Acta Math. Univ. Comenianae
Vol. LXXI, 2(2002), pp. 147–155

CRITICAL POINT THEORY FOR NONSMOOTH ENERGY FUNCTIONALS AND APPLICATIONS

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Abstract. In this paper we prove an abstract result about the minimization of nonsmooth functionals. Then we obtain some existence results for Neumann problems with discontinuities.

1. Introduction

In this paper we consider elliptic problems with multivalued nonlinear boundary conditions. We do not assume that the right-hand side is Carathéodory but we impose some monotonicity conditions. We set the energy functional which is not defined everywhere and it is not locally Lipschitz. Let us introduce the problem.

$$\left\{\begin{array}{l} -div(\|Dx(z)\|^{p-2}Dx(z)) = f(z,x(z)) \text{ a.e. on } Z \\ -\dfrac{\partial x}{\partial n_p}(z) \in \partial j(z,\tau(x)(z)) \text{ a.e. on } \Gamma, 2 \le p < \infty. \end{array}\right. \tag{1}$$

Here $D = grad$, $\frac{\partial x}{\partial n_p}(z) = \|Dx(z)\|^{p-2}(Dx(z), n(z))_{\mathbb{R}^{\mathbb{N}}}$, where $n(z)$ denotes the exterior normal vector to Γ at z.

Many authors have considered elliptic problems with discontinuous nonlinearities. Most of them studied Dirichlet problems (see for example Stuart-Tolland [**13**], Ambrosetti-Badialle [**1**]). As far as we know this is the first result of this type for Neumann problems with multivalued boundary conditions.

First we give an abstract minimization result and then we state and prove the existence theorems. At section 2 we give some definitions and we prove the minimization theorem.

2. Preliminaries and abstract results

Let X be a real Banach space and Y be a subset of X. A function $f : Y \to \mathbb{R}$ is said to satisfy a Lipschitz condition (on Y) provided that, for some nonnegative

Received February 3, 2002.
2000 *Mathematics Subject Classification.* Primary 35J25, 35J60 .
Key words and phrases. Discontinuous nonlinearities, countable number of discontinuities, critical point theory, mountain pass theorem.
This work supported partially by a postdoctoral scholarship from the State Scholarship Foundation (I. K. Y.) of Greece.

scalar K, one has

$$|f(y) - f(x)| \leq K\|y - x\|$$

for all points $x, y \in Y$. Let f be Lipschitz near a given point x, and let v be any other vector in X. The generalized directional derivative of f at x in the direction v, denoted by $f^o(x; v)$ is defined as follows:

$$f^o(x; v) = \limsup_{\substack{y \to x \\ t \downarrow 0}} \frac{f(y + tv) - f(y)}{t}$$

where y is a vector in X and t a positive scalar. If f is Lipschitz of rank K near x then the function $v \to f^o(x; v)$ is finite, positively homogeneous, subadditive and satisfies $|f^o(x; v)| \leq K\|v\|$. In addition f^o satisfies $f^o(x; -v) = (-f)^o(x; v)$. Now we are ready to introduce the generalized gradient which is denoted by $\partial f(x)$ as follows:

$$\partial f(x) = \{w \in X^* : f^o(x; v) \geq < w, v > \text{ for all } v \in X\}.$$

Some basic properties of the generalized gradient of locally Lipschitz functionals are the following:

(a) $\partial f(x)$ is a nonempty, convex, weakly compact subset of X^* and $\|w\|_* \leq K$ for every w in $\partial f(x)$.

(b) For every v in X, one has

$$f^o(x; v) = max\{< w, v >: w \in \partial f(x)\}.$$

If f_1, f_2 are locally Lipschitz functions then

$$\partial(f_1 + f_2) \subseteq \partial f_1 + \partial f_2.$$

Moreover, $(x, v) \to f^o(x; v)$ is upper semicontinuous and as function of v alone, is Lipschitz of rank K on X.

Let us mention the mean-value theorem of Lebourg.

Theorem 1 (Lebourg)**.** *Let x and y be points in X, and suppose that f is Lipschitz on an open set containing the line segment $[x, y]$. Then there exists a point $u \in (x, y)$ such that*

$$f(y) - f(x) \in < \partial f(u), y - x > . \tag{2}$$

Let $R : X \to \mathbb{R} \cup \{\infty\}$ be such that $R = \Phi + \psi$ where $\Phi : X \to \mathbb{R}$ be a locally Lipschitz functional while $\psi : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous, convex but not defined everywhere functional.

A point x in X is said to be a critical point of R if $x \in D(\psi)$ and if it satisfies the inequality

$$\Phi^o(x; y - x) + \psi(y) - \psi(x) \geq 0 \text{ for every } y \in X. \tag{3}$$

A number $c \in \mathbb{R}$ is said critical value if $R^{-1}(c)$ contains a critical point. Following Szulkin [**16**] we use the same notation for:

$$K = \{x \in X : x \text{ is a critical point}\},$$

$$R_c = \{x \in X : R(x) \leq c\}, \;\; K_c = \{x \in K : R(x) = c\}.$$

Proposition 1. *If R is as above, each local minimum is necessarily a critical point of R.*

Proof. Let x be a local minimum of R. Using convexity of ψ, it follows

$$0 \leq R((1-t)x+ty) - R(x) = \Phi(x+t(y-x)) - \Phi(x) + \\ +\psi((1-t)x+ty) - \psi(x) \leq \Phi(x+t(y-x)) - \Phi(x) + t(\psi(y) - \psi(x)).$$

Divide now with t and letting $t \to 0$ we obtain (3). Note that Φ is locally Lipschitz so

$$\lim_{t\to 0} \frac{\Phi(x+t(y-x)) - \Phi(y)}{t} \leq \Phi^o(x; y-x).$$

□

Definition 1. We say that $R : X \to \mathbb{R} \cup \{\infty\}$ with $R = \Phi + \psi$ satisfies H_1 if Φ is locally Lipschitz and ψ proper, convex and lower semicontinuous.

Let us now state the formulation of our (PS) condition.

(PS) If$\{x_n\}$is a sequence such that $R(x_n) \to c$ and

$$\Phi^o(x_n; y - x_n) + \psi(y) - \psi(x_n) \geq -\varepsilon_n \|y - x_n\| \text{ for every } y \in X \tag{4}$$

where $\varepsilon_n \to 0$, then $\{x_n\}$ has a convergent subsequence.

Proposition 2. *Suppose that R satisfies $H_1, (PS)$. Then, K_c is compact.*

Proof. Following Szulkin [**16**] it remains to show that if $x_n \to x$ in X then we have $\lim(\Phi^o(x_n, y - x_n) - \Phi^o(x; y-x)) \leq 0$. This is easy to prove since $(x, v) \to \Phi^o(x; v)$ is upper semicontinuous. □

We are ready now to prove our first abstract result.

Theorem 2. *If R is bounded below and satisfies $(H)_1$ and (PS), then*

$$c = \inf_{x\in X} R(x)$$

is a critical value.

Proof. Again by following Szulkin [**16**] we have that we can find a sequence $\{x_n\}$ such that $R(x_n) \leq c + \frac{1}{n}$ and

$$R(w) - R(x_n) \geq (-\frac{1}{n})\|w - x_n\| \text{ for all } w \in X.$$

Set $w = (1-t)x_n + tv, t \in (0,1)$. Since ψ is convex,

$$\Phi(x_n + t(v - x_n)) - \Phi(x_n) + t(\psi(v) - \psi(x_n)) \geq (-\frac{1}{n})\|v - x_n\|.$$

Dividing by t and letting $t \to 0$ we obtain

$$\Phi^o(x_n, v - x_n) + \psi(v) - \psi(x_n) \geq (-\frac{1}{n})\|v - x_n\|.$$

So by (PS) and proposition (2) $x_n \to x \in K_c$. □

3. Existence Results

Let $f : Z \times \mathbb{R} \to \mathbb{R}$, then we can define

$$f_1(z,x) = \liminf_{x' \to x} f(z,x'), f_2(z,x) = \limsup_{x' \to x} f(z,x').$$

Let $x \in W^{1,p}(Z)$ satisfies the boundary conditions. Then

Definition 2. We say that $x \in W^{1,p}(Z)$ is a solution of type I of problem (1) if there exists some $w \in W^{1,p}(Z)^*$ such that

$$w(z) \in [f_1(z,x(z)), f_2(z,x(z))]$$

and

$$-div(\|Dx(z)\|^{p-2}Dx(z)) = w(z) \text{ for almost all } z \in Z.$$

Definition 3. We say that $x \in W^{1,p}(Z)$ is a solution of type II of problem (1) if

$$-div(\|Dx(z)\|^{p-2}Dx(z)) = f(z,x(z)) \text{ for almost all } z \in Z.$$

Let us state our hypotheses for the function f of problem (1).

$\mathbf{H(f)_1}$: $f : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) is N-measurable (i.e. for every $x : Z \to \mathbb{R}$ measurable, $z \to f_{1,2}(z,x(z))$ is measurable too).

(ii) there exists $h : Z \times \mathbb{R} \to \mathbb{R}$ such that for almost all $z \in Z$ $h(z,x) \to \infty$ as $x \to \infty$ and there exists $M > 0$ such that $-F(z,x) \geq h(z,|x|)$ for $|x| \geq M$ with $F(z,x) = \int_o^x f(z,r)dr$.

(iii) for almost all $z \in Z$ and for all $x \in \mathbb{R}$ $|f(z,x)| \leq a(z) + c|x|^{\mu-1}$, $\mu < p$ $a \in L^{\mu'}(Z), c > 0, (\frac{1}{\mu} + \frac{1}{\mu'} = 1)$ and moreover $x \to f(z,x)$ is nonincreasing.

H(j): $j : Z \times \mathbb{R} \to \overline{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ is a measurable function such that for almost all $z \in Z$ $j(z,\cdot)$ is proper, convex and lower semicontinous (i.e. $j(z,\cdot) \in \Gamma_o(\mathbb{R})$).

Theorem 3. *If hypotheses $H(f)_1$ holds, then problem (1) has a solution x of type I.*

Proof. Let $\Phi, \psi : W^{1,p}(Z) \to \mathbb{R}$ defined as follows: $\Phi(x) = -\int_Z F(z,x(z))dz$, $\psi(x) = \frac{1}{p}\|Dx\|_p^p + \int_\Gamma j(z,\tau(x(z))d\sigma$. Here $d\sigma$ denotes the surface (Hausdorff) measure on Γ and τ is the trace operator. Then the energy functional is $R(x) = \Phi(x) + \psi(x)$.

It is clear that Φ is locally Lipschitz and it is easy to prove that ψ is lower semicontinuous, convex and proper. So $R = \Phi + \psi$ satisfies condition $(H)_1$.

Claim 1: $R(\cdot)$ satisfies the (PS)-condition.

Indeed, let $\{x_n\}_{n\geq 1} \subseteq W^{1,p}(Z)$ such that $R(x_n) \to c$ as $n \to \infty$ and we shall prove that this sequence is bounded in $W^{1,p}(Z)$. Suppose not. Then $\|x_n\| \to \infty$. Let $y_n(z) = \frac{x_n(z)}{\|x_n\|}$. Then clearly we have $y_n \xrightarrow{w} y$ in $W^{1,p}(Z)$. From the choice of the sequence we have

$$\Phi(x_n) + \frac{1}{p}\|Dx_n\|_p^p \leq M \tag{5}$$

(recall that $j(z,\cdot) \geq 0$). Dividing with $\|x_n\|^p$ the last inequality, we have

$$-\int_Z \frac{F(z,x(z))}{\|x_n\|^p}dz + \frac{1}{p}\|Dy_n\|_p^p \leq \frac{M}{\|x_n\|^p}.$$

By virtue of hypothesis $H(f)_1(iii)$ we have that $\frac{F(z,x_n(z))}{\|x_n\|_p^p} \to 0$.

So $\limsup \|Dy_n\|_p^p \to 0$. Thus, $\|Dy\| = 0$. So we infer that $y = c \in \mathbb{R}$. Since $\|y_n\| = 1$, $c \neq 0$. So we have that $|x_n(z)| \to \infty$. Going back to (5) and using hypothesis $H(f)_1(ii)$ we have a contradiction. So $\|x_n\|$ is bounded, i.e $x_n \xrightarrow{w} x$ in $W^{1,p}(Z)$. It remains to show that $x_n \to x$ in $W^{1,p}(Z)$.

Recall that from the choice of the sequence we have that

$$\Phi^o(x_n; y - x_n) + \psi(y) - \psi(x_n) \geq -\varepsilon_n\|y - x_n\| \text{ for all } y \in W^{1,p}(Z).$$

Choose $y = x$. Then we have:

$$\Phi^o(x_n; x - x_n) + \psi(x) - \psi(x_n) \geq -\varepsilon_n\|x - x_n\|$$

$$\Rightarrow \Phi^o(x_n; x - x_n) + \frac{1}{p}(\|Dx\|_p^p - \|Dx_n\|_p^p)$$

$$+\int_\Gamma j(z,\tau x(z))d\sigma - \int_\Gamma j(z,\tau x_n(z))d\sigma \geq -\varepsilon_n\|x - x_n\|. \tag{6}$$

So in the limit (in fact lim inf) we have that

$$\liminf_{n\to\infty} \Phi^o(x_n; x - x_n) \leq \limsup_{n\to\infty} \Phi^o(x_n; x - x_n) \leq 0$$

(note that $(x,v) \to \Phi^o(x;v)$ is upper semicontinuous). Moreover,

$$\int_\Gamma j(z,\tau x(z))d\sigma - \limsup_{n\to\infty}\int_\Gamma j(z,\tau x_n(z)) \leq$$

$$\leq \int_\Gamma j(z,\tau x(z))d\sigma - \liminf_{n\to\infty}\int_\Gamma j(z,\tau x_n(z)) = 0.$$

Thus finally we obtain

$$\limsup \|Dx_n\|_p^p \leq \|Dx\|_p^p.$$

On the other hand since $Dx_n \xrightarrow{w} Dx$ in $L^p(Z,R^N)$, from the weak lower semicontinuity of the norm, we have

$$\begin{aligned} &\liminf \|Dx_n\|_p \geq \|Dx\|_p \\ \Rightarrow\quad &\|Dx_n\|_p \to \|Dx\|_p. \end{aligned}$$

The space $L^p(Z,\mathbb{R}^{\mathbb{N}})$ being uniformly convex, has the Kadec-Klee property (see Hu-Papageorgiou [**9**], definition I.1.72(d)) and so $x_n \to x$ in $W^{1,p}(Z)$.

Claim 2 $R(\cdot)$ is bounded from below.

Suppose not. Then there exists some sequence $\{x_n\}_{n\geq 1}$ such that $R(x_n) \leq -n$. Then we have

$$\Phi(x_n) + \frac{1}{p}\|Dx_n\|_p^p \leq -n$$

(recall that $j(z,\cdot) \geq 0$.) By virtue of the continuity of $\Phi, \|Dx\|_p$ we have that $\|x_n\| \to \infty$ (because if $\|x_n\|$ is bounded then $R(x_n)$ is bounded). Dividing with $\|x_n\|^p$ and letting $n \to \infty$ we have as before a contradiction (by virtue of hypothesis $H(f)_1(iii)$). Therefore $R(\cdot)$ is bounded from below.

So by Theorem 2 we have that there exists $x \in W^{1,p}(Z)$ such that (3) holds. Let $\psi_1(x) = \frac{\|Dx\|^p}{p}$ and $\psi_2(x) = \int_\Gamma j(z,\tau(x)(z))d\sigma$. Then let $\widehat{\psi}_1 : L^p(Z) \to \mathbb{R}$ the extension of ψ_1 in $L^p(Z)$. Then $\partial\psi_1(x) = \partial\widehat{\psi}_1(x)$ (see Showalter [**14**], proposition 5.2 p. 194-195). Let $A : W^{1,p}(Z) \to W^{1,p}(Z)^*$ such that

$$< Ax, y >= \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))dz \text{ for all } y \in W^{1,p}(Z).$$

It is easy to prove that the nonlinear operator $\widehat{A} : D(\widehat{A}) \subseteq L^p(Z) \to L^q(Z)$ such that

$$< \widehat{A}x, y >= \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))dz \text{ for all } y \in W^{1,p}(Z)$$

with $D(\widehat{A}) = \{x \in W^{1,p}(Z) : Ax \in L^q(Z)\}$, satisfies $\widehat{A} = \partial\widehat{\psi}_1$. Indeed, first we show that $\widehat{A} \subseteq \partial\widehat{\psi}_1$ and then it suffices to show that $\widehat{A}$ is maximal monotone.

$$\begin{aligned}
< \widehat{A}x, y - x > &= \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z) - Dx(z))_{R^N}dz \\
&= \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N}dz - \int_Z \|Dx(z)\|^p dz \\
&\leq \int_Z (\frac{\|Dx(z)\|^{q(p-2)}\|Dx(z)\|^q}{q} + \frac{\|Dy(z)\|^p}{p})dz - \|Dx\|_p^p \\
&= \frac{\|Dx\|_p^p}{q} - \|Dx\|_p^p + \frac{\|Dy\|_p^p}{p} \\
&= \widehat{\psi}_1(y) - \widehat{\psi}_1(x).
\end{aligned}$$

The monotonicity part is obvious using the following inequality,

$$\sum_{j=1}^N (a_j(\eta) - a_j(\eta'))(\eta_j - \eta_j') \geq C|\eta - \eta'|^p.$$

for $\eta, \eta' \in R^N$, with $a_j(\eta) = |\eta|^{p-2}\eta_j$.

The maximality needs more work. Let $J : L^p(Z) \to L^q(Z)$ be defined as $J(x) = |x(\cdot)|^{p-2}x(\cdot)$. We will show later that $R(\widehat{A} + J) = L^q(Z)$. Assume for the moment that this holds. Let $v \in L^p(Z), v^* \in L^q(Z)$ be such that

$$(\widehat{A}(x) - v^*, x - v)_{pq} \geq 0$$

for all $x \in D(\widehat{A})$. By assumption $R(\widehat{A} + J) = L^q(Z))$, so there exists $x \in D(\widehat{A})$ such that $\widehat{A}(x) + J(x) = v^* + J(v)$. Using this in the above inequality we have that

$$(J(v) - J(x), x - v)_{pq} \geq 0.$$

But J is strongly monotone. Thus we have that $v = x$ and $\widehat{A}(x) = v^*$. Therefore $\widehat{A}$ is maximal monotone. It remains to show that $R(\widehat{A} + J) = L^q(Z)$. Note that $\widehat{J} = J\ |_{W^{1,p}(Z)}$: $W^{1,p}(Z) \to W^{1,p}(Z)^*$ is maximal monotone, because is demicontinuous and monotone. So $A + \widehat{J}$ is maximal monotone. But it is easy to see that the sum is coercive. So is surjective. Therefore, $R(A + \widehat{J}) = W^{1,p}(Z)^*$. Then for every $g \in L^q(Z)$, we can find $x \in W^{1,p}(Z)$ such that $A(x) + \widehat{J}(x) = g \Rightarrow A(x) = g - \widehat{J}(x) \in L^q(Z) \Rightarrow A(x) = \widehat{A}(x)$. Thus, $R(\widehat{A} + J) = Ł^q(Z)$.

Thus, we have

$$R(y) - R(x) \geq 0$$

for all $y \in W^{1,p}(Z)$.

But, note that R is convex, so we have that $0 \leq \partial R(x)$. So, we can say that

$$\int_Z w(z)y(z) = \int_Z \|Dx(z)\|^{p-2}(Dx(z), Dy(z))dz + \int_\Gamma v(z)y(z)d\sigma \tag{7}$$

with $w(z) \in [f_1(z, x(z)), f_2(z, x(z))]$ and $v(z) \in \partial j(z, \tau(x(z)))$, for every $y \in W^{1,p}(Z)$ (see Chang [**3**]). Let $y = \phi \in C_o^\infty(Z)$. Then we have

$$\int_Z w(z)\phi(z)dz = \int_Z \|Dx(z)\|^{p-2}(Dx(z), D\phi(z))dz.$$

But $div(\|Dx(z)\|^{p-2}Dx(z)) \in L^q(Z)$ because $w(z) \in L^q(Z)$. Then we have

$$-div(\|Dx(z)\|^{p-2}Dx(z)) \in [f_1(x(z)), f_2(x(z))] \text{ a.e. on } Z.$$

Going back to (12) and letting $y = C^\infty(Z)$ and finally using the Green formula 1.6 of Kenmochi [**11**], we have that $-\frac{\partial x}{\partial n_p} \in \partial j(z, \tau(x)(z))$ a.e. on Γ. So $x \in W^{1,p}(Z)$ and is of type I. □

Now, with stronger hypotheses on f we are going to have an existence result of type II.

$\mathbf{H(f)_2}$: Satisfies $H(f)_1$ and depends only on x, not on z itself.

Theorem 4. *If hypotheses $H(f)_2, H(j)$ holds, then problem (1) has a solution x of type II.*

Proof. From theorem 3 we know that there exists $x \in W^{1,p}(Z)$ such that $0 \leq R(y) - R(x)$ for all $y \in W^{1,p}(Z)$. That means

$$(-\Phi)(y) - (-\Phi)(x) \leq \psi(y) - \psi(x).$$

Note that $-\Phi, \psi$ are convex, so for every $w \in \partial(-\Phi)(x)$ we have that $w \in \partial\psi(x)$.

So,

$$< w, y > \leq < Ax, y > + < v, y >$$
$$\text{for all } y \in W^{1,p}(Z) \text{ and all } w \in \partial(-\Phi)(x)$$

with $w(z) \in [f_1(x(z)), f_2(x(z))]$. Choosing now $y = s$ and $y = -s$ with $s \in W^{1,p}(Z)$ we have $< w, s >=< Ax, s > + < v, s >$ for all $s \in W^{1,p}(Z)$ and all $w \in L^q(Z)$ such that $w(z) \in [f_1(x(z)), f_2(x(z))]$.

We will show that $\lambda\{z \in Z : x(z) \in \mathbf{d}(f)\} = 0$ with $\mathbf{d}(f) = \{x \in R : f(x^+) > > f(x^-)\}$, that is the set of upward-jumps.

So let $w \in \partial(-\Phi(x))$ and for any $t \in \mathbf{d}(f)$, set

$$\rho^{\pm}(z) = [1 - \chi_t(x(z))]w(z) + \chi_t(x(z))[f(x(z)^{\pm})] \tag{8}$$

where

$$\chi_t(s) = \begin{cases} 1 & \text{if} \quad s = t \\ 0 & \quad \text{otherwise} \end{cases} \tag{9}$$

Then $\rho^{\pm} \in L^p(Z)$ and $\rho^{\pm} \in [f_1(x), f_2(x)]$. So

$$\int_Z \rho^{\pm}(z)y(z)dz = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), Dy(z))_{R^N} dz + \int_\Gamma v(z)y(z)d\sigma$$

for all $y \in W^{1,p}(Z)$.

So for $y = \phi \in C_o^\infty(Z)$ we have

$$\int_Z \rho^{\pm}(z)\phi(z)dz = \int_Z (\|Dx(z)\|^{p-2}(Dx(z), D\phi(z))_{R^N} dz.$$

Thus, $\rho^+ = \rho^-$ for almost all $z \in Z$. From this it follows that $\chi_t(x(z)) = 0$ for almost all $z \in Z$. Since $\mathbf{d}(f)$ is countable, and

$$\chi(x(z)) = \sum_{t \in D(f)} \chi_t(x(z))$$

it follows that $\chi(x(z)) = 0$ almost everywhere, (with $\chi(t) = 1$ if $t \in \mathbf{d}(f)$ and $\chi(t) = 0$ otherwise).

Now it is clear that $x \in W^{1,p}(Z)$ solves problem (1). □

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