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# STRICTLY ERGODIC PATTERNS AND ENTROPY FOR INTERVAL MAPS

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Abstract. Let $\mathcal{M}$ be the set of all pairs $(T, g)$ such that $T \subset \mathbb{R}$ is compact, $g : T \to T$ is continuous, $g$ is minimal on $T$ and has a piecewise monotone extension to $\operatorname{conv} T$. Two pairs $(T, g), (S, f)$ from $\mathcal{M}$ are equivalent – $(T, g) \sim (S, f)$ – if the map $h : \operatorname{orb}(\min T, g) \to \operatorname{orb}(\min S, f)$ defined for each $m \in \mathbb{N}_0$ by $h(g^m(\min T)) = f^m(\min S)$ is increasing on $\operatorname{orb}(\min T, g)$. An equivalence class of this relation is called a minimal (oriented) pattern. Such a pattern $A \in \mathcal{M}_\sim$ is strictly ergodic if for some $(T, g) \in A$ there is exactly one $g$-invariant normalized Borel measure $\mu$ satisfying $\operatorname{supp} \mu = T$. A pattern $A$ is exhibited by a continuous interval map $f : I \to I$ if there is a set $T \subset I$ such that $(T, f|T) = (T, g) \in A$. Using the fact that for two equivalent pairs $(T, g), (S, f) \in A$ their topological entropies $\operatorname{ent}(g, T)$ and $\operatorname{ent}(f, S)$ equal we can define the lower topological entropy $\operatorname{ent}_{\mathrm{L}}(A)$ of a minimal pattern $A$ as that common value. We show that the topological entropy $\operatorname{ent}(f, I)$ of a continuous interval map $f : I \to I$ is the supremum of lower entropies of strictly ergodic patterns exhibited by $f$.

## 0. Introduction

Let $F\colon X \to X$ be a continuous map of a compact metric space $X$ into itself, $\mu$ be a Borel probability $F$-invariant measure. Denote by $\operatorname{ent}_\mu(F)$, resp. $\operatorname{ent}(F)$ the measure-theoretic (with respect to $\mu$), resp. topological entropy of $F$. The following variational principle plays an important role in the theory of dynamical systems.

**Theorem 0.1.** [**DGS**] *Let $X$ and $F$ be as above. Then*

$$\operatorname{ent}(F) = \sup\{\operatorname{ent}_\mu(F)\colon \mu \textit{ is } F\textit{-invariant and ergodic }\}.$$

It is known that in some particular cases Theorem 0.1 can be strengthened. For instance, in his paper [**G**] Ch. Grillenberger has proved that the topological entropy of symbolic dynamics on $t$ symbols can be achieved by measure-theoretic entropies of strictly ergodic subsystems (measures) – we recall this result in Proposition Ap.2. Since there exist minimal dynamical systems with positive entropy that are not strictly ergodic [**DGS**], Grillenberger's Theorem does not hold in general.

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Our goal is to show that in the case of one-dimensional dynamics given by an interval map such a generalization works. Using Proposition Ap.2, Misiurewicz's Theorem describing the connection between the entropy and horseshoes for interval maps – see Proposition Ap.4 – and our result on minimal patterns exhibited by an interval map – Proposition Ap.3 – we are able to show that the topological entropy of an interval map is the supremum of lower entropies (cf. Definition 2.3) of strictly ergodic patterns.

Our main result is the following (see Section 1 for definitions).

**Theorem 3.1.** *If $g$ is from $C(I)$ then*

$$\mathrm{ent}(g, I) = \sup\{\ \mathrm{ent}_{\mathrm{L}}(A)\colon A \in \mathcal{E}_{\sim} \text{ is exhibited by } g\ \}.$$

The paper is organized as follows:

In Section 1 we give some basic notation and definitions.

Section 2 is devoted to the lemmas used throughout the paper.

In Section 3 we prove the main result Theorem 3.1.

Finally, in Appendix we recall the known needed propositions useful when proving our result.

## 1. Notation and definitions

By $\mathbb{R}, \mathbb{N}, \mathbb{N}_0$ we denote the sets of real, natural and nonnegative integer numbers respectively. We also use the notation $N_s = \{0, 1, \dots, s-1\}$, $s \in \mathbb{N}$.

We work with dynamical systems $(X, F)$ where $X$ is a compact metric space and $F$ is mapping $X$ into itself continuously. The set of all such maps will be denoted by $C(X)$. For $F \in C(X)$ we define $F^n$ inductively by $F^0 = F$ and (for $n \geq 1$) $F^n = F \circ F^{n-1}$.

Let $(X, F)$ be a dynamical system (or briefly a system). A set $J \subset X$ is $F$-invariant if $F(J) \subset J$. As usually, the orbit $\mathrm{orb}(J, F)$ of $J$ is a set $\{F^i(J)\colon i \in \mathbb{N}_0\}$ and the $\omega$-limit set $\omega(J, F)$ of $J$ is equal to $\bigcap_{m \geq 0} \overline{\bigcup_{n > m} F^i(J)}$. We write $\mathrm{orb}(x, F)$, resp. $\omega(x, F)$ if $J = \{x\}$. A point $x \in X$ is called periodic (fixed) if $F^n(x) = x$ for some $n \in \mathbb{N}$ ( $n = 1$ ). The orbit $\mathrm{orb}(x, F)$ of a periodic $x$ is called a cycle. A set $J \subset X$ is minimal if for each $x \in J$, $\omega(x, F) = J$.

We say that a dynamical system $(T, g)$ is a pair if $T \subset \mathbb{R}$. By $\mathrm{conv}\, T$ we denote the convex hull of a set $T$. For a pair $(T, g)$ we define a map $g_T \in C(\mathrm{conv}\, T)$ by $g_T|T = g$ and $g_T|J$ affine for any interval $J \subset \mathrm{conv}\, T$ such that $J \cap T = \emptyset$. A pair $(T, g)$ is said to be piecewise monotone if there are $k \in \mathbb{N}$ and points $\min T = c_0 < c_1 < \cdots < c_k < c_{k+1} = \max T$ such that $g_T$ is monotone on each $[c_i, c_{i+1}]$, $i \in N_{k+1}$.

In the sequel we use some notions from ergodic theory [**DGS**]. Let $(X, F)$ be a system and $\mu$ be a Borel probability measure on $X$. We say that $\mu$ is $F$-invariant if $\mu(F^{-1}(K)) = \mu(K)$ for any Borel $K \subset X$ and we denote $\mathfrak{M}(F)$ the set of all $F$-invariant measures. A measure $\mu \in \mathfrak{M}(F)$ is called ergodic if for any Borel set $K \subset X$ satisfying $F(K) \subset K$ we have either $\mu(K) = 0$ or $\mu(K) = 1$. The set of all ergodic measures from $\mathfrak{M}(F)$ is denoted by $\mathfrak{M}_e(F)$. The support of a measure $\mu$, denoted by $\operatorname{supp}\mu$, is the smallest closed set $K \subset X$ such that $\mu(K) = 1$. A system $(X, F)$ will be called strictly ergodic if $\mathfrak{M}(F) = \mathfrak{M}_e(F) = \{\mu\}$ and $\operatorname{supp}\mu = X$.

We denote $\mathcal{M}$ the set of all piecewise monotone minimal pairs. All strictly ergodic pairs in $\mathcal{M}$ ( see Lemma 2.1(ii) ) are denoted by $\mathcal{E}$.

**Minimal pattern** Pairs $(T, g), (S, f) \in \mathcal{M}$ are said to be equivalent – we denote this equivalence by $\sim$ – if the map $h : \operatorname{orb}(\min T, g) \to \operatorname{orb}(\min S, f)$ defined for each $m \in \mathbb{N}_0$ by $h(g^m(\min T)) = f^m(\min S)$ is increasing on $\operatorname{orb}(\min T, g)$. An equivalence class $A \in \mathcal{M}_\sim$ of this relation is called a **minimal** (**oriented**) **pattern** or briefly a pattern. If $A$ is a pattern and $(T, g) \in A$ we say that the pair $(T, g)$ has pattern $A$ and we use the symbol $[(T, g)]$ to denote the pattern $A$. If $(T, g)$ is a cycle then $[(T, g)]$ is called a periodic pattern.

In the sequel we denote $I$ a compact subinterval of $\mathbb{R}$. A map $f \in C(I)$ has a pair $(T, g) \in \mathcal{M}$ if $f|T = g$. In this case we say that $f$ **exhibits** the pattern $A = [(T, g)]$ and we often write $(T, f) \in A$.

**Definition 1.1.** [**Bow**] Let $(X, F)$ be a dynamical system, let $\rho$ be a metric on $X$. A set $E \subset X$ is $(n, \varepsilon)$-separated (with respect to $F$) if, whenever $x, y \in E$, $x \neq y$ then $\max_{0 \le i \le n-1} \rho(F^i(x), F^i(y)) > \varepsilon$.

For a closed set $K \subset X$ we denote $s(n, \varepsilon, K)$ the largest cardinality of any $(n, \varepsilon)$-separated subset of $K$. Put

$$\operatorname{ent}(F, K) = \lim_{\varepsilon \to 0^+} \limsup_{n \to \infty} \frac{1}{n} \log s(n, \varepsilon, K).$$

The quantity $\operatorname{ent}(F, K)$ is called the topological entropy of $F$ with respect to $K$. In general $\operatorname{ent}(F, K) \le \operatorname{ent}(F, X)$. In the case when $K = X$ we briefly speak about topological entropy of $F$.

We use **symbolic dynamics** [**DGS**]. For $s \in \mathbb{N}$ consider the set $N_s$ as a space with the discrete topology, denote by $\Gamma_s$ the infinite product metric space $\prod_{i=0}^{\infty} X_i$, where $X_i = N_s$ for all $i$. The continuous shift map $\sigma \colon \Gamma_s \to \Gamma_s$ is defined by $(\sigma(\gamma))_i = \gamma_{i+1},\ i \in \mathbb{N}_0$. It is well known [**DGS**, Prop. 16.11] that for $\Gamma \subset \Gamma_s$ closed

$$\operatorname{ent}(\sigma, \Gamma) = \lim_{n \to \infty} \frac{1}{n} \log \# \Gamma(n),$$

where $\Gamma(n) = \{\gamma(n) = (\gamma_0, \dots, \gamma_{n-1}) \colon \gamma \in \Gamma\}$. In particular, $\operatorname{ent}(\sigma, \Gamma_s) = \log s$.

## 2. Lemmas

In the first lemma we recall needed properties of minimal systems. The statement(i) holds for any compact metric space $X$ and $F \in C(X)$.

**Lemma 2.1.**

(i) *If $(X, F)$ is minimal and $\mu \in \mathfrak{M}(F)$ then either $X$ is finite and then $\mu$ is atomic or $X$ is infinite and then $\mu$ is nonatomic. In any case* $\operatorname{supp}\mu = X$.

(ii) *Let $(T,g)$ be a strictly ergodic pair with $\mathfrak{M}(g)=\{\mu\}$. Then it is minimal and for any $a,b\in T$ with $a\le b$ and $x\in T$,*

$$\mu([a,b])=\lim_{n\to\infty}\frac{\#\{i\in N_n\colon g^i(t)\in[a,b]\}}{n}.$$

*Proof.* See [**DGS**]. □

**Lemma 2.2.** *Let $(T,g),(S,f)\in\mathcal{M}$ be equivalent pairs. The following is true.*
(i) *$(T,g)$ is strictly ergodic iff $(S,f)$ is strictly ergodic.*
(ii) $\operatorname{ent}(g,T)=\operatorname{ent}(f,S)$.

*Proof.* The conlusion is clear for cycles. Thus, suppose that both pairs are infinite. First we prove (i). Denote $x=\min T$, resp. $y=\min S$. Consider four sequences $\{m_k\}$, $\{n_k\}$, $\{o_k\}$, $\{p_k\}$ of positive integers such that $\{g^{m_k}(x)\}$, $\{g^{o_k}(x)\}$ are increasing, $\{g^{n_k}(x)\}$, $\{g^{p_k}(x)\}$ are decreasing,

$$a=\lim_{k\to\infty}g^{m_k}(x)\le\lim_{k\to\infty}g^{n_k}(x)=b and\ c=\lim_{k\to\infty}g^{o_k}(x)\le\lim_{k\to\infty}g^{p_k}(x)=d.$$

and $(a,b)\cap T=\emptyset$. Since $(T,g)$ and $(S,f)$ are equivalent, the map $h:\operatorname{orb}(x,g)\to\operatorname{orb}(y,f)$ defined for each $m\in\mathbb{N}_0$ by $h(g^m(x))=f^m(y)$ is increasing on $\operatorname{orb}(x,g)$ hence we have for suitable $\alpha,\beta,\gamma,\delta\in S$ the relations

$$\alpha=\lim_{k\to\infty}f^{m_k}(y)\le\lim_{k\to\infty}f^{n_k}(y)=\beta and\ \gamma=\lim_{k\to\infty}f^{o_k}(y)\le\lim_{k\to\infty}f^{p_k}(y)=\delta$$

and $(\alpha,\beta)\cap S=\emptyset$. Moreover, the equivalence of $(T,g)$ and $(S,f)$ gives that for each $t\in[a,b]$, $s\in[\alpha,\beta]$ and $n\in\mathbb{N}$

$$\#\{i\in N_n\colon g^i(t)\in[c,d]\}=\#\{i\in N_n\colon f^i(s)\in[\gamma,\delta]\}.$$

Now the conclusion follows from Lemma 2.1(ii).

It was shown in [**Bo**, Lemmas 2.3–4] that the pairs $(T,g)$ and $(S,f)$ have a common $(2,1)$-factor. Using Proposition Ap.1 we obtain the property (ii). □

Recall that by $\mathcal{E}$ we denote the subset of $\mathcal{M}$ of all strictly ergodic pairs. Using Lemma 2.2 we can define the following.

**Definition 2.3.** A pattern $A\in\mathcal{E}_\sim$ is called a **strictly ergodic pattern**. For $A\in\mathcal{M}$ the value $\operatorname{ent_L}(A)=\operatorname{ent}(g,T)$, $(T,g)\in A$ is called the **lower topological entropy of pattern** $A$.

**Remark.** In the literature [**ALM**], the topological entropy $\operatorname{ent}(A)$ of a periodic pattern $A$ is defined as the value $\operatorname{ent}(g_T)$ for any cycle $(T,g)\in A$. By this definition there are periodic patterns with positive entropy. Using the results from [**Bo**], the same approach can be applied in order to define $\operatorname{ent}(A)$ for any $A\in\mathcal{M}_\sim$. Our Definition 2.3 of lower entropy differs; for every $A\in\mathcal{M}_\sim$ we have $\operatorname{ent}(A)\ge$ $\ge\operatorname{ent}_L(A)$ and $\operatorname{ent}_L(A)=0$ whenever $A$ is periodic.

As above, we denote $I$ a compact subinterval of $\mathbb{R}$.

**Definition 2.4.** *Let $f\in C(I)$ and $s\in\mathbb{N}$. We say that $f$ has an $s$-horseshoe if there are pairwise disjoint closed subintervals $J_0,\dots,J_{s-1}$ of $I$ such that $\bigcap_{i=0}^{s-1}f(J_i)\supset\bigcup_{i=0}^{s-1}J_i$.*

For two closed sets $K, L \subset \mathbb{R}$ we write $K < L$ if $\max K < \min L$. Let $f \in C(I)$ has an $s$-horseshoe for some $s \in \mathbb{N} \setminus \{1\}$. Then there are pairwise disjoint closed subintervals $[a_i, b_i]$ of $I$ such that

$$[a_0, b_0] < \cdots < [a_{s-1}, b_{s-1}] \textit{and} \bigcap_{i=0}^{s-1} f([a_i, b_i]) \supset \bigcup_{i=0}^{s-1} [a_i, b_i].$$

Clearly for each $i \in N_s$ we can find points $x_i, y_i \in [a_i, b_i]$ such that $f(x_i) = a_0$ and $f(y_i) = b_{s-1}$. In general we do not know if $x_i < y_i$ or $x_i > y_i$. But the following easy lemma is true. We let its proof to the reader.

**Lemma 2.5.** *Let $f \in C(I)$ and $s \in \mathbb{N}\setminus\{1\}$, put $t = [\frac{s}{2}]$. If $f$ has an $s$-horseshoe then $f$ has also a $t$-horseshoe created by intervals $[x_0, y_0] < \cdots < [x_{t-1}, y_{t-1}]$ satisfying $f(x_i) = x_0$ and $f(y_i) = y_{t-1}$ for each $i \in N_t$. In particular, $x_0, y_{t-1}$ are fixed points of $f$.*

For $t \in \mathbb{N} \setminus \{1\}$ let us consider a system $(Y_t, G)$ defined as follows: let $X \subset \mathbb{R}$ be a union of $t$ pairwise disjoint closed intervals $J_0 = [x_0, y_0] < \cdots < J_{t-1} = = [x_{t-1}, y_{t-1}]$. Consider a map $F$ affine on each $J_i$ and satisfying $F(x_i) = x_0$ and $F(y_i) = y_{t-1}$ for each $i \in N_t$. If we put

$$Y_t = \bigcap_{i=0}^{\infty} F^{-i}(J_0 \cup \cdots \cup J_{t-1}) \textit{and } G = F|Y_t,$$

then $(Y_t, G)$ is a dynamical system. The following lemma can be considered to belong to folklore knowledge. For the sake of completeness we present its proof. For the definition of topological conjugacy – see Appendix.

**Lemma 2.6.** *The systems $(Y_t, G)$ and $(\Gamma_t, \sigma)$ are topologically conjugate.*

*Proof.* By its definition, $F(J_i) = X$ for each $i \in N_t$. We know that $F$ is affine on each $J_i$. Clearly, the slope of $F$ on each $J_i$ has to be greater than 1. It implies that to each $\gamma \in \Gamma_t$ there exists exactly one point $x(\gamma) \in Y_t \subset X$ satisfying $F^j(x(\gamma)) = G^j(x(\gamma)) \in J_{\gamma_j}$, $j \in \mathbb{N}_0$. Using the properties of $F$, it is easy to verify that the map $h$ defined by $h(\gamma) = x(\gamma)$ is a conjugacy of $(Y_t, G)$ and $(\Gamma_t, \sigma)$. $\square$

## 3. Main result

Our goal in this section is to use lemmas developed in the previous section and results from Appendix to prove the main theorem. As above, we denote $I$ a compact subinterval of $\mathbb{R}$.

**Theorem 3.1.** *If $g$ is from $C(I)$ then*

$$\mathrm{ent}(g, I) = \sup\{\ \mathrm{ent}_{\mathrm{L}}(A) \colon A \in \mathcal{E}_\sim \textit{ is exhibited by } g\ \}.$$

*Proof.* If $\mathrm{ent}(g, I) = 0$ then from Definition 1.1 follows $\mathrm{ent}_{\mathrm{L}}(A) = 0$ for any $A \in \mathcal{M}_\sim$ exhibited by $g$. Thus, the conclusion holds in this case.

Assume that $\mathrm{ent}(g,I)>0$ is finite. Fix an $\varepsilon$ positive. By Proposition Ap.4 for suitable $s,n\in\mathbb{N}$ the map $f=g^n$ has an $s$-horseshoe satisfying

$$\frac{1}{n}\log\left[\frac{s}{2}\right]>\mathrm{ent}(g,I)-\varepsilon. \tag{1}$$

Put $t=[\frac{s}{2}]$. By virtue of Lemma 2.5 there is a $t$-horseshoe of $f$ created by pairwise disjoint closed subintervals $J_i$ of $I$ such that

$$J_0=[x_0,y_0]<\cdots<J_{t-1}=[x_{t-1},y_{t-1}] \text{ and } \bigcap_{i=0}^{t-1}f(J_i)\supset\bigcup_{i=0}^{t-1}J_i,$$

$f(x_i)=x_0$ and $f(y_i)=y_{t-1}$ for each $i\in N_t$. It implies that if we put $S=\bigcup_{i=0}^{t-1}\{x_i,y_i\}$ then $f(S)=\{x_0,y_{t-1}\}\subset S$.

Hence, Proposition Ap.3 can be applied to $f=g^n$ and the finite $f$-invariant set $S$. Moreover, combining (1) and Proposition Ap.2 we can consider a strictly ergodic set $\Phi\subset\Gamma_t$ such that

$$\frac{1}{n}\mathrm{ent}(\sigma,\Phi)>\mathrm{ent}(g,I)-\varepsilon. \tag{2}$$

Put $X=J_0\cup\cdots\cup J_{t-1}$ and define a map $F\colon X\to X$ by $F|J_i=f_S$. Using the same procedure as before Lemma 2.6 we obtain a system $(Y_t,G)$. We know from Lemma 2.6 that the systems $(Y_t,G)$ and $(\Gamma_t,\sigma)$ are topologically conjugate. If $h\colon\Gamma_t\to Y_t$ is a corresponding conjugacy, the set $h(\Phi)=T\subset Y_t\subset X$ is strictly ergodic in $(Y_t,G)$ and $\mathrm{ent}(\sigma,\Phi)=\mathrm{ent}(G,T)$. Since by the previous $G|T=F|T=f_S|T$, the pair $(T,f_S)$ is also strictly ergodic. Using Proposition Ap.3 and Lemma 2.2 we can see that there is a set $T^\star\subset\mathrm{conv}\,S$ such that the strictly ergodic pair $(T^\star,f=g^n)$ is from $[(T,f_S)]$. Fix $x\in T^\star$ and consider the $\omega$-limit set $\omega(x,g)$. It can be easily verified that $T^\star\subset\omega(x,g)$ and the pair $(\omega(x,g),g)$ is strictly ergodic. Denote $A=[(\omega(x,g),g)]\in\mathcal{E}_\sim$. Summarizing, from Definitions 1.1 and 2.3, the equality $n\,\mathrm{ent}(g,\omega(x,g))=\mathrm{ent}(g^n,T^\star)$ [**DGS**], Lemma 2.2, the equality $\mathrm{ent}(G,T)=\mathrm{ent}(\sigma,\Phi)$ and the property (2) we obtain the relations

$$\begin{aligned}\mathrm{ent}(g,I)\ge\mathrm{ent}_L(A)=\mathrm{ent}(g,\omega(x,g)) &= \frac{1}{n}\mathrm{ent}(g^n,T^\star)\\ &= \frac{1}{n}\mathrm{ent}(G,T)>\mathrm{ent}(g,I)-\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary the conclusion of our theorem for $\mathrm{ent}(g)\in(0,\infty)$ follows.

Let $\mathrm{ent}(g)=\infty$. By virtue of Proposition Ap.4, for fixed $K\in\mathbb{R}$ we can consider $s,n\in\mathbb{N}$ such that $\frac{1}{n}\log\left[\frac{s}{2}\right]>K$. Now we can proceed analogously as above. This proves the theorem. □

## 4. Appendix

We refer the reader to Introduction for other notions used here.

A dynamical system $(S,f)$ **is a factor of** $(T,g)$ if there is a continuous surjective factor map $h\colon T\to S$ such that $h\circ g=f\circ h$. That factor is a $(n,1)$-factor for $n\in\mathbb{N}$ if for each $s\in S$ it holds $1\le\#h^{-1}(s)\le n$. In the case when $n=1$ we say that **systems** $(T, g)$ **and** $(S, f)$ **are topologically conjugate** and $h$ **is a conjugacy**.

The following theorem is well known.

**Proposition Ap.1.** [**Bow**] *If* $(S, f)$ *is a factor of* $(T, g)$ *then*

$$\mathrm{ent}(f, S) \leq \mathrm{ent}(g, T) \leq \mathrm{ent}(f, S) + \sup_{s \in S} \mathrm{ent}(g, h^{-1}(\{s\})).$$

*In particular, if it is an* $(n, 1)$*-factor for some* $n \in \mathbb{N}$ *then* $\mathrm{ent}(g, h^{-1}(\{s\})) = 0$ *for each* $s \in S$ *hence* $\mathrm{ent}(g, T) = \mathrm{ent}(f, S)$.

The following proposition presents a strong result concerning the topological entropy of strictly ergodic subsystems of $\Gamma_t$.

**Proposition Ap.2.** [**G**] *Let* $t \in \mathbb{N}$. *For any* $\delta$ *positive there is a strictly ergodic set* $\Phi$ *in* $\Gamma_t$ *such that* $\mathrm{ent}(\sigma, \Phi) > -\delta + \log t$.

In order to investigate properties of minimal patterns exhibited by interval maps we need some method that will help us to recognize that a fixed map $f \in C(I)$ exhibits a minimal pattern $A \in \mathcal{M}_{\sim}$. The following statement satisfies this requirement.

**Proposition Ap.3.** [**Bo**] *Let* $f \in C(I)$*, assume there is an* $f$*-invariant compact set* $S \subset I$. *Then for* $f_S \in C(\mathrm{conv}\, S)$ *and* $T \subset \mathrm{conv}\, S$ *such that* $(T, f_S) \in \mathcal{M}$ *there is* $T^* \subset \mathrm{conv}\, S$ *for which* $(T^*, f) \in [(T, f_S)]$.

The last result of this section has a crucial meaning when proving an analog of Proposition Ap.2 for interval maps. Concerning fixed $f \in C(I)$ with positive topological entropy one can ask the presence of horseshoe. The following is true.

**Proposition Ap.4.** [**M**] *Let* $f \in C(I)$. *If* $\mathrm{ent}(f, I) \in (0, \infty]$ *then there exist sequences* $\{n_l\}_{l=1}^{\infty}$ *and* $\{s_l\}_{l=1}^{\infty}$ *of positive integers such that the map* $f^{n_l}$ *has an* $s_l$*-horseshoe and*

$$\lim_{l \to \infty} \frac{1}{n_l} \log s_l = \mathrm{ent}(f, I).$$

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