Acta Math. Univ. Comenianae
Vol. LXXI, 2(2002), pp. 157–161

A MOORE-LIKE BOUND FOR GRAPHS OF DIAMETER 2 AND GIVEN DEGREE, OBTAINED AS ABELIAN LIFTS OF DIPOLES

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Abstract. In this note we prove a Moore-like bound for graphs of diameter two and given degree which arise as lifts of dipoles with loops and multiple edges, with voltage assignments in Abelian groups.

1. Introduction

The well known and widely studied *degree-diameter problem* is to determine the largest number $n_{d,k}$ of vertices in a graph of maximum degree d and diameter at most k. Literature concerning this problem is abundant and we refer to the recent survey article [**5**] for history, background, and latest development. Here we just mention the obvious inequality $n_{d,k} \leq 1 + d + d(d-1) + \cdots + d(d-1)^{k-1}$, the right-hand side of which is called *Moore bound* and denoted $M_{d,k}$. In particular for diameter two we have $M_{d,2} \leq d^2 + 1$, with equality if and only if $d = 2, 3, 7$ and possibly 57 [**3**].

In connection with network design applications the search for large graphs of given degree and diameter has often been confined to vertex-transitive and Cayley graphs. Let $vt_{d,k}$ be the largest number of vertices of a vertex-transitive graph of degree d and diameter k. Despite the fact that vertex-transitivity is a rather restrictive property, in general there is no better upper bound on $vt_{d,k}$ than the Moore bound. For diameter two, the current best lower bound [**4**] is $vt_{d,2} \geq \geq \frac{8}{9}(d+\frac{1}{2})^2$ for all d such that $d = (3q-1)/2$, where q is a prime power congruent to 1 (mod 4). The *McKay-Miller-Širáň* graphs that meet this lower bound are vertex-transitive but non-Cayley; the graph corresponding to the value $q = 5$ is the Hoffman-Singleton graph, and for $q = 9$ the corresponding graph of diameter 2 and degree 13 has 162 vertices, which is only 8 less than the Moore bound $M_{13,2} = 170$.

The McKay-Miller-Širáň graphs were constructed as Abelian lifts of complete bipartite graphs $K_{q,q}$ with $(q-1)/4$ loops at each vertex. A simplified construction in terms of Abelian lifts of dipoles with loops and multiple edges was presented in [**7**] using a characterization of composition of regular coverings [**6**].

Received March 1, 2002.
2000 *Mathematics Subject Classification.* Primary 05C99.

Regular Abelian lifts of two-vertex graphs (dipoles) are just one step away from Abelian Cayley graphs, which are regular lifts of one-vertex graphs with voltage assignments in Abelian groups. It is therefore natural to ask about upper bounds on the number of vertices in graphs of degree d and diameter 2, obtained as Abelian lifts of a dipole of degree d. The purpose of this note is to present such an upper bound; for large d it has the form $c \cdot d^2$, where $c = 4(10 + \sqrt{2})/49 \approx 0.932$. This is obviously an improvement of the Moore bound $M_{d,2} = 1 + d^2$. Note that the number of vertices of McKay-Miller-Širáň graphs is approximately $0.889d^2$.

Another motivation for studying Abelian lifts of dipoles comes from a considerable number of papers devoted to constructing large Abelian Cayley graphs of given degree and diameter; see [**1**] and references therein. The combination of algebraic and geometric methods presented in the recent deep study of degree-diameter problem restricted to Abelian Cayley graphs [**2**] seem to work quite well for small degree and large diameter but not vice versa; our interest is in the case of diameter two and large degree.

The main result is presented in Section 2, after a brief introduction to voltage assignments and lifts. In Section 3 we discuss a few related topics.

2. Results

For an undirected graph G let $D(G)$ denote the set of all darts (oriented edges). Each edge can be viewed as a pair of opposite darts. Let Γ be a group. A mapping $\alpha : D(G) \to \Gamma$ such that $\alpha(e^{-1}) = (\alpha(e))^{-1}$ for each dart of G is called a *voltage assignment* on G in the group Γ. A *lift* of G, denoted by G^α, is a graph whose vertex set is $V(G^\alpha) = V(G) \times \Gamma$ and the dart set is $D(G^\alpha) = D(G) \times \Gamma$. A dart e_g in G^α emanates from u_g and terminates at v_h if and only if e is a dart in G from u to v, and $h = g\alpha(e)$. The reverse of the dart e_g is the dart $(e^{-1})_{g\alpha(e)}$. This pair of darts form an undirected edge in G^α, and so the lift is undirected. The set $\{v_g; g \in \Gamma\}$ is a *fibre* in G^α above a vertex v in G. A sequence $e_1e_2 \dots e_t$ of darts in G such that the terminal vertex of e_i coincides with the initial vertex of e_{i+1} $(1 \le i < t)$ is a *walk* in G of length t. If $W = e_1e_2 \dots e_t$ is a walk in G then we set $\alpha(W) = \alpha(e_1)\alpha(e_2) \cdots \alpha(e_t)$. Examining walks in the base graph we are able to determine the diameter of a lift as stated in the following Lemma [**4**].

Lemma 1. *Let α be a voltage assignment on a graph G in a group Γ. Then, $diam(G^\alpha) \le k$ if and only if for each ordered pair of vertices u, v (possibly, $u = v$) of G and for each $g \in \Gamma$ there exists a $u \to v$ walk of length $\le k$ of net voltage g.*

We denote by $D_{m,l}$ the graph with two vertices u, v, joined by m parallel edges, and with l loops attached to each vertex. For brevity we call the graph $D_{m,l}$ a *dipole*. We first present an upper bound on the number of vertices of a diameter two Abelian lift of a dipole.

Proposition 1. *Let α be a voltage assignment on a dipole $D_{m,l}$ in an Abelian group Γ such that the lift $D^\alpha_{m,l}$ has diameter two. Then the number of vertices in $D^\alpha_{m,l}$ is at most $w(m, l)$, where*

$$w(m,l) = 2\ \min\{1 + m(m-1) + 2l(l+1), m(4l+1)\}. \tag{1}$$

Proof. Let $D^\alpha_{m,l}$ be of diameter 2. It follows from Lemma 1 that the number of vertices in the fibre above u cannot exceed the number of distinct voltages on the $u \to u$ walks in $D_{m,l}$ of length at most 2. Using the fact that the voltage group is Abelian this number is easily seen to be $1 + 2l(l+1) + m(m-1)$. On the other hand the number of vertices in the same fibre is also bounded above by $m(1+4l)$, which corresponds to the number of distinct voltages on $v \to u$ walks in $D_{m,l}$ of length at most 2 (note that this step does not use commutativity). Since there are two fibres in the lift, the number of vertices of $D^\alpha_{m,l}$ satisfies $|V(D^\alpha_{m,l})| \le w(m,l)$, where

$$w(m,l) = 2\ \min\{1 + m(m-1) + 2l(l+1), m(4l+1)\}.$$

$\square$

Now we state and prove our main result bounding the number of vertices of an Abelian lift of a dipole of given degree and diameter two.

Theorem 1. *For an arbitrary $d \ge 3$ let D be a dipole of degree d, and let α be a voltage assignment on D in an Abelian group Γ such that the lift D^α is of diameter two. Then*

$$|V(D^\alpha)| \le \frac{4(10+\sqrt{2})}{49}(d+0.34)^2. \tag{2}$$

Proof. According to Proposition 1, for $d = m+2l$ the bound (1) can be expressed in terms of d, which is

$$w(d) = 2\ \max\min\{6l^2 + 4(1-d)l + d^2 - d + 1, -8l^2 + 2(2d-1)l + d\},$$

the maximum being taken over all integer l such that $1 \le l < \frac{d}{2}$.

To find the maximum we need to know the vertices and points of intersection of the two parabolas $p_1(l) = 6l^2+4(1-d)l+d^2-d+1$ and $p_2(l) = -8l^2+2(2d-1)l+d$. The first coordinates of the vertices of $p_1(l)$ and $p_2(l)$ are $x_1 = \frac{d-1}{3}$, $x_2 = \frac{2d-1}{8}$, respectively. The parabolas intersect at $l_1 = (4d-3-\sqrt{2d^2+4d-5})/14$, and $l_2 = (4d-3+\sqrt{2d^2+4d-5})/14$. For $d \ge 3$ we have $0 < l_1 < x_2 < x_1 < l_2$. It means that the vertex of the concave down parabola, p_2, precedes the vertex of the concave up parabola, p_1. From the symmetry of the parabolas it follows that the maximum is attained at l_1 (when taken over all real l such that $0 < l < \frac{d}{2}$). As $l_1 \ge 1$ only if $d \ge 7$ the bound for $d \ge 7$ is

$$w(d) \le \frac{2}{49}(20d^2 + (2d-5)\sqrt{2d^2+4d-5} + 19d + 13). \tag{3}$$

A routine calculation shows that for $d \ge 7$ the right-hand side of (3) is smaller than

$$\frac{4(10+\sqrt{2})}{49}(d+0.34)^2.$$

For $d < 7$ the maximum is achieved at $l = 1$ evaluated for $p_1(l)$. The corresponding values are $w(3) = 10,\ w(4) = 14,\ w(5) = 22,\ w(6) = 34$. $\square$

Theorem 1 has the following consequence whose obvious proof is left to the reader.

Corollary 1. *In the notation of Theorem 1, we have*

$$\limsup_{d\to\infty} \frac{|V(D^\alpha)|}{d^2} \le \frac{4(10+\sqrt{2})}{49} \approx 0.932. \tag{4}$$

3. Remarks

We begin with commenting on Proposition 1. The McKay-Miller-Širáň graphs were constructed as lifts of dipoles $D_{m,l}$, where m is a prime power $\equiv 1 \pmod 4$, and $l = (m-1)/4$. It is therefore interesting to compare the number of vertices, $2m^2$, of McKay-Miller-Širáň graphs with the bound obtained in Proposition 1, which for the above values of m and l gives

$$w(m,(m-1)/4) = 2\ \min\{\frac{1}{8}(9m^2-6m+5), m^2\} = 2m^2$$

for $m \ge 5$. It follows that for $m \equiv 1 \pmod 4$ and $l = (m-1)/4$ the McKay-Miller-Širáň graphs are the largest possible graphs that can be obtained by this method.

In Theorem 1 we presented an upper bound on $w(d)$. A comparison of this bound with the lower bound given by McKay-Miller-Širáň graphs and the Moore bound is presented in Table 1 for some selected values of the degree d of the form $d = (3m-1)/2$, where m is a prime power $\equiv 1 \pmod 4$. The values in the third column are rounded down to the nearest even number.

Degree d	$\frac{8}{9}(d+\frac{1}{2})^2$	$0.932(d+0.34)^2$	d^2+1
25	578	598	626
61	3362	3506	3722
151	20402	21346	22802
253	57122	59816	64010
541	260642	273120	292682
1261	1414562	1482792	1565002
5221	24234722	25408548	27258842

Table 1. Comparison of lower and upper bounds on $|V(D^\alpha)|$.

Finally, note that our asymptotic bound (4) gives better result than the Moore bound $M_{d,2} = d^2+1$, and compares well with the lower bound $\approx 0.889d^2$ obtained from the graphs of McKay-Miller-Širáň.

Acknowledgement. The author acknowledges partial support from the VEGA grant No. 1/9176/02.

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