By Carmona and Molchanov, [**4**], $\liminf_{t\to\infty} \frac{1}{t}\log u(t,x) \geq \lambda(\kappa)$. By Corollary 2.12 however

$$\limsup_{t\to\infty} \frac{1}{t} \log u(t,x) \leq \limsup_{t\to\infty} \frac{1}{t} \log E[e^{\int_0^t dB_{X(t-s)}(s)}] = \lambda_1(\kappa) = \lambda(\kappa).$$

□

We now state our result on the asymptotics of of $\lambda(\kappa)$.

**Theorem 2.14.**

$$\lim_{\kappa\downarrow 0} \lambda(\kappa) \log \frac{1}{\kappa} = \frac{\alpha^2}{4}$$

The proof of Theorem 2.14 will be given in the next section. Some of the results needed for the proof of Theorem 2.14 will be used in the proof of Theorem 2.11, which is given in the final section.

## 3. Asymptotics of the Lyapunov exponent

The proof of Theorem 2.14 is based on Borell's inequality and Fernique-Talagrand's theorem for Gaussian random fields together with a classical large deviation result for Poisson processes.

Recall from Theorem 1.1, that $\alpha > 0$ satisfies $\alpha = \sup_n \frac{1}{n} E_Q[\vec{A}_{[0,n],n}] \leq C$.

**Lemma 3.1.** *Let any $a > 0$ be fixed. For every $\epsilon > 0$*

$$Q\left(\exists n_0(\omega) : (\alpha-\epsilon)\sqrt{mn} \leq \vec{A}_{[0,n],m} \leq (\alpha+\epsilon)\sqrt{nm} \quad, \forall m \geq an, \quad \forall n \geq n_0\right) = 1.$$

*Proof.* Applying Theorem 1.1, we have

$$Q(|\vec{A}^0_{[0,n],n} - E_Q[\vec{A}^0_{[0,n],n}]| > \frac{\epsilon n}{2}) \leq 2e^{-\epsilon^2 n/8}.$$

Since $E_Q[\vec{A}^0_{[0,n],n}] \sim \alpha n$ by Liggett [**9**], it holds that for large $n$

$$Q(|\vec{A}^0_{[0,n],n} - \alpha n| > \epsilon n) \leq 2e^{-\epsilon^2 n/8},$$

which becomes

$$Q(|\vec{A}^0_{[0,n],m} - \alpha\sqrt{nm}| > \epsilon\sqrt{nm}) \leq 2e^{-\epsilon^2 m/8},$$

by the Browinian scaling invariance;

$$\vec{A}_{[0,n],m} \stackrel{\mathcal{L}}{=} \sqrt{\frac{n}{m}}\vec{A}_{[0,m],m}.$$

Finally use the Borel-Cantelli lemma to complete the proof of Lemma 3.1. □

**Lemma 3.2.** *(i) Let $F : [0,\infty) \mapsto \mathbb{R}$ be a continuous function satisfying*

$$\limsup_{x\to\infty} \frac{F(x)}{x\log x} < 1.$$

*Then*

$$\lim_{n\to\infty} \frac{1}{n} \log E_0[e^{nF(\frac{N(X,n)}{n})}] = \sup_{x\geq 0}\{F(x) - x\log\frac{x}{\kappa} + x - \kappa\}.$$

*(ii) For $0 \leq a < b \leq \infty$*

$$\lim_{n\to\infty} \frac{1}{n} \log E_0[e^{\left(\alpha\sqrt{nN(X,n)}-\beta N(X,n)\right)} : an \leq N(X,n) \leq bn]$$
$$= \sup_{a\leq x\leq b}\{\alpha\sqrt{x} - \beta x - x\log\frac{x}{\kappa} + x - \kappa\}.$$

*Proof.* Since $N(X,n)$ has a Poisson distribution with mean $\kappa n$, it follows from a large deviation theorem by Cramer that for any bounded continuous function $F : [0,\infty) \mapsto \mathbb{R}$

$$\lim_{n\to\infty} \frac{1}{n} \log E_0[e^{nF(\frac{N(X,n)}{n})}] = \sup_{x\geq 0}\{F(x) - x\log\frac{x}{\kappa} + x - \kappa\}.$$

Furthermore it is easy to extend it for any $F$ satisfying the assumption. (ii) follows from (i) immediately. □

**Lemma 3.3.** *(i) For $\theta > 0$,*

$$\sup_{x\geq 0}\{\theta\sqrt{x} - x\log\frac{x}{\kappa} + x - \kappa\} \sim \frac{\theta^2}{4\log\frac{1}{\kappa}} \quad (\kappa \downarrow 0).$$

*(ii) For $c > 0$, $\theta > 0$, $0 < q < 1$ and $\kappa > 0$,*

$$\sup_{\kappa^q\leq x\leq b}\left(\theta\sqrt{x} + (\log c\kappa^q)x - x\log\frac{x}{\kappa} + x - \kappa\right) \sim \frac{\theta^2}{4(1+q)}\frac{1}{\log\frac{1}{\kappa}} \quad (\kappa \downarrow 0).$$

*Proof.* For (ii) let

$$f(x) = \theta\sqrt{x} + (\log c\kappa^q)x - x\log\frac{x}{\kappa} + x - \kappa.$$

Then,

$$f'(x) = \frac{\theta}{2\sqrt{x}} + (\log c\kappa^q) - \log\frac{x}{\kappa}$$

is a decreasing function and $f'(\kappa^q) > 0$ and $f'(b) < 0$ for a sufficiently small $\kappa > 0$, so that $f$ has a unique maximum at $x(\kappa) \in (\kappa^q, b)$ such that

$$\frac{\theta}{2\sqrt{x(\kappa)}} + (\log c\kappa^q) - \log\frac{x(\kappa)}{\kappa} = 0.$$

This implies

$$\frac{\theta}{2\sqrt{x(\kappa)}} \sim (1+q)\log\frac{1}{\kappa} \quad (\kappa \downarrow 0).$$

Hence

$$\sup_{\kappa^q \le x \le b} f(x) = f(x(\kappa)) \sim \frac{\theta^2}{4(1+q)}\frac{1}{\log\frac{1}{\kappa}}.$$

□

*Proof of the upper bound*

By Lemma 3.1, we can suppose that for $a > 0$ and $\epsilon > 0$,

$$\vec{A}_{[0,n],m} \le (\alpha+\epsilon)\sqrt{mn} \quad (\forall m \ge an) \quad Q-a.s..$$

Then

$$\begin{aligned} (3.1)\quad E_0[e^{\int_0^n dB_{X(s)}(s)} : N(X,n) \ge an] &\le \sum_{j=an}^{\infty} e^{\vec{A}_{[0,n],j}} P_0(N(X,n)=j) \\ &\le \sum_{j=an}^{\infty} e^{(\alpha+\epsilon)\sqrt{jn}} P_0(N(X,n)=j) \\ &\le E_0[e^{(\alpha+\epsilon)\sqrt{nN(X,n)}}]. \end{aligned}$$

Noting that $\vec{A}_{[0,n],m}$ is increasing in $m$ we have

$$E_0[e^{\int_0^n dB_{X(s)}(s)} : N(X,n) \le an] \le e^{\vec{A}_{[0,n],an}} \le e^{(\alpha+\epsilon)\sqrt{a}n}.$$

Combining ([**13**]), ([**7**]) together with Lemma 3.2, we have

$$\lambda(\kappa) \le \sup\{(\alpha+\epsilon)\sqrt{x} - x\log\frac{x}{\kappa} - x + \kappa\} + (\alpha+\epsilon)\sqrt{a}.$$

Choosing a suitable $a = a(\kappa)$, by using Lemma 3.3 (i) we obtain

$$\limsup_{\kappa\searrow 0} \lambda(\kappa)\log\frac{1}{\kappa} \le \frac{(\alpha+\epsilon)^2}{4},$$

which gives the upper bound of Theorem 2.14.

*Proof of the lower bound*

For $\gamma \in \Gamma^{\epsilon}_{[0,n],m}$ let $t_1 < t_2 < \cdots < t_m < n$, be the jump times of $\gamma$ during $[0,n]$, and let $I_i^{\gamma}$ be the interval $[\frac{j_i \epsilon}{3}, \frac{(j_i+1)\epsilon}{3}]$ with $j_i \in \mathbf{Z}$, including $t_i$. If $\eta \in \Gamma^{\epsilon}_{[0,n],m}$ has $s_1 < \cdots < s_m < n$ as its jump times during $[0,n]$, $I_i^{\eta} = I_i^{\gamma}$ and $\gamma(t_i) = \eta(s_i)$ for every $1 \le i \le m$, we denote $\eta \sim \gamma$. Then it is easy to see that if $\eta, \gamma \in \Gamma^{\epsilon}_{[0,n],m}$ satisfy $\eta \sim \gamma$,

$$|\int_0^n dB_{\gamma(s)}(s) - \int_0^n dB_{\eta(s)}(s)| < \omega(\gamma),$$

where

$$\omega(\gamma) = \sum_{i=1}^{m} \left( \omega(B_{\gamma(t_i)} : I_i) + \omega(B_{\gamma(t_{i-1})} : I_i) \right),$$

$$\omega(B_x : I) = \sup_{t,s \in I} |B_x(t) - B_x(s)|.$$

**Lemma 3.4.** *For any $\delta > 0$, $a > 0$ and $b > a$ there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ and for all $an \le m \le bn$,*

*(i) there is a $K > 0$ such that*

$$Q(\omega(\gamma) > cn) \le K^m e^{-\frac{cn}{\sqrt{\epsilon}}} \textit{ for all } \gamma \in \Gamma^{\epsilon}_{[0,n],m}.$$

*(ii)*

$$Q\left( \sup_{\gamma \in \Gamma^{\epsilon}_{[0,n],m}} \omega(\gamma) \ge \delta \sqrt{nm} \right) \le ce^{-n}.$$

*(iii) Moreover,*

$$Q\left( \exists n_0(\omega) : \sup_{\gamma \in \Gamma^{\epsilon}_{[0,n],m}} \omega(\gamma) \le \delta \sqrt{nm} \quad, \forall m \in [an, bn], \quad \forall n \ge n_0 \right) = 1\,.$$

*Proof.* We first count the number of increasing sequences $j_1 < j_2 < \cdots < j_m$ of integers which determine the intervals $I_i^{\epsilon}$. We must have $j_m < \frac{3n}{2\epsilon}$. Thus, there are fewer than

$$\begin{aligned} \binom{\frac{3n}{2\epsilon}}{m} &\le c_1 \frac{(\frac{3n}{2\epsilon})^{\frac{3n}{2\epsilon}+\frac{1}{2}}}{m^{m+\frac{1}{2}}(n(\frac{3}{2\epsilon} - \frac{m}{n}))^{n(\frac{3}{\epsilon}-1)+\frac{1}{2}}} \\ &\le c_1(\frac{c_2}{\epsilon})^m, \end{aligned}$$

where $c_1, c_2$ are positive constants. For each such sequence, $j_1 < j_2 < \cdots < j_m$, there are $2^n$ possible sequences of sites visited. This means there are $C_1(\frac{2c_2}{\epsilon})^m$ possible variables $2(\omega(B_{\gamma(t_i)}, I_i^{\epsilon}) + \omega(B_{\gamma(t_{i-1})}, I_{i-1}^{\epsilon}))$ to consider. For each such sequence,

$$\omega(\gamma) \stackrel{\mathcal{L}}{=} \sqrt{\epsilon} \sum_{i=1}^{m} Y_i$$

where $Y_i$ are *i.i.d.* random variables satisfying

$$E_Q[e^{Y_i}] = K < \infty.$$

Then for fixed $\gamma \in \Gamma^{0,\epsilon}_{[0,n],m}$,

$$Q(\omega(\gamma) > cn) \leq K^m e^{-\frac{cn}{\sqrt{\epsilon}}},$$

which proves (i). Therefore,

$$\begin{aligned} Q(\sup_{\gamma\in\Gamma^{0,\epsilon}_{[0,n],n}} \omega(\gamma) > c\sqrt{nm}) &\leq c_1(\frac{c_2}{\epsilon})^m K^m e^{-\frac{c\sqrt{nm}}{\sqrt{\epsilon}}} \\ &\leq c_1 e^{-cn}, \end{aligned}$$

provided $\epsilon$ is sufficiently small. The third statement follows easily from the second. □

**Lemma 3.5.** *For any $\delta > 0$, and all $a > 0$ and $b > a$ there exists $\epsilon_0 > 0$ such that for $0 < \epsilon < \epsilon_0$*

$$Q\left(\exists n_0(\omega) : |A^\epsilon_{[0,n],m} - \alpha\sqrt{nm}| \leq \delta\sqrt{nm} \quad, \forall m \in [an, bn], \quad \forall n \geq n_0\right) = 1.$$

*Proof.* By Borell's inequality,

$$Q\left(|\vec{A}^\epsilon_{[0,n],m} - E_Q[\vec{A}^\epsilon_{[0,n],m}]| \geq \frac{\delta\sqrt{nm}}{2}\right) \leq 2e^{-\delta^2 m/8}.$$

Since

$$E_Q[\vec{A}^\epsilon_{[0,n],m}] = \sqrt{\frac{n}{m}} E_Q[\vec{A}^\epsilon_{[0,m],m}] \sim \sqrt{nm}\alpha$$

uniformly in $m \in [an, bn]$ as $n \to \infty$, we have

$$Q\left(|\vec{A}^\epsilon_{[0,n],m} - \alpha\sqrt{nm}| \geq \delta\sqrt{nm}\right) \leq 2e^{-\delta^2 m/8}.$$

□

By Lemmas 3.6 and 3.7 we can assume that the Brownian motions $\{B_x(t)\}$ satisfy the following two conditions:
For every $\delta > 0$, $a > 0$ and $b > a$, there exists $\epsilon_0 > 0$ such that if $0 < \epsilon < \epsilon_0$,

$$\exists n_0 \text{ such that } \sup_{\gamma\in\Gamma^\epsilon_{[0,n],m}} \omega(\gamma) \leq \delta\sqrt{nm} \quad \text{for all } m \in [an, bn], \quad \text{for all } n \geq n_0,$$

and

$$\exists n_0 \text{ such that } |\vec{A}^\epsilon_{[0,n],m} - \alpha\sqrt{nm}| \leq \delta\sqrt{nm} \quad \text{for all } m \in [an, bn], \quad \text{for all } n \geq n_0.$$

Then,

$$\begin{aligned} & E_0[e^{\int_0^n dB_{X(s)}(s)}] \\ \geq & \sum_{m=an}^{bn} E_0[e^{\int_0^n dB_{X(s)}(s)} : N(X,n) = m] \\ = & \sum_{m=an}^{bn} e^{A^\epsilon_{[0,n],m}} E_0[e^{\left(\int_0^n dB_{X(s)}(s) - \int_0^n dB_{\gamma^{n,m}(s)}(s)\right)} : N(X,n) = m] \\ \equiv & (*) \end{aligned}$$

where $\gamma^{n,m} \in \Gamma^{\epsilon}_{[0,n],m}$ is defined by

$$\vec{A}^{\epsilon}_{[0,n],m} = \int_0^n dB_{\gamma^{n,m}(s)}(s).$$

Thus,

$$\begin{aligned} (*) \quad &\geq \sum_{m=an}^{bn} e^{\left(\vec{A}^{\epsilon}_{[0,n],m} - \omega(\gamma^{n,m})\right)} P_0\left(X \sim \gamma^{n,m}, N(X,n) = m\right) \\ &\geq \sum_{m=an}^{bn} e^{(\alpha-2\delta)\sqrt{nm}} P_0\left(X \sim \gamma^{n,m}, N(X,n) = m\right). \end{aligned}$$

Since

$$P_0\left(X \sim \gamma^{n,m} | N(X,n) = m\right) = \frac{m!}{n^m}\left(\frac{\epsilon}{6}\right)^m,$$

supposing $a = \kappa^q$ with $0 < q < 1$, we have some $c > 0$ satisfying

$$P_0\left(X \sim \gamma^{n,m} | N(X,n) = m\right) \geq (c\epsilon\kappa^q)^m,$$

so that

$$\begin{aligned} &E_0[e^{\int_0^n dB_{X(s)}(s)}] \\ \geq\ &E_0\big[e^{\left((\alpha-2\delta)\sqrt{nN(X,n)} + \log(c\epsilon\kappa^q)N(X,n)\right)} \\ &\quad \kappa^q n \leq N(X,n) \leq bn\big]. \end{aligned}$$

Hence by Lemma 3.3 (ii), we obtain that for every $\delta > 0$ and $0 < q < 1$

$$\liminf_{\kappa \downarrow 0} \lambda(\kappa) \log\frac{1}{\kappa} \geq \frac{(\alpha - 2\delta)^2}{4(1+q)},$$

which gives the lower bound of the Theorem 2.14, letting $q \searrow 0$ and $\kappa \searrow 0$.

## 4. Proof of **Theorem 2.11**

Before making the next definition we remark that by Lemma 3.2 with $a = M$, $b = \infty$, $\alpha$ replaced by $\alpha + \epsilon$, $\beta = 0$ then if $M$ is sufficiently large,

$$\begin{aligned} E_x[e^{\int_0^{N_0} dB_{X(s)}(s)} 1_{[MN_0,\infty)}(N(X,N_0))] &\leq \sum_{m=MN_0}^{\infty} e^{\vec{A}_{[0,N_0],m}} P(N(X,N_0) = m) \\ &\leq \sum_{m=MN_0}^{\infty} e^{(\alpha+\epsilon)\sqrt{mN_0}} P(N(X,N_0) = m) \\ &= E_x[E^{(\alpha+\epsilon)\sqrt{mN_0}} : MN_0 \leq N(X,N_0)] \\ &\leq e^{(1+\delta)N_0((\alpha+\epsilon)\sqrt{M} - M\log\frac{M}{\kappa} + M - \kappa)} \\ &\leq e^{-cMN_0}, \text{ for some positive } c. \end{aligned}$$

For the remainder of the section, we assume that the value of $M$ is large enough to make the above estimate valid. This means that we no longer need to worry about the random walk paths that make too many jumps (i.e. more than $MN_0$

in time $N_0$,) their contribution to the expectation is negligible. We introduce a notation for the time-shifted Brownian motion field by setting

$$B_x^j(t) = B_x(t+j).$$

**Definition 4.1.** Break up space-time into disjoint blocks of the form $((i-1)MN_0, (i+1)MN_0] \times [(j-1)N_0, jN_0]$ where $i \in \mathbf{Z}, j \in \mathbf{Z}^+$. satisfy $i+j \equiv 0$ mod 2. Label the block $((i-1)MN_0, (i+1)MN_0] \times [(j-1)N_0, jN_0]$ by $(i,j)$. We say $(i,j)$ is a good block (denoted $(i,j) \in G$) if $\forall x \in ((i-1)MN_0, (i+1)MN_0]$,

$$E_x[e^{\int_0^{N_0} dB^j_{X(s)}(s)} I_{[0,MN_0]}(N(X,N_0))] \le e^{(\lambda_1(\kappa)+\frac{\epsilon}{3})N_0}.$$

Next we introduce a discretization scheme for paths in $\Gamma^0_{[0,t],Mt}$ which will allow us via simple counting arguments to derive some estimates on our basic functionals of $\{B_x\}$. For a path $\gamma \in \Gamma^0_{[0,t],Mt}$ we say that its $N_0$-skeleton (or skeleton since the value $N_0$ will be 'understood') is the sequence $\lambda(0), \gamma(N_0), \gamma(2N_0), \ldots, \gamma(t)$, where we assume without loss of generality that $t$ is a multiple of $N_0$. We claim

**Lemma 4.2.** *Given $\epsilon > 0$ there is an $N$ so that for $N_0 > N$, the number of distinct skeletons corresponding to paths in $\Gamma^0_{[0,t],Mt}$ is bounded by $e^{\frac{\epsilon t}{10^6}}$.*

*Proof.* Each skeleton determines a sequence $\{i_j : j = 0, \ldots, \frac{t}{N_0}\}$ with $i_j + j \equiv 0$ mod 2 such that $\gamma(jN_0) \in [(i_j-1)MN_0, (i_j+1)MN_0]$ for each $j$. We shall call the sequence $\{i_j : j = 0, \ldots, \frac{t}{N_0}\}$ the trace of $\gamma$. Letting $A(i_1, i_2, \ldots, i_{\frac{t}{N_0}})$ denote the set of skeletons of elements of $\Gamma^0_{[0,t],Mt}$ with trace $\{i_j : j = 0, \ldots, \frac{t}{N_0}\}$, it is easy to see that if $N_0$ is sufficiently large,

$$|A(i_1, i_2, \ldots, i_{\frac{t}{N_0}})| \le (2MN_0)^{\frac{t}{N_0}}.$$

Furthermore the number of traces of elements of $\Gamma^0_{[0,t],Mt}$ is constrained by the requirement that the corresponding path have no more than $Mt$ jumps. In terms of the trace, this translates into the bound

$$\sum_j (|i_j - i_{(j-1)}| - 1)_+ \le \frac{t}{N_0}.$$

There are at most $2^{3\frac{t}{N_0}}$ such sequences. Thus, the total number of skeletons of elements of $\Gamma^0_{[0,t],Mt}$ is bounded by

$$(2MN_0)^{\frac{t}{N_0}} 2^{3\frac{t}{10^6}} \le e^{\frac{\epsilon t}{N_0}},$$

provided $N_0$ is chosen sufficiently large. □

**Lemma 4.3.** *For any $\delta > 0$,and $M < \infty$ as above, there is an $N_1$ so that for $N_0 \ge N_1$, we have*

$$Q((i,j) \in G) \ge 1 - \delta.$$

*Proof.* Without loss of generality we may treat the case $j = 0$. Fix an $\eta$ with $0 < \eta << \frac{\epsilon}{M}$. Then, $\forall x \in (-MN_0, MN_0] \cap \mathbf{Z}$,

$$E_x[e^{\int_0^{N_0} dB_{X(s)}(s)} 1_{[0,MN_0]}(N(X,N_0))] \leq$$
$$\leq \sum_{l=-2MN_0}^{2MN_0} E_x[e^{\int_0^{\eta N_0} dB_{X(s)}(s)}] E_l[e^{\int_0^{(1-\eta)N_0} dB^{\eta N_0}_{X(s)}(s)}].$$

An easy argument using Chebychev's inequality and the fact that

$$E_Q E_x[e^{\int_0^{\eta N_0} dB_{X(s)}(s)}] = E_x E_Q[e^{\int_0^{\eta N_0} dB_{X(s)}(s)}] = e^{\frac{\eta N_0}{2}},$$

shows that the event

$$C(N_0) = \cap_{x=-MN_0}^{MN_0} \{E_x[e^{\int_0^{\eta N_0} dB_{X(s)}(s)}] \leq \frac{e^{\frac{\epsilon N_0}{12}}}{4MN_0}\}$$

has $\lim_{N_0 \to \infty} Q(C(N_0)) = 1$. Moreover, on the event $C(N_0)$, we have

$$\sup_{x \in [-MN_0, MN_0]} E_x[e^{\int_0^{N_0} dB_{X(s)}(s)} 1_{[0,MN_0]}(N(X,N_0))]$$
$$\leq \quad e^{\frac{\epsilon N_0}{12}} \sup_{l \in [-2MN_0, 2MN_0]} E_l[e^{\int_0^{(1-\eta)N_0} dB^{\eta N_0}_{X(s)}(s)}].$$

The right hand side in this inequality can be bounded by a supremum over a finite number of terms, where that finite number is independent of $N_0$. To do this, take points $x_1, x_2, \ldots, x_R$ in $[-2MN_0, 2MN_0]$, separated by $\frac{\eta\epsilon N_0}{2}$ so that $R \cong \frac{8M}{\eta\epsilon}$. For a fixed $x_i$ and any $l$ with $|x_i - l| \leq \frac{\eta\epsilon N_0}{2}$,

$$E_{x_i}[e^{\int_0^{N_0} dB_{X(s)}(s)}] \geq E_{x_i}[e^{\int_0^{\eta N_0} dB_{X(s)}(s)} 1_l(X(\eta N_0))] E_l[e^{\int_0^{(1-\eta)N_0} dB^{\eta N_0}_{X(s)}(s)}].$$

Now we claim

$$\lim_{N_0 \to \infty} Q( \inf_{|l-x_i| \leq \frac{\eta\epsilon N_0}{2}} E_{x_i}[e^{\int_0^{\eta N_0} dB_{X(s)}(s)} 1_l(X(\eta N_0))] \geq e^{-\frac{\epsilon N_0}{12}}) = 1.$$

Before proving this claim we proceed to show how it proves the lemma. Assuming the claim,

$$\lim_{N_0 \to \infty} Q( \sup_{1 \leq i \leq R} E_{x_i}[e^{\int_0^{N_0} dB_{X(s)}(s)}] \geq$$

$$\geq e^{-\frac{\epsilon N_0}{12}} \sup_{l \in [-2MN_0, 2MN_0]} E_l[e^{\int_0^{(1-\eta)N_0} dB^{\eta N_0}_{X(s)}(s)}]) = 1.$$

Thus, with $Q$ probability approaching 1 as $N_0 \to \infty$ we have for all $x \in$ $\in [-MN_0, MN_0]$,

$$\begin{aligned}
&E_x[e^{\int_0^{N_0} dB_{X(s)}(s)} 1_{[0,MN_0]}(N(X,N_0))] \leq \\
&\leq e^{\frac{\epsilon N_0}{12}} \sup_{l\in[-2MN_0,2MN_0]} E_l[e^{\int_0^{(1-\eta)N_0} dB^{\eta N_0}_{X(s)}(s)}] \\
&\leq e^{\frac{\epsilon N_0}{6}} \sup_{1\leq i\leq R} E_{x_i}[e^{\int_0^{N_0} dB_{X(s)}(s)}] \\
&\leq e^{\frac{\epsilon N_0}{6}} e^{(\lambda(\kappa)+\frac{\epsilon}{6})N_0},\ R \text{ independent of } N_0, \\
&= e^{(\lambda(\kappa)+\frac{\epsilon}{3})N_0},\ \text{as desired.}
\end{aligned}$$

Now back to the proof of the claim. First, given $l$ with $|x_i - l| \leq \frac{\eta\epsilon N_0}{2}$, take an $x_i - to - l$ path $\gamma$ taking exactly $|x_i - l|$ steps and evenly spaced jump times. Using the notation of the previous section (see material preceding Lemma 3.6,) then making sure $\epsilon$ is taken so that $\kappa\epsilon < 1$, we have

$$\begin{aligned}
P_{x_i}(X \sim \gamma) &= 2^{-|x_i-l|}\frac{(\lambda\eta N_0)^{|x_i-l|}}{|x_i-l|!}e^{-\lambda\eta N_0}|x_i-l|!(\eta N_0)^{-|x_i-l|}(\epsilon)^{|x_i-l|} \\
&\geq 2^{-\frac{\eta\epsilon N_0}{2}}(\lambda\epsilon)^{\frac{\eta\epsilon N_0}{2}}e^{-\lambda\eta N_0} \\
&\geq e^{-c\eta N_0}, \text{for some positive c.}
\end{aligned}$$

Then using Lemma 3.6, we have for some $c > 0$,

$$\begin{aligned}
&Q(E_{x_i}[e^{\int_0^{\eta N_0} dB_{X(s)}(s)} 1_l(X(s))] \leq e^{-\frac{\epsilon N_0}{12}}) \\
&\leq Q(E_{x_i}[e^{\int_0^{\eta N_0} dB_{X(s)}(s)}; X\sim\gamma] \leq e^{-\frac{\epsilon N_0}{12}}) \\
&\leq Q(e^{\int_0^{\eta N_0} dB_{\lambda(s)}(s)+\omega(\gamma)}P_{x_i}(X\sim\gamma) \leq e^{-\frac{\epsilon N_0}{12}}) \\
&\leq Q(e^{\int_0^{\eta N_0} dB_{\gamma(s)}(s)+c\eta N_0}P_{x_i}(X\sim\gamma) \leq e^{-\frac{\epsilon N_0}{12}}) + e^{-c\eta N_0} \\
&\leq Q(e^{\int_0^{\eta N_0} dB_{\gamma(s)}(s)} \leq e^{-\frac{\epsilon N_0}{12}-c\eta N_0}) + e^{-c\eta N_0} \\
&= Q(B_0(\eta N_0) \leq -\frac{\epsilon N_0}{12} - c\eta N_0) + e^{-c\eta N_0},
\end{aligned}$$

which clearly tends to 0 as $N_0$ tends to $\infty$. This is enough to establish the claim and that finishes the proof of the lemma. $\square$

With this lemma we can now conclude that

**Lemma 4.4.** *For any $\eta > 0$, there is an $N_1$ such that $\forall N_0 > N_1$,*

$$Q(\sup \sum_{j=0}^{\frac{t}{N_0}-1} 1_{G^c}(i_j, j) \geq \eta t/N_0) \leq e^{-\frac{8t}{N_0}}.$$

*where the* sup *is taken over all traces compatible with paths starting at zero and having less than $Mt$ jumps in time interval $[0,t]$.*

*Proof.* This follows from elementary large deviations for the binomial random variables and Lemma 4.2. □

The next lemma says that the contribution of any collection of a restricted number of blocks can not be very big with high probability.

**Lemma 4.5.** *Given $\epsilon > 0$ and $\eta > 0$ as in Lemma 4.2, for all $N_0$ sufficiently large,*

$$Q(\exists (x_{i_1}, x_{i_2}, \ldots, x_{i_{\frac{t}{N_0}}}) \text{ compatible with } \leq Mt \text{ jumps, } J \subset \{1, 2, \ldots, \frac{t}{N_0}\}$$
$$\text{with } |J| \leq \frac{\eta t}{N_0}, :$$
$$\prod_J E_{x_{i_j}}[e^{\int_0^{N_0} dB^{jN_0}_{X(s)}(s)}] \geq e^{\frac{\epsilon t}{10}}) \leq e^{-c\epsilon t}.$$

*Proof.* First fix a skeleton $(x_{i_1}, x_{i_2}, \ldots, x_{i_{\frac{t}{N_0}}})$. For a given subset $J \subset \{1, 2, \ldots, \ldots, \frac{t}{N_0}\}$ with $|J| \leq \frac{\eta t}{N_0}$ and skeleton $(x_{i_1}, x_{i_2}, \ldots, x_{i_{\frac{t}{N_0}}})$, we have by Chebychev's inequality,

$$Q(\prod_J E_{x_{i_j}}[e^{\int_0^{N_0} dB^{jN_0}_{X(s)}(s)}] \geq e^{\frac{\epsilon t}{10}}) \leq e^{|J|/2} e^{-\epsilon t/10}.$$

We now complete the proof by first summing over the set of possible $J$ and then the set of possible skeletons $(x_{i_1}, x_{i_2}, \ldots, x_{i_{\frac{t}{N_0}}})$, using Lemma 4.1. □

In the same way we have,

**Lemma 4.6.** *Given $\epsilon > 0$ as in Lemma 4.2, for all $N_0$ sufficiently large,*

$$Q(\exists (x_{i_1}, x_{i_2}, \ldots, x_{i_{\frac{t}{N_0}}}) \text{ compatible with } \leq Mt \text{ jumps, any } J \subset \{1, 2, \ldots, \frac{t}{N_0}\}$$
$$\prod_J E_{x_j}[e^{\int_0^{N_0} dB^{jN_0}_{X(s)}(s)} I_{[MN_0, \infty)}(N(X, N_0))] \geq e^{\frac{\epsilon t}{10}}) \leq e^{-c\epsilon t}.$$

We can now give the proof of Theorem 2.11.

*Proof.* (of Theorem 2.11) For a given skeleton $(x_0, x_1, \ldots, x_{\frac{t}{N_0}-1})$, compatible with a path starting at zero with no more than $Mt$ jumps, we have that

$$\begin{aligned} E_0[e^{\int_0^t dB_{X(s)}(s)} \prod_{j=0}^{\frac{t}{N_0}-1} 1_{x_{i_j}}(X(jN_0))] &\leq \prod_{j=0}^{\frac{t}{N_0}-1} E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)}] \\ &= \prod_B E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)}] \prod_G E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)}], \end{aligned}$$

where $G$ is the set of $j$ so that block $(i_j, j)$ is a good block, B is the set of $j$ where this condition fails. By Lemma 4.3 we have that outside of probability

$e^{-8t/N_0}$ that for every possible skeleton $|B| < \eta t/N_0$. By Lemma 4.4 ( outside of probability $e^{-c\epsilon t}$ ) we have for all skeletons and all choices of B, with $|B| < \frac{\eta}{N_0}t$,

$$\prod_B E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)}] \ < \ e^{\epsilon t/10}.$$

Moreover we can split up $\prod_G E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)}]$ as

$$\prod_G (E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)} 1_{(MN_0,\infty)}(N(X,N_0))] + E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)} 1_{[0,MN_0]}(N(X,N_0))])$$

$$= \sum_{I\subset G}\prod_{j\in I} E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)} 1_{[0,MN_0]}(N(X,N_0))]\times$$

$$\times \prod_{j\notin I} E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)} 1_{(MN_0,\infty)}(N(X,N_0))].$$

By Lemma 4. 5, (outside of probability $e^{-c\epsilon t}$) for all skeletons and subsets I, simultaneously, the term $\prod_{j\notin I} E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)} 1_{(MN_0,\infty)}(N(X,N_0))] < e^{\epsilon t/10}$. So with large probability,

$$\begin{aligned}\prod_G E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)}] \ &\leq \ e^{\epsilon t/10}\sum_{I\subset G}\prod_{j\in I} E_{x_j}[e^{\int_0^t dB^{jN_0}_{X(s)}(s)} 1_{[0,MN_0]}(N(X,N_0))] \\ &\leq \ e^{\epsilon t/10} 2^{t/N_0} e^{t(\lambda(\kappa)+\epsilon/3)}\end{aligned}$$

by definition of a good block. That is we have shown that provided $N_0$ was fixed sufficiently large, then outside probability $e^{-\epsilon ct} + e^{-t/N_0}$ for every compatible skeleton

$$E_0[e^{\int_0^t dB_{X(s)}(s)} \prod_{j=0}^{\frac{t}{N_0}-1} 1_{x_{i_j}}(X(jN_0)))] \leq e^{\epsilon t/10} e^{\epsilon t/10} 2^{t/N_0} e^{t(\lambda(\kappa)+\epsilon/3)}.$$

By Lemma 4. 1, the number of skeletons compatible with no more than $Mt$ jumps is bounded by $e^{\epsilon t/10^6}$. Thus we have outside the above probability

$$E[e^{\int_0^t dB_{X(s)}(s)} I_{[0,Mt]}(N(X,t))] \leq e^{t(\lambda(\kappa)+2\epsilon/3)}.$$

The Theorem now follows from the estimate

$$E[e^{\int_0^t dB_{X(s)}(s)} I_{(Mt,\infty)}(N(X,t))] \leq e^{-cMt}.$$

□

## 5. Moment Lyapunov exponents

In this section we outline a relation between moment and sample Lyapunov exponents. Let

$$u(t,x) = E_x[e^{\int_0^t dB_{X(t-s)}(s)}]$$

as before. For $p \in \mathbb{R}$, set

$$M_p(t) = E_Q[u^p(t,x)],$$

which is finite for any $p \in \mathbb{R}$. Using Jensen's inequality, one can easily see that for $p \in [0,1]$

$$M_p(t+s) \geq M_p(t)M_p(s) \quad (t, s \geq 0),$$

and for $p \in \mathbb{R} \setminus [0,1]$,

$$M_p(t+s) \leq M_p(t)M_p(s) \quad (t, s \geq 0).$$

Thus, there exists a moment Lyapunov exponent $\Lambda(p)$ as defined by

$$\Lambda(p) = \lim_{t\to\infty} \frac{1}{t} \log M_p(t) = \sup_{t>0} \frac{1}{t} \log M_p(t) \quad (p \in [0,1]),$$

and

$$\Lambda(p) = \lim_{t\to\infty} \frac{1}{t} \log M_p(t) = \inf_{t>0} \frac{1}{t} \log M_p(t) \quad (p \in \mathbb{R} \setminus [0,1]).$$

Recall that $\lambda(\kappa) = \lambda_1(\kappa)$ denotes the sample Lyapunov exponent, and by Theorem 2.11, given $\epsilon > 0$ there is a positive constant $c(\epsilon)$ such that

$$Q(u(t,x) \geq e^{(\lambda(\kappa)+\epsilon)t}) \leq e^{-c(\epsilon)t}.$$

Applying Theorem 2.11 and the results of section three of [**7**] we obtain the following result.

**Theorem 5.1.** *$\Lambda(p)$ is differentiable at $p = 0$ and*

$$\lambda(\kappa) = \Lambda'(0).$$

*Proof.* For the proof we summarize several facts about $\Lambda(p)$ from [**7**].

**Fact 1.** $\Lambda(p)$ is concave for $p \in [0,1]$ and convex for $p \in \mathbf{R} \setminus [0,1]$ with $\Lambda(0) = 0$. Thus, for every $p$, the right derivative $\Lambda'(p+)$ and the left derivative $\Lambda'(p-)$ exist. Moreover it holds that

$$\Lambda'(0+) \geq \Lambda'(0-).$$

**Fact 2.**

$$\lim_{t\to\infty} E_Q[|\frac{1}{t} \log u(t,x) - \Lambda'(0-)|] = 0.$$

Combining **Fact 2** and Corollary 2.13 we have

**Fact 3.** $\lambda(\kappa) = \Lambda'(0-)$.

Applying Theorem 2.11 and the Schwarz inequality we see that for $p > 0$

$$\begin{aligned} M_p(t) &= E_Q[u^p(t,x) : u(t,x) < e^{(\lambda(\kappa)+\epsilon)t}] \\ &\quad + E_Q[u^p(t,x) : u(t,x) \geq e^{(\lambda(\kappa)+\epsilon)t}] \\ &\leq e^{p(\lambda(\kappa)+\epsilon)t} + M_{2p}(t)^{1/2} Q(u(t,x) \geq e^{(\lambda(\kappa)+\epsilon)t})^{1/2} \\ &\leq e^{p(\lambda(\kappa)+\epsilon)t} + M_{2p}(t)^{1/2} e^{-c(\epsilon)t/2}, \end{aligned}$$

from which it follows that

$$\Lambda(p) \leq \max\{p(\lambda(\kappa)+\epsilon), \frac{1}{2}(\Lambda(2p) - c(\epsilon))\}.$$

Dividing both sides by $p > 0$, and letting $p \searrow 0$, we have that for every $\epsilon > 0$

$$\Lambda'(0+) \leq \lim_{p\searrow 0} \max\{\lambda(\kappa)+\epsilon, \frac{1}{2p}(\Lambda(2p) - c(\epsilon))\} = \lambda(\kappa) + \epsilon.$$

Therefore

$$\Lambda'(0+) \leq \lambda(\kappa) = \Lambda'(0-),$$

which completes the proof of the theorem. $\square$

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M. Cranston, Department of Mathematics, University of Rochester, Rochester, New York 14627, USA, *e-mail*: `cran@math.rochester.edu`

T. S. Mountford, DMA, EPFL, CH-1015, Switzerland, *e-mail*: `thomas.mountford@epfl.ch`

T. Shiga, Mathematics Tokyo Inst. of Tech., Tokyo,152-8551, Japan, *e-mail*: `tshiga@math.titech.ac.jp`