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ON STANDARD BASIS AND MULTIPLICITY OF $(X^a - Y^b, X^c - Y^d)$

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ABSTRACT. Let $I = (X^a - Y^b, X^c - Y^d) \cdot k[X, Y]$ be an ideal of dimension zero in polynomial ring in two variables. In this note a formula for standard basis of I with respect of anti-graded lexicographic order is derived. As a consequence the discussion on the common points of the plane curves $V(X^a - Y^b)$ and $V(X^c - Y^d)$ is given.

INTRODUCTION

Let $A = k[X_1, X_2, ..., X_n]$ be a polynomial ring over a field k and let $>$ be a linear ordering on A (precisely on the set of monomials $x^\alpha =: X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}$). Any total ordering on A that is compatible with multiplication and that satisfies $1 > X_i$ for all $i = 1, \dots, n$ is called a *local order* on A. One of the local orderings on A is the *anti-graded lexicographic order,* (alex, for short) which is defined as follows:

Let $\alpha, \beta \in Z^n_{\geq 0}$. We say that $x^\alpha >_{\text{alex}} x^\beta$ if

$$|\alpha| = \sum_{i=1}^{n} \alpha_i < |\beta| = \sum_{i=1}^{n} \beta_i$$

or

$|\alpha| = |\beta|$ and in the difference $\alpha - \beta$ the left-most nonzero entry is positive.

Let now $f = \sum_\alpha c_\alpha x^\alpha$ be a polynomial in A, where $c_\alpha \in k$. The leading term of f with respect to the alex ordering on A is the product $c_\alpha x^\alpha$ where x^α is the largest monomial of f. We shall use the notation $\mathrm{Lt}(f)$ for the leading term of f. Suppose that I is an ideal of A. Consider the ideal $\mathrm{Lt}(I)$ generated by leading terms from I. The ideal $\mathrm{Lt}(I)$ will be said to be a *leading ideal* of I. One of the intriguing questions, which can be raised here, is to find a basis of $\mathrm{Lt}(I)$. In Theorem 2 we give a solution to this problem.

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Definition. *Let I be an ideal of $A = k[X_1, X_2, \ldots, X_n]$ which is contained in $M =: (X_1, X_2, \ldots, X_n)A$. Let $>$ denote a local order in A. A standard basis of I is the set $\{g_1, g_2, \ldots, g_t\}$ of polynomials of I such that*

$$\mathrm{Lt}(I) = (\mathrm{Lt}(g_1), \mathrm{Lt}(g_2), \ldots, \mathrm{Lt}(g_t))A.$$

There is an effective method how to find a standard basis for an ideal in terms of S-polynomials (Buchberger's Criterion):

Having nonzero polynomials f and g in $M =: (X_1, X_2, \ldots, X_n)A$, we can consider $\mathrm{Lt}(f) = cx^\alpha$, $\mathrm{Lt}(g) = dx^\beta$ and x^γ, the least common multiple of x^α and x^β.

An S-polynomial of f and g is the polynomial

$$S(f,g) =: \frac{x^\gamma}{\mathrm{Lt}(f)} f - \frac{x^\gamma}{\mathrm{Lt}(g)} g.$$

Now let $\Gamma = \{f_1, f_2, ..., f_t\}$ be a set of polynomials of $A = k[X_1, X_2, ..., X_n]$ (with the local order). Then every polynomial $f \in A$ can be expressed as:

$$f = a_1 f_1 + a_2 f_2 + ... + a_t f_t + r$$

with $a_i, r \in A$ and either $r = 0$ or no monomial of r is divisible by any $\mathrm{Lt}(f_1), \mathrm{Lt}(f_2), \ldots, \mathrm{Lt}(f_t)$. The polynomial r is called a *remainder* of f on division by Γ and marked by $\overline{f}^\Gamma$.

Now we can formulate the well-known Buchberger's Criterion for standard basis of any ideal:

A set $\Gamma = \{g_1, g_2, \ldots, g_t\}$ of polynomials of I is a standard basis of I if and only if for all pairs g_i, g_j of Γ it holds

$$\overline{S(g_i, g_j)}^\Gamma = 0.$$

see [**C**, Chap. 4],

In the next part of this note we use Buchberger's Criterion for producing standard basis of one class of zero dimensional ideals.

Standard basis of the ideal $(X^a - Y^b, X^c - Y^d)$.

Assume that $(X^a - Y^b, X^c - Y^d) \cdot R$ is an ideal of dimension zero in polynomial ring $R := k[X, Y]$. (Dimension zero is equivalent to the statement $ad \neq bc$.) We shall derive a standard basis of $(X^a - Y^b, X^c - Y^d) \cdot R$ in anti-graded lexicographic order of R.

Without loss of generality we can suppose $a \geq b$ and $d > c$, or $a > b$ and $c > d$.

Theorem 1. *Let $q = (X^a - Y^b, X^c - Y^d)$ be an ideal in R of dimension zero. Let $a \geq b$ and $d > c$. Then a standard basis of q is $\{Y^b - X^a, X^c - Y^d\}$.*

Proof. Set $F = Y^b - X^a$, $G = Y^d - X^c$ and $\Gamma = \{F, G\}$. For the S-polynomial of F and G it holds

$$S(F,G) = Y^{b+d} - X^{a+c} \text{ and hence } \overline{S(F,G)}^\Gamma = 0.$$

$\square$

Let us continue with the rest case $a > b$ and $c > d$. In addition, we can assume that $b > d$. If k denotes the integral part of $\frac{b}{d}$, then we get $0 \leq \frac{b}{d} - k < 1$. It is easy to see that for the decimal part $\frac{b}{d} - k$ the following statement holds:

Lemma 1. *There is only one integer $n \in N, n \geq 2$ such that*

a) $\frac{1}{n+1} < \frac{b}{d} - k \leq \frac{1}{n}$, *or*

b) $1 - \frac{1}{n} < \frac{b}{d} - k \leq 1 - \frac{1}{n+1}$.

Before formulating the main result we bring a list of important polynomials:

$$\begin{aligned}
&F = X^a - Y^b,\\
&G = X^c - Y^d,\\
&H = X^{kc}Y^{b-kd} - X^a,\\
&K = X^{(n+1)a-nkc} - X^{(k+1)c}Y^{(n+1)b-(k(n+1)+1)d},\\
&T = X^{(k+1)c} - X^{na-(n-1)kc}Y^{(nk+1)d-nb},\\
&U = X^{(n-1)(k+1)c-(n-2)a}Y^{nb-(nk+(n-1))d} - X^{2a-kc},\\
&V = X^{n(k+1)c-(n-1)a} - X^{2a-kc}Y^{((n+1)k+n)d-(n+1)b},\\
&W = X^a \cdot Y^{(k+1)d-b} - X^{(k+1)c}.
\end{aligned} \tag{$*$}$$

Theorem 2. *Let $q = (X^a - Y^b,\ X^c - Y^d)$ be an ideal in R of dimension zero, and let $a > b,\ c > d,\ b > d$. Assume that k is the integral part of $\frac{b}{d}$ and that n is the integer assigned to $\frac{b}{d} - k$ as given in Lemma 1. Then there are five possibilities for a standard basis of q. More precisely,*

(i) *If $bc < ad$, then the standard basis of q is*

$$(F, G, H, T) \qquad or \qquad (F, G, H, U, V, W)^1$$

according to integer n satisfying the relation a) or b) in Lemma 1, respectively.

(ii) *If $ad < bc$Then the standard basis of q is*

$$(F, G, H),$$

whenever $a - kc \leq b - kd$. In addition, if $a - kc > b - kd$, then the standart basis if q is

$$(F, G, H, K, T) \qquad or \qquad (F, G, H, U, W)$$

with respect to integer n given by respective cases a) or b) in Lemma 1.

Proof. Part (i). Suppose that integer n satisfies the following relation

$$\frac{1}{n+1} < \frac{b}{d} - k \leq \frac{1}{n}.$$

We shall apply Buchberger's Criterion. Let us calculate S-polynomials and their remainders of all pairs of $\{F, G, H, T\} = \Gamma$.

[1] Se the list $(*)$ of polynomials

For F and G we have $S(F,G) = X^cY^{b-d} - X^a$. Since $d < b$ for all $i = 1, \dots, k-1$ and $b - kd < a - kc$, we get

$$S(F,G) = G(X^cY^{b-2d} + X^{2c}Y^{b-3d} + \dots + X^{(k-1)c}Y^{b-kd}) + H,$$

so $\overline{S\,(F,G)}^{\Gamma} = 0$.

Let us take F and H.

$$\begin{aligned} S(F,H) &= X^aY^{kd} - X^aX^{kc} \\ &= G(Y^{(k-1)d} + X^cY^{(k-2)d} + \dots \\ &\quad +X^{(k-1)c})X^a\,, \end{aligned}$$

and therefore $\overline{S\,(F,H)}^{\Gamma} = 0$. Consider$G$ and H.

$$\begin{aligned} S(G,H) &= X^aY^{(k+1)d-b} - X^{(k+1)c} \\ &= H(X^{a-kc}Y^{(2k+1)d-2b} + X^{2a-2kc}Y^{(3k+1)d-3b} + \dots \\ &\quad +X^{(n-1)a-(n-2)kc}Y^{(nk+1)d-nb}) + T\,, \end{aligned}$$

so $\overline{S\,(G,H)}^{\Gamma} = 0$. Now, take H and T.

$$\begin{aligned} S(H,T) &= X^{na-(n-1)kc}(Y^{((n-1)k+1)d-(n-1)b} - X^{((n-1)k+1)c-(n-1)a}) \\ &= H(X^{na-nkc}Y^{(nk+1)d-nb}) + T(X^{a-kc}) \end{aligned}$$

and $\overline{S\,(T,H)}^{\Gamma} = 0$. Because the leading terms of pairs F, T and G, T are relatively prime, it holds

$$\overline{S\,(F,T)}^{\Gamma} = \overline{S\,(G,T)}^{\Gamma} = 0.$$

and Γ is a standard basis.

Consider now $1 - \frac{1}{n} < \frac{b}{d} - k \le 1 - \frac{1}{n+1}$, and $\Gamma =: \{F, G, H, U, V, W\}$.

$\overline{S\,(F,G)}^{\Gamma} = \overline{S\,(F,H)}^{\Gamma} = \overline{S\,(G,H)}^{\Gamma} = 0$ by the same argument as above.

$\overline{S\,(F,V)}^{\Gamma} = \overline{S\,(G,V)}^{\Gamma} = 0$ because the leading terms are relatively prime. For the rest pairs we have

$$\begin{aligned} S(F,W) &= X^{(k+1)c}Y^{2b-(k+1)d} - X^{2a} \\ &= H(X^cY^{b-d} + X^{a-(k-1)c}Y^{(k-1)d} + X^{2a-(2k-1)c}Y^{(2k-1)d-b} + \dots \\ &\quad +X^{ma-(mk-1)c}Y^{(mk-1)d-(m-1)b})+ \\ &\quad +W(X^{ma-(mk-1)c}Y^{((m-1)k-2)d-(m-2)b}+ \\ &\quad +X^{(ma-((m-1)k-2)c)-a}Y^{((m-2)k-3)d-(m-3)b} + \dots) \end{aligned}$$

$m = k$, so $\overline{S\,(F,W)}^{\Gamma} = 0$.

$$\begin{aligned} S(F,U) &= X^{2a-kc}Y^{(n(k+1)-1)d-(n-1)b} - X^{(k+1)(n-1)c-(n-3)a} \\ &= G(X^{2a-kc}Y^{(n(k+1)-2)d-(n-1)b} + \dots \\ &\quad +X^{2a-(k-(m-2))c}Y^{(n(k+1)-m)d-(n-1)b})+ \\ &\quad +W(X^{a-(k-(m-1))c}Y^{((n-1)(k+1)-m)d-(n-2)b}+ \\ &\quad +X^{mc}Y^{((n-2)(k+1)-m)d-(n-3)b} + \dots) \end{aligned}$$

$m = k+1$, so $\overline{S\,(F,U)}^{\Gamma} = 0$.

$$S(G,W) = X^{(k+1)c}Y^{b-kd} - X^{a+c} = HX^c$$

so $\overline{S(G,W)}^{\Gamma} = 0.$

$$\begin{aligned} S(G,U) &= X^{2a-kc}Y^{n((k+1)d-b)} - X^{(nk+n-k)c-(n-2)a} \\ &= W(X^{a-kc}Y^{(n-1)(k+1)d-(n-1)b} + X^cY^{(n-2)(k+1)d-(n-2)b}+ \\ &\quad +X^{(k+2)c-a}Y^{(n-3)(k+1)d-(n-3)b} + \dots) \end{aligned}$$

so $\overline{S(G,U)}^{\Gamma} = 0.$

$$\begin{aligned} S(H,W) &= X^{(k+1)c}Y^{2b-2kd-d} - X^{2a-kc} \\ &= W(X^{(k+1)c-a}Y^{3b-3kd-2d}) + U \end{aligned}$$

so $\overline{S(H,W)}^{\Gamma} = 0.$

$$\begin{aligned} S(H,U) &= X^{2a-kc}Y^{(n-1)((k+1)d-b)} - X^{(nk-2k+n-1)c-(n-3)a} \\ &= W(X^{a-kc}Y^{(n-2)((k+1)d-b)} + X^cY^{(n-3)((k+1)d-b)}+ \\ &\quad +X^{(k+2)c-a}Y^{(n-4)((k+1)d-b)} + \dots) \end{aligned}$$

so $\overline{S(H,U)}^{\Gamma} = 0.$

$$\begin{aligned} S(H,V) &= X^{2a-kc}Y^{n((k+1)d-b)} - X^{(nk+n-k)c-(2-n)a} \\ &= W(X^{a-kc}Y^{(n-1)((k+1)d-b)} + X^cY^{(n-2)((k+1)d-b)}+ \\ &\quad +X^{(k+2)c-a}Y^{(n-3)((k+1)d-b)} + \dots) \end{aligned}$$

so $\overline{S(H,V)}^{\Gamma} = 0.$

$$S(W,U) = X^{n(k+1)c-(n-1)a} - Y^{((n+1)k+n)d-(n+1)b}X^{2a-kc} = V$$

so $\overline{S(W,U)}^{\Gamma} = 0.$

$$S(W,V) = W(X^{a-kc}Y^{((n+1)k+n)d-(n+1)b}) + YX^{(k+1)c-a}$$

so $\overline{S(W,V)}^{\Gamma} = 0.$

$$S(U,V) = X^{2a-kc}Y^{(k+1)d-b} - X^{a+c} = W(X^{a-kc})$$

so $\overline{S(U,V)}^{\Gamma} = 0.$

The remainders of all S-polynomials are equal zero and this completes proof of the part (i).

The proof of the part (ii) is similar. □

One of the applications of standard basis relates to the multiplicity theory. Let Q be any primary ideal in R_M belonging to a maximal ideal in R_M (R_M denotes the localization of polynomial ring R by the ideal $M =: (X,Y)\cdot R$). Let $\mathrm{e}_0(Q;R_M)$ denote the leading term of the Hilbert-Samuel polynomial $\dim_k(R_M/Q^{n+1}\cdot R_M)$. For the Samuel multiplicity of Q in R_M (see [**Z-S**]). Since ring R_M is a two-dimensional Cohen-Macaulay local ring, for the ideal Q generated by two elements (parameter ideal) holds

$$\mathrm{e}_0(Q;R_M) = \dim_k(R_M/Q\cdot R_M).$$

We can apply these notions to our ideal I. $R_M = (X^a - Y^b, X^c - Y^d) \cdot R_M$. For the calculation of Samuel multiplicity of $(X^a - Y^b, X^c - Y^d) \cdot R_M$ is important this proposition:

Lemma 2. *Let $J \in R := k[X,Y]$ be an ideal contained in maximal ideal $M = (X,Y) \cdot R$. Let $\mathrm{Lt}(J)$ be the leading ideal of J with respect to the local order in R. Then*

$$\dim_k(R_M/J \cdot R_M) = \dim_k(R_M \mathrm{Lt}(J) \cdot R_M).$$

Proof. See [**C**, Chap. 4, \$ 3, Cor. 4. 5]. □

As an application we give another proof of the main theorem of [**B-O**].

Theorem 3. $\mathbf{e}_0((X^a - Y^b, X^c - Y^d) \cdot R_M; R_M) = \min\{a \cdot d, b \cdot c\}$.

Proof. By Lemma 4 and the formula above the Samuel multiplicity of $(X^a - Y^b, X^c - Y^d) \cdot R_M$ is equal to the length of the (M –primary) leading ideal $q = \mathrm{Lt}((X^a - Y^b, X^c - Y^d) \cdot R_M)$. In the case of Theorem 1 is $q = (X^c, Y^b)$ and therefore $l(q) = bc < ad$.

In the case of Theorem 3.

(i) if $bc < ad$, then

$$q = (Y^d, X^{kc}Y^{b-kd}, X^{(k+1)c})$$

for n according a) of Lemma 1 or

$$q = (Y^d, X^{kc}Y^{b-kd}, X^aY^{(k+1)d-b}, X^{(n-1)(k+1)c-(n-2)a}Y^{nb-(nk+(n-1))d}, X^{n(k+1)c-(n-1)a})$$

for n according b) of Lemma 1.

(ii) if $ad < bc$, then

$$q = (Y^d, X^a)$$

if $a - kc \leq b - kd$, or

$$q = (Y^d, X^{kc}Y^{b-kd}, X^{na-(n-1)kc}Y^{(nk+1)d-nb}, X^{(n+1)a-nkc})$$

if $a - kc > b - kd$ for n according a) of Lemma 1 or

$$q = (Y^d, X^{kc}Y^{b-kd}, X^aY^{(k+1)d-b}, X^{2a-kc})$$

if $a - kc > b - kd$ for n according a) of Lemma 1.

For the calculation of the length of q we have

$$\begin{aligned} l(q) &= l((X^{(k+1)c}, Y^d, X^{kc}Y^{b-kd})) \\ &= (k+1)c(b-kd) + kdc - kc(b-kd) \\ &= b \cdot c \end{aligned}$$

or

$$\begin{aligned} l(q) &= l((X^{n(k+1)c-(n-1)a}, Y^d, X^{(n-1)(k+1)c-(n-2)a}Y^{nb-(nk+(n-1))d}, \\ &\quad X^aY^{(k+1)d-b}X^{kc}Y^{b-kd})) \\ &= (n(k+1)c-(n-1)a)(nb-(nk+(n-1))d)+ \\ &\quad +((n-1)(k+1)c-(n-2)a) \\ &\quad ((k+1)d-b)+a(b-kd)+dkc-((n-1)(k+1)c-(n-2)a) \\ &\quad (nb-(nk+(n-1))d)-a((k+1)d-b-kc(b-kd) \\ &= bc \end{aligned}$$

in (i) and

$$l(q) \quad = \quad l((Y^d, X^a)) = ad$$

or

$$\begin{aligned} l(q) &= l((X^{(n+1)a-nkc}, Y^d, X^{kc}Y^{b-kd}, X^{na-(n-1)kc}Y^{(nk+1)d-nb})) \\ &= l((X^{(n+1)a-nkc}, Y^d, X^{kc}Y^{b-kd})) - l((X^{(n+1)a-nkc}, \\ &\quad Y^d, X^{kc}Y^{b-kd}) : (X^{na-(n-1)kc}Y^{(nk+1)d-nb})) \\ &= l((X^{(n+1)a-nkc}, Y^d, X^{kc}Y^{b-kd}))- \\ &\quad -l((X^{a-kc}, Y^{(n+1)b-d(k(n+1)+1)})) \\ &= (((n+1)a-nkc)(b-kd)+dkc-(a-kc)((n+1)b-d(k(n+1)+1))) \\ &= ad \end{aligned}$$

or

$$\begin{aligned} l(q) &= l((X^{2a-kc}, Y^d, X^{kc}Y^{b-kd}, X^aY^{(k+1)d-b})) \\ &= l((X^{2a-kc}, Y^d, X^{kc}Y^{b-kd}))- \\ &\quad -l((X^{2a-kc}, Y^d, X^{kc}Y^{b-kd}) : (X^aY^{(k+1)d-b})) \\ &= l((X^{2a-kc}, Y^d, X^{kc}Y^{b-kd})) - l((X^{a-kc}, Y^{2b-(2k+1)d}) \\ &= ((2a-kc)(b-kd)+dkc-(kc(b-kd)))-((a-kc)(2b-(2k+1)d)) \\ &= ad \end{aligned}$$

in (ii) (see [**B-S**] and [**L**]). □

COMMON POINTS OF $V(X^a - Y^b)$ AND $V(X^c - Y^d)$.

Let now $R := k[X, Y]$ be a polynomial ring over an algebraic closed field k. In addition, let V and W be plane algebraic curves of E^2 defined over k by equations $X^a - Y^b = 0$ and $X^c - Y^d = 0$, respectively. Suppose that the intersection $V \cap W$ consists only of isolated points (equivalently the ideal $I = (X^a - Y^b, X^c - Y^d) \cdot R$ is of dimension zero).

How many times meet the curves V and W at origin O? Let us denote this multiplicity number as $i(O; V, W)$. By well known Bezout's Theorem $i(O; V, W) =$ $= \mathrm{e}_0(I.R_M; R_M)$ (see [**V**] for details).

So, as consequence of the main theorem

$$i(O; V, W) = \min\{a \cdot d, b \cdot c\}.$$

which is the multiplicity of $V \cap W$ at origin O.

Another question concerns the number of all common points of $V \cap W$ in E^2. By the construction of Gröbner Basis of I (with respect to any monomial order in R), the leading ideal of I and the length of this ideal we can prove this conjecture:

Conjecture 1. *The number of all common points of the curves V and W in E^2 is equal to* $\max\{a \cdot d, b \cdot c\}$.

Last question relates to the common asymptotic directions (points in infinity) of studying curves. If $\overline{V}$ and $\overline{W}$ are projective closures of V and W in extended Euclidean plane $\overline{E^2}$, than we can formulate the classical Bezout's Theorem: $\max\{a, b\} \cdot \max\{c, d\}$ = number of common points of $\overline{V}$ and $\overline{W}$ in $\overline{E^2}$. So we are able formulate the last conjecture:

Conjecture 2. *The number of all common asymptotic directions of the curves V and W in E^2 is equal to* $\max\{a, b\} \cdot \max\{c, d\} - \max\{a \cdot d, b \cdot c\}$.

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