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CONNECTIVITY OF PATH GRAPHS

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ABSTRACT. The aim of this paper is to lower bound the connectivity of k-path graphs. From the bounds obtained, we give conditions to guarantee maximum connectivity. Then, it is shown that those maximally connected graphs satisfying the previous conditions are also super-λ. While doing so, we derive some properties about the girth and the diameter of path graphs. Finally, the results are extended to path graphs resulting from the iteration of the k-path graph operator.

1. Introduction

The k-path graph corresponding to a graph G has for vertices the set of all paths of length k in G. Two vertices are connected by an edge whenever the intersection of the corresponding paths forms a path of length $k-1$ in G, and their union forms either a cycle or a path of length $k+1$ in G. Intuitively, this means that the vertices are adjacent if and only if one can be obtained from the other by 'shifting' the corresponding paths in G. Following the notation used by Knor and Niepel, the k-path graph of G will be denoted as $P_k(G)$. Path graphs were introduced by Broersma and Hoede in [**3**] as a natural generalization of line graphs. Indeed, for every graph G, the graph $P_1(G)$ coincides with the line graph of G. A characterization of P_2-path graphs is given in [**3**] and [**9**], some important structural properties of path graphs are presented in [**1**], [**11**], [**12**], and [**13**], while distance properties of path graphs are studied in [**2**] and [**7**]. The edge connectivity and super edge-connectivity of line graphs was studied by Jixiang Meng [**14**]. The connectivity of path graphs was studied by Xueliang Li [**10**] and later by Knor, Niepel and Mallah [**6, 8**]. Note that the path graph can be thought of as an operator on graphs, and therefore, we can study graphs arising from the iteration of the k-path graph operator. Indeed, the s-iterated k-path graph of G is the graph $P_k^s(G)$ defined as $P_k(G)$ if $s=1$, and $P_k(P_k^{s-1}(G))$ if $s>1$. Given a path $u_0, u_1, \ldots, u_k$ in G, the corresponding vertex in $P_k(G)$ will be denoted by $U = u_0u_1 \ldots u_k$.

We recall next the definition of some concepts related to the connectivity of a graph and refer the reader to [**4, 5**] for additional information. A graph G is called *connected* if every pair of vertices is joined by a path. An *edge cut* in a graph G is

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a set A of edges such that $G - A$ is not connected. The *edge-connectivity* $\lambda(G)$ of a graph G is the cardinality of a minimal edge cut of G. Since $\lambda(G) \leq \delta(G)$, a graph G is said to be *maximally edge-connected* when $\lambda(G) = \delta(G)$. A minimal edge cut $(C, \overline{C})$ is called *trivial* if $C = \{v\}$ or $\overline{C} = \{v\}$ for some vertex v with $\deg(v) = \delta(G)$. A maximally edge-connected graph is called *super*-λ if every edge cut $(C, \overline{C})$ of cardinality $\delta(G)$ is trivial. The *superconnectivity* of a graph is denoted by $\lambda_1(G)$ and it is defined as $\lambda_1(G) = \min\{|(C, \overline{C})|, (C, \overline{C})$ is a non trivial edge cut$\}$. Then, a graph G is super-λ if and only if $\lambda_1(G) > \delta(G)$.

Following the notation of [**6**], for a graph G and two integers k and t, $k \geq 2$ and $0 \leq t \leq k-2$, by $P^*_{k,t}$ we denote an induced tree in G with diameter $k+t$ and a diametric path $(x_t, x_{t-1}, \ldots, x_1, v_0, v_1, \ldots v_{k-t}, y_1, y_2, \ldots, y_t)$ such that all the endvertices of $P^*_{k,t}$ are at distance no greater than t from v_0 or v_{k-t}, the degrees of $v_1, v_2 \ldots v_{k-t-1}$ are 2 in $P^*_{k,t}$ and no vertex in $V(P^*_{k,t}) - \{v_1, v_2 \ldots v_{k-t-1}\}$ is adjacent with a vertex in $V(G) - V(P^*_{k,t})$. The path $v_1, v_2 \ldots v_{k-t-1}$ is the *base* of $P^*_{k,t}$, and for a path A of length k we say that $A \in P^*_{k,t}$ if and only if the base of $P^*_{k,t}$ is a subpath of A.

Theorem A. [**6**] Let G be a connected graph with girth at least $k+1$. Then, $P_k(G)$ is disconnected if and only if G contains a $P^*_{k,t}$, $0 \leq t \leq k-2$, and a path A of length k, such that $A \notin P^*_{k,t}$.

After a section presenting sufficient conditions for an iterated k-path graph to be connected, Section 3 is devoted to the study of the edge connectivity and superconnectivity of connected path graphs and the results are extended to iterated path graphs, when possible.

The main results in Section 2 are:

Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then,

a. $P_k G$ is connected. *(Theorem 2.4)*
b. If G has diameter D, then the diameter of $P_k(G)$ is at most $D + 2k$. *(Theorem 2.6)*

Regarding measures of the connectivity, the main results in Section 3 are:

a. Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, $\lambda(P_k(G)) \geq 2(\delta - (k-1))$. *(Theorem 3.4)*
b. Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 2(k-1)$. Then,
   1. $P_k(G)$ is maximally connected. *(Corollary 3.5)*
   2. $P_k(G)$ is super-λ. *(Theorem 3.7)*

2. Connected k-path graphs

Knor and Niepel [**6**] provided in Theorem A a characterization of connected k-path graphs $P_k(G)$ of graphs with large girth. In this section, we present a lower bound on the minimum degree of a connected graph which suffices to assure that its k-path graph is connected. Thus, this result complements the previous one, and as it will be shown, gives a condition that is preserved under the path graph iteration.

Lemma 2.1. *Let k be a positive integer and let G be a graph with minimum degree $\delta > 2(k-1)$. If U and V are two vertices in $P_k(G)$ determined by paths in G which share an endvertex, then there is a path of length $2k$ joining U and V.*

Proof. Let the vertices U and V be determined by the paths $U = u_0u_1 \dots u_k$ and $V = v_0v_1 \dots v_k$ in G. Since the paths $u_0, u_1, \dots, u_k$ and $v_0, v_1, \dots, v_k$ share an endvertex, whitout loss of generality we can assume $u_0 = v_0$. Since $\delta > 2(k-1)$, there exists a vertex $x_k \in N(u_0) \setminus \{u_1, \dots, u_{k-1}, v_1, \dots, v_{k-1}\}$, and as a consequence, there exist vertices $U_1 = x_ku_0u_1 \dots u_{k-1}$ and $V_1 = x_kv_0v_1 \dots v_{k-1}$ in $P_k(G)$, $U_1 \in N(U)$ and $V_1 \in N(V)$. Notice that the paths $u_0, u_1, \dots, u_k$ and $v_0, v_1, \dots, v_k$ could eventually share other vertices in addition to an endvertex. Indeed, we can repeat the previous reasonament with U_1 and V_1 and obtain a vertex $x_{k-1} \in N(x_k) \setminus \{u_0, \dots, u_{k-2}, v_0, \dots, v_{k-2}\}$ and vertices $U_2 = x_{k-1}x_ku_0u_1 \dots u_{k-2}$ and $V_2 = x_{k-1}x_kv_0v_1 \dots v_{k-2}$ in $P_k(G)$, $U_2 \in N(U_1)$ and $V_2 \in N(V_1)$. Repeating this procedure k times we will obtain two paths in $P_k(G)$, $U_k \dots U_1U$ and $VV_1 \dots V_k$, where $U_k = x_1 \dots x_ku_0$ and $V_k = x_1 \dots x_kv_0$. Then, $U_k = V_k$ because $u_0 = v_0$, and we have a path in $P_k(G)$, the path $U, U_1, \dots, U_{k-1}, U_k, V_{k-1}, \dots, V_1$, V, joining U and V. Clearly, the length of that path is $2k$. □

The above lemma can be extended to two vertices in $P_k(G)$ whose corresponding paths in G share a vertex, which is not neccessarily and endvertex. We are going to prove it, but to simplify the writing we first introduce some notation.

Let $P = a_0, a_1, \dots, a_r$ be a walk in G. If $r \geq k$ and no two vertices at distance smaller than or equal to k in P coincide, there exist vertices $a_0a_1 \dots a_k$ and $a_{r-k}a_{r-k+1} \dots a_r$ in $P_k(G)$. Moreover, the path P induces a path in $P_k(G)$ between them, which is going to be denoted as $I_k(P)$ or equivalently, $I_k(a_0, a_1, \dots, a_r)$. Note that $I_k(P)$ has length $r - k$.

Lemma 2.2. *Let k be a positive integer and let G be a graph with minimum degree $\delta > 2(k-1)$. If U and V are two vertices in $P_k(G)$ determined by paths in G which share a vertex, then there is a path of length at most $2k$ joining U and V.*

Proof. Let the vertices U and V be determined by the paths $U = u_0u_1 \dots u_k$ and $V = v_0v_1 \dots v_k$ in G. If the paths $u_0, u_1, \dots, u_k$ and $v_0, v_1, \dots, v_k$ share an endvertex it suffices to apply Lemma 2.1. If not, there exist vertices u_s and v_t such that $u_s = v_t$ and $\{u_0, \dots, u_{s-1}\} \cap \{v_0, \dots, v_{t-1}\} = \emptyset$. Without loss of generality we can assume $s \geq t$. Then, proceeding as in the proof of Lemma 2.1, since $\delta > 2(k-1)$ it is possible to construct a path $x_k, \dots, x_{s+1}, u_0$ which gives rise to the paths $I_k(x_k, \dots, x_{s+1}, u_0, \dots, u_k)$ and $I_k(x_k, \dots, x_{s+1}, u_0, \dots, u_s, v_{t+1}, \dots, v_k)$ in $P_k(G)$. The union of these two paths determines a path in $P_k(G)$ joining U and the vertex $u_{s-t} \dots u_{s-1}v_t \dots v_k$. At the same time, the vertex $u_{s-t} \dots u_{s-1}v_t \dots v_k$ is connected to V. Indeed, if we procceed as before, since $\delta > 2(k-1)$ we can find a path $v_k, y_0, \dots, y_{t-1}$ in G from which arise the paths $I_k(u_{s-t} \dots u_{s-1}v_t \dots v_k, y_0, \dots, y_{t-1})$ and $I_k(v_0 \dots v_k, y_0, \dots, y_{t-1})$ in $P_k(G)$.

Thus, the union of those paths connects the vertex $u_{s-t} \dots u_{s-1} v_t \dots v_k$ with V. As a consequence, there is path joining U and V obtained from the union of the previous paths. Furthermore, the lengths of the four original paths used to connect U and V are respectively $k-s$, $k-t$, t and t, so their union has length $2k-s+t$ and since $s \geq t$, it is at most $2k$. $\square$

Lemma 2.3. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. If U and V are two vertices in $P_k(G)$ whose corresponding paths in G do not share any vertex, then there is a of length at most $2k + D(G)$ path joining U and V.*

Proof. Let the vertices U and V be determined by the paths $U = u_0 u_1 \dots u_k$ and $V = v_0 v_1 \dots v_k$ in G. Let us assume that the shortest path between $\{u_0, \dots, u_k\}$ and $\{v_0, \dots, v_k\}$ is the shortest path between the vertices u_s and v_t, denoted by $u_s = z_0, z_1, \dots, z_d = v_t$. Note that because of this choice, $\{u_0, \dots, u_k\} \cap \{z_1, \dots, z_{d-1}\} = \emptyset$ and $\{v_0, \dots, v_k\} \cap \{z_1, \dots, z_{d-1}\} = \emptyset$. Since $\delta > 2(k-1)$ there exist paths $x_k, \dots, x_{s+1}, u_0$ and $v_k, y_0, \dots, y_{t-1}$, in such a way that there are also paths $I_k(x_k, \dots, x_{s+1}, u_0, \dots, u_k)$, $I_k(x_k, \dots, x_{s+1}, u_0, \dots, u_s, z_1, \dots, z_{d-1}, v_t, \dots, v_k, y_0, \dots, y_{t-1})$ and $I_k(v_0, \dots, v_k, y_0, \dots, y_{t-1})$ in $P_k(G)$. The union of those three paths forms a path joining U and V. Besides, the lengths of those three paths are respectively $k-s$, $d+k-t$ and t. Therefore, the total length will be $2k+d-s$. Since $s \geq 0$ and $d \leq D(G)$, we conclude that the length of the path between U and V is at most $2k + D(G)$. $\square$

As a direct consequence of the previous lemmas we can obtain the following theorem.

Theorem 2.4. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then $P_k(G)$ is connected.*

Corollary 2.5. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, for every $s \geq 1$, $P_k^s G$ is connected.*

Proof. It is enough to see that the minimum degree of $P_k(G)$ is lower bounded by $2(\delta - (k-1))$ which is greater than $2(k-1)$ because $\delta > 2(k-1)$. The proof can be then completed by induction on s using Theorem 2.4. $\square$

Also considering the statements about the length of the paths obtained in the previous lemmas we can establish the following results regarding the diameter.

Theorem 2.6. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then $D(P_k(G)) \leq D(G) + 2k$.*

As above, the following corollary can be proved by induction.

Corollary 2.7. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, for every $s \geq 1$, $D(P_k^s(G)) \leq D(G) + 2sk$.*

3. Connectivity and Superconnectivity

This section is devoted to measure the connectivity of connected path graphs.

Lemma 3.1. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > k$. Then, there exist $\delta - k$ edge disjoint paths of length 3 between any two adjacent vertices in $P_k(G)$.*

Proof. Let $U = u_0u_1 \dots u_k$ and $V = u_1 \dots u_ku_{k+1}$ be two adjacent vertices in $P_k(G)$. Since $\delta > k$, there exist $b_i \in N(u_k) - \{u_1 \dots u_{k-1}, u_{k+1}\}$, $i = 1, \dots, \delta - k$ and $c_j \in N(u_1) - \{u_0, u_2 \dots u_k\}$, $j = 1, \dots, \delta - k$, and as a consequence, there exist vertices $u_1 \dots u_kb_i$ and $c_ju_1 \dots u_k$ in $P_k(G)$. Let γ be an isomorphism in the integer set $\{1, \dots, \delta - k\}$, then we can assign each b_i with one particular $c_{\gamma(i)}$. Therefore, there is a path $\mathcal{P}_i : u, u_1 \dots u_kb_i, c_{\gamma(i)}u_1 \dots u_k, v$ in $P_k(G)$, which obviously has length 3 and joins U and V. Let us see that these paths are edge disjoint. In fact, the only case in which two different paths $\mathcal{P}_i$ and $\mathcal{P}_j$ could share a vertex is if $u_1 \dots u_kb_i = c_{\gamma(i)}u_1 \dots u_k$, which implies in particular $u_2 = u_1$, that is not possible because $u_0, u_1, \dots u_k$ is a path in G. Therefore, we have obtained the $\delta - k$ paths between U and V claimed. □

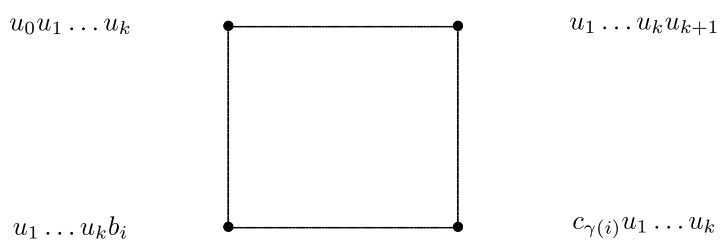

Figure 1. Paths given in Lemma 3.1.

Since cycles of length at least $k + 1$ are fixed points under the k-path graph operator, the previous lemma gives rise to the following corollary, which illustrates an interesting property of k-path graphs.

Corollary 3.2. *Let k be an integer and let G be a connected graph with minimum degree $\delta > k$. Then, the girth of $P_k(G)$ is 3 if $k = 1$ or 2 and G has triangles, and 4 otherwise.*

The above corollary shows that Theorem A does not work for iterated path graphs if $k \geq 4$.

Lemma 3.3. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k - 1)$. Then, there exist $\delta - (k - 1)$ edge disjoint paths of length $3k - 1$ between any two adjacent vertices in $P_k(G)$.*

Proof. Let $U = u_0u_1 \dots u_k$ and $V = u_1 \dots u_ku_{k+1}$ be two adjacent vertices in $P_k(G)$. Since $\delta \geq k$, for each $i = 1, \dots, \delta - (k - 1)$ there exists a choice of vertices

$a_2^i, \ldots, a_k^i$ and $b_2^i, \ldots, b_{k+1}^i$ which determine the walks in G

$$P_i : a_2^i, \ldots, a_k^i, u_0, u_1, b_2^i, \ldots, b_{k+1}^i \text{ and } Q_i : b_{k+1}^i, \ldots, b_2^i, u_1, \ldots, u_{k+1}.$$

Moreover, since $\delta \geq 2(k-1)$, for any integers i, j, $1 \leq i, j \leq \delta - (k-1))$ we can choose the vertices so that $a_k^i \neq a_k^j$ and $b_2^i \neq b_2^j$ if $i \neq j$. Let γ be an isomorphism in the integer set $\{1, \ldots, \delta - (k-1)\}$, then we can associate each path P_i with a path $Q_{\gamma(i)}$. Now, the union of the paths $I_k(P_i)$ and $I_k(Q_{\gamma(i)})$ determines a path between U and V in $P_k(G)$. Observe that the resulting paths are disjoint, because the choice of P_i and $Q_{\gamma(i)}$ guarantees that they do not share any subsequence of length k and the vertices that $I_k(P_i)$ and $I_k(Q_{\gamma(i)})$ have in common correspond to common subpaths of length k. □

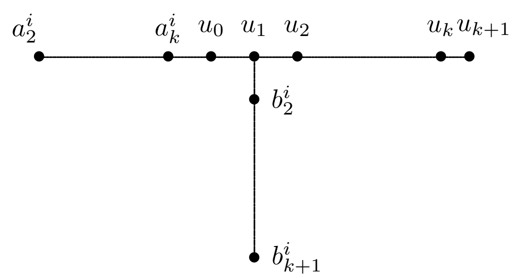

Figure 2. Paths given in Lemma 3.3.

Theorem 3.4. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, $\lambda(P_k(G)) \geq 2(\delta - (k-1))$.*

Proof. As a consequence of Lemma 3.1 and Lemma 3.3, there are $\delta - k$ internally disjoint paths of length 3 and $\delta - (k-1)$ internally disjoint paths of length $3k-1$ joining any two adjacent vertices U and V in $P_k(G)$. Let us show that a path of length 3 cannot share an internal vertex with a path of length $3k-1$. Let $U = u_0u_1 \ldots u_k$ and $V = u_1 \ldots u_k u_{k+1}$, a path P_s of length 3 can be written as

$$P_s : U, u_1 \ldots u_k b_s, c_{\gamma(s)} u_1 \ldots u_k, V, \text{ for every } s = 1, \ldots \delta - k$$

where b_i and $c_{\gamma(i)}$ are chosen as in Theorem 3.1. Analogously, a path Q_t of length $3k-1$ can be expressed as

$$Q_t : U, a_k^t u_0 \ldots u_{k-1}, \ldots, a_2^t \ldots a_k^t u_0 u_1, a_3^t \ldots a_k^t u_0 u_1 b_2^t, \ldots,$$

$$u_1 b_2^t \ldots b_{k+1}^t, \ldots, u_k \ldots u_1 b_2^t, V, \text{ for every } t = 1, \ldots \delta - (k-1)$$

where $a_2^t, \ldots, a_k^t$ and $b_2^t, \ldots, b_{k+1}^t$ are chosen as in Theorem 3.3.

Therefore, the general expression for an internal vertex in Q_t is either

$$a_j^t \ldots a_k^t u_0 \ldots u_{j-1}^t \text{ for every } j = 2, \ldots k$$

or

$$a_j^t \ldots a_k^t u_0 u_1 b_2^t \ldots b_{j-1}^t \text{ for every } j = 3, \ldots k$$

or

$$u_j \dots u_1 b_2^t \dots b_{k+2-j}^t \text{ for every } j = 1, \dots k\,.$$

Let us see that there are no possible values of s, t and j for which a vertex in the above form could coincide with any of the two internal vertices of P_s. Reasoning by contradiction, let us suppose that such s, t and j exist. For the vertex $u_1 \dots u_k b_s$ we consider the following cases:

1. If $a_j^t \dots a_k^t u_0 \dots u_{j-1}^t = u_1 \dots u_k b_s$, then $u_{j-2}^t = u_k$ and since $2 \leq j \leq k$ this implies that $0 \leq j-2 \leq k-2$ so $u_0 \dots u_k$ cannot be a path in G.
2. If $a_j^t \dots a_k^t u_0 u_1 b_2^t \dots b_{j-1}^t = u_1 \dots u_k b_s$, then $a_j^t = u_1$ and since $3 \leq j \leq k$ the vertex u_1 appears at least twice in the sequence $a_j^t \dots a_k^t u_0 u_1 b_2^t \dots b_{j-1}^t$, which must determine a path in G.
3. If $u_j \dots u_1 b_2^t \dots b_{k+2-j}^t = u_1 \dots u_k b_s$, then $u_j = u_1$ and since $j \neq 1$ the sequence $u_0 \dots u_k$ cannot determine a path in G.

Analogously, for the vertex $c_{\gamma(s)} u_1 \dots u_k$ we consider the following cases:

1. If $a_j^t \dots a_k^t u_0 \dots u_{j-1}^t = c_{\gamma(s)} u_1 \dots u_k$, then $u_{j-1}^t = u_k$ and since $2 \leq j \leq k$ the sequence $u_0 \dots u_k$ is not a path in G.
2. If $a_j^t \dots a_k^t u_0 u_1 b_2^t \dots b_{j-1}^t = c_{\gamma(s)} u_1 \dots u_k$, since $3 \leq j \leq k$ the vertex u_1 already appears in the sequence $u_1 \dots u_k$, which is not possible because it is a path in G.
3. If $u_j \dots u_1 b_2^t \dots b_{k+2-j}^t = c_{\gamma(s)} u_1 \dots u_k$, then it must be $j = 2$ and j where $b_2^t = u_0$ which is not possible because of the choice of b_2^t in Theorem 3.1.

The previous paths, together with the edge between U and V form a set of $2(\delta - (k-1))$ edge disjoint paths joining U and V. As a consequence, for each edge (U, V) contained in an edge cut, there must be at least $2(\delta - (k-1)) - 1$ more edges in the edge cut, in order to disconnect the pair of vertices U and V. Then, follows immediately that $\lambda(P_k(G)) \geq 2(\delta - (k-1))$. □

Note that in general $\delta(P_k(G)) \geq 2(\delta - (k-1))$, but if G is δ-regular, then $\delta(P_k(G)) = 2(\delta - (k-1))$, so we can derive the following result.

Corollary 3.5. *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 2(k-1)$. Then, $P_k(G)$ is maximally connected.* □

Since the condition $\delta > 2(k-1)$ is preserved under the path graph iteration, by induction, Theorem 3.4 can be extended to iterated path graphs.

Corollary 3.6. *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 2(k-1)$. Then, for every positive integer s, $P_k^s(G)$ is maximally connected.*

The study of the superconnectivity of a graph is interesting once it is known that the graph is maximally connected. For that reason, and as a consequence of Corollary 3.5, we restrict the study of the superconnectivity to connected δ-regular graphs G where $P_k(G)$ is connected.

Theorem 3.7. *Let k be a positive integer and let G be a connected δ-regular graph with $\delta > 2(k-1)$. Then, $P_k(G)$ is super-λ.*

Proof. Let A be a non trivial edge cut in $P_k(G)$, we will show that $|A|$ must be greater than $\delta(P_k(G))$. Suppose that $|A| \leq \delta(P_k(G))$. Since $P_k(G)$ is maximally connected, it can only be $|A| = \delta(P_k(G))$. As a consequence of Lemma 3.1 and Lemma 3.3, A must contain at least one edge in each of the $2(\delta - k + 1)$ edge disjoint paths joining the endpoints of every in A. But precisely because those paths are edge disjoint and the set A is not trivial, the set A must contain at least $\delta(P_k(G)) + 1$ more edges. $\square$

The previous theorem gives rise to a trivial lower bound for the superconnectivity of path graphs, which is $\lambda_1(P_k(G)) \geq 2(\delta - k + 1) + 1$.

As before, by induction we can obtain the following corollary.

Corollary 3.8. *Let k be a positive integer and let G be a connected graph with minimum degree $\delta > 2(k-1)$. Then, for every positive integer s, $P_k^s(G)$ is super-λ and $\lambda_1(P_k^s(G)) \geq 2(\delta - (k-1)) + 1$.* $\square$

Notice that the proofs for Lemmas 3.1 and 3.3, as well as the proofs for Theorem 3.4 and its corollaries and Theorem 3.7 also work if the bound on the degree, $\delta > 2(k-1)$, is replaced by a more relaxed one, $\delta > k$, but together with a lower bound on the girth g, which must be $g \geq k + 1$. However, as a consequence of Lemma 3.1, the new results cannot be extended to iterated path graphs if $k \geq 4$.

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