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SOME DOUBLE INTEGRAL INEQUALITIES AND APPLICATIONS

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Abstract. Some double integral inequalities are established. These inequalities give upper and lower error bounds for the well-known mid-point and trapezoid quadrature rules. Some inequalities for convex and concave functions are derived. Applications in numerical integration are also given.

1. Introduction

Recently, P. Cerone and S. S. Dragomir [**1**] obtained the following double integral inequality.

Theorem 1. *Let $f : [a, b] \to R$ be a twice differentiable mapping on (a, b) and suppose that $\gamma \le f''(t) \le \Gamma$ for all $t \in (a, b)$. Then we have the double inequality:*

$$\frac{\gamma(b-a)^2}{24} \le \frac{1}{b-a}\int_a^b f(t)dt - f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(b-a)^2}{24}. \tag{1.1}$$

In this paper we derive new upper and lower bounds for the quantity $(b-a)^{-1}\int_a^b f(t)dt - f(\frac{a+b}{2})$. We show that these new bounds can be better than the bounds in (1.1).

In [**2**] the same authors proved the next result.

Theorem 2. *Under the assumptions of Theorem 1 we have*

$$\frac{\gamma(b-a)^2}{12} \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{\Gamma(b-a)^2}{12}. \tag{1.2}$$

We here derive new upper and lower bounds for the quantity $\frac{f(a)+f(b)}{2} - (b-a)^{-1}\int_a^b f(t)dt$ and show that these new bounds can be better than the bounds in (1.2).

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Note that the inequalities (1.1) and (1.2) give upper and lower error bounds for the well-known mid-point and trapezoid quadrature rules, respectively.

We also consider a result obtained in [**5**] and compare this result with a result obtained in this paper.

In Section 3 we derive some inequalities for convex and concave functions (similar to the well-known Hermite-Hadamard inequalities). In Section 4 applications in numerical integration are given. For example, we derive a perturbed composite trapezoidal quadrature rule. It is slight different than the classical composite trapezoidal rule. In fact, only one additional term appears in the perturbed rule. We show that the perturbed rule has two times better estimation of error than the classical one.

2. Main results

Theorem 3. *Let the assumptions of Theorem 1 hold. Then we have*

$$\frac{3S-2\Gamma}{24}(b-a)^2 \le \frac{1}{b-a}\int_a^b f(t)dt - f(\frac{a+b}{2}) \le \frac{3S-2\gamma}{24}(b-a)^2, \tag{2.1}$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. We define

$$p(t) = \begin{cases} \frac{(t-a)^2}{2}, & t \in \left[a, \frac{a+b}{2}\right] \\ \frac{(t-b)^2}{2}, & t \in \left(\frac{a+b}{2}, b\right] \end{cases} . \tag{2.2}$$

Then we have

$$\int_a^b p(t)dt = \int_a^{\frac{a+b}{2}} \frac{(t-a)^2}{2}dt + \int_{\frac{a+b}{2}}^b \frac{(t-b)^2}{2}dt = \frac{(b-a)^3}{24}. \tag{2.3}$$

Integrating by parts, we have

$$\begin{aligned} &\int_a^b p(t)f''(t)dt \\ = & -\int_a^{\frac{a+b}{2}} (t-a)f'(t)dt - \int_{\frac{a+b}{2}}^b (t-b)f'(t)dt \\ = & \int_a^b f(t)dt - f(\frac{a+b}{2})(b-a). \end{aligned} \tag{2.4}$$

From (2.3) and (2.4) we get

$$\int_a^b p(t)\,[f''(t)-\gamma]\,dt = \int_a^b f(t)dt - f(\frac{a+b}{2})(b-a) - \gamma\frac{(b-a)^3}{24}. \tag{2.5}$$

We also have

$$\int_a^b p(t)\,[f''(t)-\gamma]\,dt \le \max_{t\in[a,b]} |p(t)| \int_a^b |f''(t)-\gamma|\,dt \tag{2.6}$$

and

$$\max_{t\in[a,b]} |p(t)| = \frac{(b-a)^2}{8}, \tag{2.7}$$

$$\begin{aligned} \int_a^b |f''(t)-\gamma|\,dt &= f'(b)-f'(a)-\gamma(b-a) \\ &= (S-\gamma)(b-a), \end{aligned} \tag{2.8}$$

since $f''(t)\ge\gamma$, $t\in[a,b]$.

From (2.5)–(2.8) it follows

$$\int_a^b f(t)dt - f(\frac{a+b}{2})(b-a) \le \frac{3S-2\gamma}{24}(b-a)^3. \tag{2.9}$$

On the other hand, we have

$$\begin{aligned} \int_a^b p(t)\,[\Gamma-f''(t)]\,dt &\le \max_{t\in[a,b]} |p(t)| \int_a^b |f''(t)-\Gamma|\,dt \\ &= \frac{(b-a)^3}{8}(\Gamma-S), \end{aligned} \tag{2.10}$$

since

$$\begin{aligned} \int_a^b |\Gamma-f''(t)|\,dt &= \Gamma(b-a)-f'(b)+f'(a) \\ &= (\Gamma-S)(b-a). \end{aligned} \tag{2.11}$$

From (2.3) and (2.4) we get

$$\int_a^b p(t)\,[\Gamma - f''(t)]\,dt \tag{2.12}$$

$$= -\int_a^b f(t)dt + f(\frac{a+b}{2})(b-a) + \Gamma\frac{(b-a)^3}{24}.$$

From (2.10) and (2.12) we have

$$\frac{3S-2\Gamma}{24}(b-a)^3 \le \int_a^b f(t)dt - f(\frac{a+b}{2})(b-a). \tag{2.13}$$

This completes the proof. □

We now show that (2.1) can be better than (1.1). For that purpose, we give the following examples.

Example 1. Let us choose $f(t) = t^k$, $k > 2$, $a = 0$, $b > 0$. Then we have

$f'(t) = kt^{k-1}$, $f''(t) = k(k-1)t^{k-2}$, $\gamma = 0$, $\Gamma = k(k-1)b^{k-2}$, $S = kb^{k-2}$.

Thus, the right-hand sides of (2.1) and (1.1) become:

$$R.H.S.(2.1) = \frac{k}{8}b^k$$

and

$$R.H.S.(1.1) = \frac{k(k-1)}{24}b^k.$$

We easily find that $R.H.S.(2.1) < R.H.S.(1.1)$ if $k > 4$. In fact, if $k \gg 4$ then (2.1) is much better than (1.1).

Example 2. Let us choose $f(t) = -t^k$, $k > 2$, $a = 0$, $b > 0$. Then we have

$f'(t) = -kt^{k-1}$, $f''(t) = -k(k-1)t^{k-2}$, $\Gamma = 0$, $\gamma = -k(k-1)b^{k-2}$, $S = -kb^{k-2}$.

Thus, the left-hand sides of (2.1) and (1.1) become:

$$L.H.S.(2.1) = -\frac{k}{8}b^k$$

and

$$L.H.S.(1.1) = -\frac{k(k-1)}{24}b^k.$$

If $2 < k < 4$ then $L.H.S.(2.1) < L.H.S.(1.1)$.

Theorem 4. *Under the assumptions of Theorem 1 we have*

$$\frac{3S-\Gamma}{24}(b-a)^2 \le \frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt \le \frac{3S-\gamma}{24}(b-a)^2, \tag{2.14}$$

where $S = \frac{f'(b)-f'(a)}{b-a}$.

Proof. We define

$$p(t) = \frac{1}{2}(t - \frac{a+b}{2})^2 - \frac{(b-a)^2}{8}. \tag{2.15}$$

Then we have

$$\int_a^b p(t)dt = -\frac{(b-a)^3}{12}. \tag{2.16}$$

Integrating by parts, we have

$$\begin{aligned} &\int_a^b p(t)f''(t)dt \\ = \ & -\int_a^b (t - \frac{a+b}{2})f'(t)dt = \int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a). \end{aligned} \tag{2.17}$$

From (2.16) and (2.17) we get

$$\begin{aligned} &\int_a^b p(t)\,[\gamma - f''(t)]\,dt \\ = \ & -\int_a^b f(t)dt + \frac{f(a)+f(b)}{2}(b-a) - \gamma\frac{(b-a)^3}{12}. \end{aligned} \tag{2.18}$$

We also have

$$\int_a^b p(t)\,[\gamma - f''(t)]\,dt \leq \max_{t\in[a,b]} |p(t)| \int_a^b |f''(t) - \gamma|\,dt. \tag{2.19}$$

and

$$\max_{t\in[a,b]} |p(t)| = \frac{(b-a)^2}{8}, \tag{2.20}$$

$$\begin{aligned} \int_a^b |f''(t) - \gamma|\,dt &= f'(b) - f'(a) - \gamma(b-a) \\ &= (S-\gamma)(b-a). \end{aligned} \tag{2.21}$$

From (2.18)–(2.21) we have

$$\frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)dt \leq \frac{3S-\gamma}{24}(b-a)^3.$$

On the other hand, we have

$$\int_a^b p(t)\left[f''(t)-\Gamma\right]dt = \int_a^b f(t)dt - \frac{f(a)+f(b)}{2}(b-a) + \Gamma\frac{(b-a)^3}{12} \tag{2.22}$$

and

$$\left|\int_a^b p(t)\left[f''(t)-\Gamma\right]dt\right| \leq \max_{t\in[a,b]}|p(t)|\int_a^b \left|f''(t)-\Gamma\right|dt = \frac{(b-a)^3}{8}(\Gamma - S). \tag{2.23}$$

From (2.22) and (2.23) it follows

$$\frac{3S-\Gamma}{24}(b-a)^3 \leq \frac{f(a)+f(b)}{2}(b-a) - \int_a^b f(t)dt.$$

This completes the proof. □

We now show that (2.14) can be better than (1.2).

Example 3. Let us choose $f(t)=t^k$, $k>2$, $a=0$, $b>0$. Then we have

$$f'(t)=kt^{k-1},\ f''(t)=k(k-1)t^{k-2},\ \gamma=0,\ \Gamma=k(k-1)b^{k-2},\ S=kb^{k-2}.$$

Thus, the right-hand sides of (2.14) and (1.2) become:

$$R.H.S.(2.14) = \frac{k}{8}b^k$$

and

$$R.H.S.(1.2) = \frac{k(k-1)}{12}b^k.$$

We easily find that $R.H.S.(2.14) < R.H.S.(1.2)$ if $k > \frac{5}{2}$. In fact, if $k \gg \frac{5}{2}$ then (2.14) is much better than (1.2).

Example 4. Let us choose $f(t)=-t^k$, $k>2$, $a=0$, $b>0$. Then we have

$$f'(t)=-kt^{k-1},\ f''(t)=-k(k-1)t^{k-2},\ \Gamma=0,\ \gamma=-k(k-1)b^{k-2},\ S=-kb^{k-2}.$$

Thus, the left-hand sides of (2.14) and (1.2) become:

$$L.H.S.(2.14) = -\frac{k}{8}b^k$$

and

$$L.H.S.(1.2) = -\frac{k(k-1)}{12}b^k.$$

If $2<k<\frac{5}{2}$ then $L.H.S.(2.14) < L.H.S.(1.2)$.

In [**5**] we can find the following result.

Theorem 5. *Let $f : I \subseteq R \to R$ be a differentiable mapping on I, $a, b \in I$, with $a < b$. If $|f'|$ is convex on $[a, b]$ then the following inequality holds:*

$$\left| \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \right| \leq \frac{|f'(b)|+|f'(a)|}{8}(b-a). \tag{2.24}$$

Remark 1. Inequality (2.14) can be much better than (2.24). For example, if f is a convex function ($f'' > 0$ such that f' is an increasing function, i.e. $f'(b) > f'(a)$) then (2.24) becomes

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{|f'(b)|+|f'(a)|}{8}(b-a), \tag{2.25}$$

while (2.14) becomes

$$0 \leq \frac{f(a)+f(b)}{2} - \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f'(b)-f'(a)}{8}(b-a). \tag{2.26}$$

If f' is large on $[a, b]$ and $f'(a)$ is close to $f'(b)$ then (2.26) is much better than (2.25).

3. Results for convex and concave functions

One of cornerstones of analysis are the well-known Hermite-Hadamard inequalities for convex functions:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}. \tag{3.1}$$

We here can obtain similar inequalities using the previous derived results.

If $f''(t) \geq 0$, $t \in [a, b]$, i.e. f is a convex function, then we can set $\gamma = 0$ in (2.1). Thus,

$$\frac{1}{b-a} \int_a^b f(t)dt \leq f\left(\frac{a+b}{2}\right) + \frac{1}{8}S,$$

where $S = [f'(b) - f'(a)](b-a)$. On the other hand, from (2.14) we get

$$\frac{1}{b-a} \int_a^b f(t)dt \geq \frac{f(a)+f(b)}{2} - \frac{1}{8}S.$$

Hence, the following result is valid.

Theorem 6. *Let the assumptions of Theorem 1 hold. If $f''(t) \geq 0$, $t \in [a,b]$ then we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{8}S \leq \frac{1}{b-a}\int_a^b f(t)dt \leq f\left(\frac{a+b}{2}\right) + \frac{1}{8}S, \tag{3.2}$$

where $S = [f'(b) - f'(a)](b-a)$.

Corollary 1. *Under the assumptions of Theorem 6 we have*

$$\frac{f(a)+f(b)}{2} - \frac{1}{8}S \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq f\left(\frac{a+b}{2}\right) + \frac{1}{8}S.$$

If $f''(t) \leq 0$, $t \in [a,b]$, i.e. f is a concave function, then we can set $\Gamma = 0$ in (2.1). Thus,

$$\frac{1}{b-a}\int_a^b f(t)dt \geq f\left(\frac{a+b}{2}\right) + \frac{1}{8}S,$$

where $S = [f'(b) - f'(a)](b-a)$. On the other hand, from (2.14) we get

$$\frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} - \frac{1}{8}S.$$

If f is a concave function then we have

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(t)dt \leq f\left(\frac{a+b}{2}\right).$$

Hence, we obtain the following result.

Theorem 7. *Let the assumptions of Theorem 1 hold. If $f''(t) \leq 0$, $t \in [a,b]$ then we have*

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8}S \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} - \frac{1}{8}S,$$

where $S = [f'(b) - f'(a)](b-a)$.

Corollary 2. *Under the assumptions of Theorem 7 we have*

$$\frac{f(a)+f(b)}{2} \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2} - \frac{1}{8}S$$

and

$$f\left(\frac{a+b}{2}\right) + \frac{1}{8}S \leq \frac{1}{b-a}\int_a^b f(t)dt \leq f\left(\frac{a+b}{2}\right).$$

4. Applications in numerical integration

Theorem 8. *Under the assumptions of Theorem 3 we have*

$$\begin{aligned} &(\frac{1}{8}S - \frac{1}{12}\Gamma)\frac{(b-a)^3}{n^2} \\ \leq\ & \int_a^b f(t)dt - \frac{b-a}{n}\sum_{i=0}^{n-1} f(a+(i+\frac{1}{2})h) \\ \leq\ & (\frac{1}{8}S - \frac{1}{12}\gamma)\frac{(b-a)^3}{n^2} \end{aligned} \tag{4.1}$$

where $h = \frac{b-a}{n}$, $n > 1$ is a positive integer.

Proof. If we apply Theorem 3 to the interval $[x_i, x_{i+1}]$, where $x_i = a + ih$, then we get

$$\frac{3S_i - 2\Gamma}{24}h^3 \leq \int_{x_i}^{x_{i+1}} f(t)dt - f(\frac{x_i + x_{i+1}}{2})h \leq \frac{3S_i - 2\gamma}{24}h^3, \tag{4.2}$$

where $S_i = \frac{f'(x_{i+1}) - f'(x_i)}{h}$, $i = 0, 1, 2, ..., n-1$. We now sum the above relation over i from 0 to $n-1$. We get

$$\begin{aligned} &\left[\frac{1}{8}\sum_{i=0}^{n-1}\frac{f'(x_{i+1}) - f'(x_i)}{h} - \frac{1}{12}n\Gamma\right]h^3 \\ \leq\ & \int_a^b f(t)dt - \frac{b-a}{n}\sum_{i=0}^{n-1} f(a+(i+\frac{1}{2})h) \\ \leq\ & \left[\frac{1}{8}\sum_{i=0}^{n-1}\frac{f'(x_{i+1}) - f'(x_i)}{h} - \frac{1}{12}n\gamma\right]h^3, \end{aligned} \tag{4.3}$$

since $\frac{x_i+x_{i+1}}{2} = a + (i+\frac{1}{2})h$. From (4.3) and the fact that $\sum\limits_{i=0}^{n-1} \frac{f'(x_{i+1})-f'(x_i)}{h} = \frac{f'(b)-f'(a)}{h}$ it follows

$$\begin{aligned} &\left[\frac{1}{8}\frac{f'(b)-f'(a)}{b-a}n - \frac{1}{12}n\Gamma\right]\frac{(b-a)^3}{n^3} \\ \leq\ & \int_a^b f(t)dt - \frac{b-a}{n}\sum_{i=0}^{n-1} f(a+(i+\frac{1}{2})h) \\ \leq\ & \left[\frac{1}{8}\frac{f'(b)-f'(a)}{b-a}n - \frac{1}{12}n\gamma\right]\frac{(b-a)^3}{n^3}. \end{aligned} \tag{4.4}$$

Inequalities (4.4) are equivalent to (4.1). This completes the proof. □

Remark 2. From the above theorem it is not difficult to get

$$\int_a^b f(t)dt = \frac{b-a}{n}\sum_{i=0}^{n-1} f(a+(i+\frac{1}{2})h) + \left[\frac{S}{8} - \frac{\Gamma+\gamma}{24}\right]\frac{(b-a)^3}{n^2} + R(f)$$

where

$$|R(f)| \leq \frac{\Gamma-\gamma}{24n^2}(b-a)^3. \tag{4.5}$$

For example, if $\gamma, \Gamma > 0$ then the classical composite mid-point quadrature rule has the estimate of error $\frac{\Gamma}{24n^2}(b-a)^3$. It is obvious that, in this case, (4.5) is better than the last estimate.

Theorem 9. *Under the assumptions of Theorem 4 we have*

$$\begin{aligned} &(\frac{1}{8}S - \frac{1}{24}\Gamma)\frac{(b-a)^3}{n^2} \\ \leq\ & \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(a+ih)\right] - \int_a^b f(t)dt \\ \leq\ & (\frac{1}{8}S - \frac{1}{24}\gamma)\frac{(b-a)^3}{n^2}, \end{aligned} \tag{4.6}$$

where $h = \frac{b-a}{n}$ and $n > 1$ is a positive integer.

Proof. The proof of this theorem is almost the same as the proof of Theorem 8. Instead of Theorem 3, we here apply Theorem 4. □

Remark 3. From the above theorem it is not difficult to get

$$\int_a^b f(t)dt = \frac{b-a}{n}\left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(a+ih)\right] - \left[\frac{S}{8} - \frac{\Gamma+\gamma}{48}\right]\frac{(b-a)^3}{n^2} + R(f)$$

where

$$|R(f)| \leq \frac{\Gamma - \gamma}{48n^2}(b-a)^3. \tag{4.7}$$

For example, if $\gamma, \Gamma > 0$ then the classical composite trapezoidal quadrature rule has the estimate of error $\frac{\Gamma}{12n^2}(b-a)^3$. It is obvious that, in this case, (4.7) is better than the last estimate. In fact, if $M_2 = \max\{|\Gamma|, |\gamma|\}$ then the classical composite trapezoidal quadrature rule has the estimate of error $\frac{M_2}{12n^2}(b-a)^3$. On the other hand, from (4.7) we get the estimate $\frac{M_2}{24n^2}(b-a)^3$. This is, in all cases, two times better estimate than the classical estimate.

References

1. Cerone P. and Dragomir S. S., *Midpoint-type Rules from an Inequalities Point of View*, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, (2000), 135–200.

2. ———, *Trapezoidal-type Rules from an Inequalities Point of View*, Handbook of Analytic-Computational Methods in Applied Mathematics, Editor: G. Anastassiou, CRC Press, New York, (2000), 65–134.

3. Dedić Lj., Pečarić J. and Ujević N., *Generalizations of Some Inequalities of Ostrowski Type*, Czechoslovak Math. J. (accepted).

4. Dragomir S. S., Cerone P. and Roumeliotis J., *A New Generalization of Ostrowski Integral Inequality for Mappings Whose Derivatives Are Bounded and Applications in Numerical Integration and for Special Means*, Appl. Math. Lett. **13** (2000), 19–25.

5. Dragomir S. S. and Agarwal R. P., *Two Inequalities for Differentiable Mappings and Applications to Special Means of Real Numbers and to Trapezoidal Formula*, Appl. Math. Lett. **11**(5) (1998), 91–95.

6. Dragomir S. S. and Agarwal R. P., *Two New Mappings Associated with Hadamard's Inequalities for Convex Functions*, Appl. Math. Lett. **11**(3) (1998), 33–38.

7. Ghizzetti A. and Ossicini A., *Quadrature Formulae, Birkhaüses* Verlag, Basel, Stuttgart, 1970.

8. Mitrinović D. S., Pečarić J. E. and Fink A. M., *Inequalities Involving Functions and Their Integrals and Derivatives*, Kluwer Acad. Publ., Dordrecht, Boston, Lancaster, Tokyo, 1991.

9. ———, *Classical and New Inequalities in Analysis*, Kluwer Acad. Publ., Dordrecht, Boston, Lancaster, Tokyo, 1993.

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