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# TWO KINDS OF CHAOS AND RELATIONS BETWEEN THEM

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Abstract. In this paper we consider relations between chaos in the sense of Li and Yorke, and $\omega$-chaos. The main aim is to show how important the size of scrambled sets is in definitions of chaos. We provide an example of an $\omega$-chaotic map on a compact metric space which is chaotic in the sense of Li and Yorke, but any scrambled set contains only two points. Chaos in the sense of Li and Yorke cannot be excluded: We show that any continuous map of a compact metric space which is $\omega$-chaotic, must be chaotic in the sense of Li and Yorke. Since it is known that, for continuous maps of the interval, Li and Yorke chaos does not imply $\omega$-chaos, Li and Yorke chaos on compact metric spaces appears to be weaker. We also consider, among others, the relations of the two notions of chaos on countably infinite compact spaces.

## 1. Introduction

In this paper we study two different (but similar) definitions of chaos and relations between them.

Chaos in the sense of Li and Yorke, briefly LYC, was introduced in 1975 by T. Y. Li and J. A. Yorke [**10**]: A continuous map $f : I \to I$, where $I$ is the unit interval, is LYC if there is an uncountable set $S \subset I$ such that trajectories of any two distinct points $x, y$ in $S$ are *proximal* and *not asymptotic*, i.e.,

$$\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0.$$

The original definition contains another condition which later appeared to be superfluous [**7**]. The requirement of uncountability of $S$ in this definition (i.e., for continuous maps of the interval, but not in a general compact metric space) is equivalent to the condition that $S$ contains two points [**7**], or that $S$ is a perfect set (i.e., nonempty, compact and without isolated points) [**12**].

The second type of chaos is an $\omega$-chaos, briefly $\omega$C, introduced in 1993 by S. Li [**9**]: A continuous map $f : I \to I$ is $\omega$C if there is an uncountable set $S$ such that for any distinct $x$ and $y$ in $S$,

$$\omega_f(x)\backslash\omega_f(y) \text{ is uncountable}, \ \omega_f(x) \cap \omega_f(y) \neq \emptyset, \text{ and } \omega_f(x) \setminus Per(f) \neq \emptyset.$$

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If $f : I \to I$ is continuous, then $\omega$C is equivalent to PTE (positive topological entropy) [**9**], and by [**12**] PTE implies LYC but not conversely. Moreover, $f$ is $\omega$C if and only if it has an $\omega$-scrambled set containing two points [**9**], and if and only if it has a perfect $\omega$-scrambled set [**13**]. However, in the case when $X$ is a general compact metric space, the size of $S$ is essential.

By a *compactum* we mean an infinite compact metric space $X$ (countable or uncountable) with a metric $d$, and all maps considered in this paper are continuous. The space of all continuous maps of $X$ is denoted by $C(X, X)$. The set of limit points of the trajectory of a point $x \in X$ under $f \in C(X, X)$, i.e., of the sequence $\{f^n(x)\}_{n=0}^{\infty}$, is called *$\omega$-limit set* of $x$ under $f$, and is denoted by $\omega_f(x)$. The set of strictly increasing sequences of positive integers is denoted by $\mathcal{A}$.

**Definition 1.1.** Let $f \in C(X, X)$, and let $S \subset X$ contain at least two points. We say that $f$ is *chaotic in the sense of Li and Yorke* (briefly, $f$ is LYC), and that $S$ is a *scrambled set* for $f$ if, for any distinct $x, y \in S$,

(1) $\limsup_{n\to\infty} d(f^n(x), f^n(y)) > 0$,

(2) $\liminf_{n\to\infty} d(f^n(x), f^n(y)) = 0$.

Stronger notions of Li and Yorke chaos are these with *infinite*, or with an *uncountable* scrambled set. To distinguish between these three types of chaos we use notation $LY_2C$, or $LY_\infty C$, or $LY_u C$, respectively. Also we say that $f$ is *completely LYC* if $S = X$.

Now we introduce several modifications of the notion of $\omega$-chaos.

**Definition 1.2.** Let $f \in C(X, X)$, and let $S \subset X$ contain at least two points. We say that $f$ is *$\omega^u$-chaotic* (briefly, $f$ is $\omega^u$C), and $S$ is an *$\omega^u$-scrambled set* for $f$ if, for any distinct $x, y \in S$,

(1) $\omega_f(x)\backslash\omega_f(y)$ is uncountable,

(2) $\omega_f(x) \cap \omega_f(y)$ is nonempty,

(3) $\omega_f(x)$ is not contained in the set of periodic points.

In particular, $f$ is *$\omega_2^u$-chaotic*, or *$\omega_\infty^u$-chaotic*, or *$\omega_u^u$-chaotic* (briefly, $\omega_2^u C$, or $\omega_\infty^u C$, or $\omega_u^u C$, respectively), if there is an $\omega^u$-scrambled set containing two, or infinitely many, or uncountable many points, respectively).

The next definition modifies the notion of $\omega$-chaos for countable compact spaces.

**Definition 1.3.** Let $f \in C(X, X)$, and let $S \subset X$ contain at least two points. We say that $f$ is *$\omega^\infty$-chaotic* (briefly, $f$ is $\omega^\infty$C), and that $S$ is an *$\omega^\infty$-scrambled set* for $f$ if, for any distinct $x, y \in S$,

(1) $\omega_f(x)\backslash\omega_f(y)$ is infinite,

(2) $\omega_f(x) \cap \omega_f(y)$ is nonempty,

(3) $\omega_f(x)$ is not contained in the set of periodic points.

In particular, the map $f$ is *$\omega_2^\infty$-chaotic*, or *$\omega_\infty^\infty$-chaotic* (briefly, $\omega_2^\infty C$, or $\omega_\infty^\infty C$, respectively), if $f$ has an $\omega^\infty$-scrambled set possessing two, or infinitely many points, respectively.

Let me note, that there are two types of indices in the previous definitions. The upper indices in the definitions concerning $\omega$-chaos denote the cardinality of the difference of $\omega$-limit sets, and the lower ones the cardinality of the scrambled set.

**Remark 1.4.** It is obvious, that $\mathrm{LY}_u\mathrm{C} \Rightarrow \mathrm{LY}_\infty\mathrm{C} \Rightarrow \mathrm{LY}_2\mathrm{C}$; the converse implications are true for continuous maps on the interval [**12**] but, on general compact metric spaces they are no more valid [**5**], [**4**]. Also $\omega_u^u\mathrm{C} \Rightarrow \omega_\infty^u\mathrm{C} \Rightarrow \omega_2^u\mathrm{C} \Rightarrow \omega_2^\infty\mathrm{C}$ and $\omega_\infty^u\mathrm{C} \Rightarrow \omega_\infty^\infty\mathrm{C} \Rightarrow \omega_2^\infty\mathrm{C}$, and again it is possible to show that the converse implications are not true in the general case. Moreover, by [**12**] there is a $\mathrm{LY}_u\mathrm{C}$ map of the interval with zero topological entropy. This map has a unique infinite $\omega$-limit set and consequently, by [**9**], it cannot be $\omega_2^\infty\mathrm{C}$. Thus, in the general case, no form of Li and Yorke chaos implies the weakest form of $\omega$-chaos.

Thus it remains to answer the question: which forms of $\omega$-chaos imply Li and Yorke chaos? In the present paper we show that any form of $\omega\mathrm{C}$ implies $\mathrm{LY}_2\mathrm{C}$, cf. the next Theorem 1.5. But the implied LYC may be very small. In fact, we show that $\omega\mathrm{C}$ map on a compact metric space may have only two points LY-scrambled sets – cf. Theorem 3.11. On the other hand, we show that even completely LYC homeomorhisms may not be $\omega\mathrm{C}$ (Theorems 2.3. and 2.5.; compare with Theorem 3.1.).

Recall that a subset $M$ of $X$ is *minimal* for a map $f$, if it is nonempty, closed and invariant, and no proper subset of $M$ has these three properties (or equivalently, $M$ is minimal for a map $f$, if and only if $\omega_f(x) = M$, for each $x \in M$), and every point belonging to a minimal subset is called *uniformly recurrent.*

**Theorem 1.5.** *Let $X$ be a compact metric space, and let $f \in C(X,X)$ be $\omega_2^\infty C$. Then $f$ is $LY_2C$. In general, any point in an $\omega^\infty$-scrambled set of $f$ forms a LYC pair with a suitable point in $X$.*

*Proof.* Let $u$, $v$ be points in $X$ from an $\omega^\infty$-scrambled set for $f$. By results of Auslander [**1**] and Ellis [**3**], in a dynamical system on a compact metric space any point is proximal to a uniformly recurrent point in its orbit closure. Let $x$ be such a uniformly recurrent point proximal to $u$. Then $x$ belongs to a minimal set $M = \omega_f(x) \subset \omega_f(u)$. But $M$ must be a proper subset of $\omega_f(u)$. For if $M = \omega_f(u)$ then $\omega_f(u) \cap \omega_f(v) \neq \emptyset$ and $\omega_f(v) \setminus \omega_f(u) \neq \emptyset$ would imply $\omega_f(u) \subset \omega_f(v)$ and consequently, $\omega_f(u) \setminus \omega_f(v) = \emptyset$ – a contradiction. Thus $u$ and $x$ are proximal points, which cannot be asymptotic since $\omega_f(x) \neq \omega_f(u)$. □

## 2. Examples on countably infinite spaces

For a set $A \subset X$, and for any nonnegative integer $n$, define the $n$-th derivative $A^n$ of $A$ by $A^0 = A$, and $A^{n+1}$ is the set of cluster points of $A^n$. Denote $X_0 = X \setminus X^1$, and $X_j = X^j \setminus X^{j+1}$ for each $j = 1, 2, \ldots$.

**Proposition 2.1.** [**5**, Proposition 2.2.] *Let $f$ be a completely LYC homeomorphism of a compactum $X$. Then, $f$ has a unique fixed point.*

**Remark 2.2.** In [**5**] there is given a construction of a countably infinite compactum $X'$ and completely LYC homeomorphism $\varphi$ on $X'$, with fixed point $p$.

The set $X'$ is contained in the plane $\mathbb{R}^2$, it can be written in the form $X' = \bigcup_{j=0}^{\infty} X_j \cup \{p\}$, where $\omega_f(x) = X^{(j+1)}$ for each $j = 0, 1, 2, \dots$ and each $x \in X_j$ (cf. [**5**, Theorem 3.1]). Note that $p \in \omega_f(x)$ for each $x \in X$.

**Theorem 2.3.** *There is a countable compactum $X$ and completely LYC homeomorphism $f : X \to X$ such that $f$ is not $\omega_2^{\infty} C$.*

*Proof.* Put $X = X'$, and $f = \varphi$ (see Remark 2.2.). From the form of $\omega_f(x)$, $x \in X$, it is easy to see that, for each $x, y \in X$, $\omega_f(x) \subset \omega_f(y)$ or $\omega_f(y) \subset \omega_f(x)$. So, the map $f$ cannot be $\omega_2^{\infty}$C. $\square$

**Proposition 2.4.** [**5**, Proposition 2.5.] *Let $f$ be a completely LYC homeomorphism of a compactum X. Then for each $x \neq y \in X$, there is $\{n_i\} \in \mathcal{A}$ such that $f^{n_i}(x) \to p$ and $f^{n_i}(y) \to p$ where $p$ is the unique fixed point of $f$ (cf. Proposition 2.1.).*

**Theorem 2.5.** *There is a countable compactum $X$ and a completely LYC homeomorphism $\varphi : X \to X$ such that $\varphi$ is $\omega_{\infty}^{\infty} C$.*

*Proof.* Let $D = \{1/n,\ n \in \mathbb{N}\} \cup \{0\}$, where $\mathbb{N}$ is the set of positive integers. By collapsing $\{p\} \times D$ in $X \times D$ into a point, we get a compact metrizable space. Denote it by $X'D$. One can think of it as a subspace of $\mathbb{R}^3$. The topology on $X'D$ is given by the metric inherited from $\mathbb{R}^3$. We can imagine the space $X'D$ as a union of slices $S_i$ with one common point $p$. (So each $S_i$ is countably infinite compactum.) There are $f_i : S_i \to S_i$ (by [**5**] and Remark 2.2.), which are completely chaotic homeomorphisms with the fixed point $p$. Let $\varphi : X'D \to X'D$ be a map such that $\varphi$ restricted to $S_i$ is equal to $f_i$, for each $i$. It is easy to see, that $\varphi$ is a homeomorphism with the fixed point $p$.

It is clear that $\liminf_{n\to\infty} d(\varphi^n(x), \varphi^n(y)) = 0$ for each $x \neq y \in X'D$ (by Proposition 2.4.). Since, for any $i \neq j$ and any neighbourhood $U$ of $p$, the distance between $S_i \setminus U$ and $S_j \setminus U$ is positive, $\limsup_{n\to\infty} d(\varphi^n(x), \varphi^n(y)) > 0$ for each $x \neq y \in X'D$. Consequently, the map $\varphi$ is completely LYC.

On the other hand, the set $S = \bigcup_{i=0}^{\infty} \{x_i\}$, where $x_i$ is an arbitrary point from $S_i \setminus \{p\}$, is $\omega^{\infty}$-scrambled for the map $\varphi$. Really, $\omega_\varphi(x_i) \setminus \omega_\varphi(x_j)$ is countably infinite, $\omega_\varphi(x_i) \cap \omega_\varphi(x_j) = \{p\}$, for each $x_i \neq x_j \in S$ (see Remark 2.2.), and $\omega_\varphi(x)$ is not contained in the set of periodic points of $\varphi$, for each $x \in S$ (note that it is singleton $\{p\}$). Thus, $\varphi$ is $\omega_{\infty}^{\infty}$C. $\square$

## 3. Examples on uncountable spaces

By a *Cantor set*, denoted by $C$, we mean a compactum which is homeomorphic to the Cantor middle third set.

**Theorem 3.1.** *There is a perfect compact set $X \subset \mathbb{R}^3$ possessing a completely LYC homeomorphism $\varphi : X \to X$, such that $\varphi$ is not $\omega_2^{u} C$.*

*Proof.* This proof is similar to the proof of Theorem 2.5., we only replace the set $D$ by the set $C$. (The set $X'C$ is perfect, since each point of $X'C$ is accumulation one.) The map $\varphi : X'C \to X'C$ is not $\omega_2^u$C, since for each $x \in S$, $\omega_\varphi(x)$ is countable. □

To conclude this section we provide an example of a map which has a two points $\omega^u$-scrambled set and has only two points LY-scrambled set. The construction of this example is based on symbolic dynamics. The standard notions and basic known results can be found, e.g., in [**6**].

Let $\Sigma_2$ denote the set of sequences $x = x_1x_2x_3\dots$ where $x_n = 0$ or 1 for each $n$, equipped with the metric of pointwise convergence. Thus, for $y = y_1y_2y_3\dots$, put $\rho(x,y) = 1/k$ if $x \neq y$, and $k = \min\{n \in \mathbb{N} : x_n \neq y_n\}$, and let $\rho(x,y) = 0$ for $x = y$. Then $\Sigma_2$ is a compactum and the "shift map" $\sigma : \Sigma_2 \to \Sigma_2$ defined by $\sigma(x_1x_2x_3\dots) = x_2x_3\dots$ is continuous.

Recall that a sequence $x = x_1x_2x_3\dots \in \Sigma_2$ is called *uniformly recurrent* if for each block $x_1x_2\dots x_l$ there is $k$, such that for each $i$ at least one of the sequences $\sigma^i(x), \sigma^{i+1}(x)\dots\sigma^{i+k}(x)$ starts with the block $x_1x_2\dots x_l$.

For the construction of our example we use the following special uniformly recurrent sequences which, among others, have all blocks periodic.

Denote by $\mathbb{N}_0$ the set of nonnegative integers, i.e., $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $\mathcal{N} = \{N_n = 2^{n-1}(1 + 2\mathbb{N}_0), n = 1, 2, \dots\}$. It is easy to verify that $\mathcal{N}$ is a decomposition of $\mathbb{N}$. Define a map $\Phi : \Sigma_2 \to \Sigma_2$ so that, for $x = x_1x_2x_3\dots \in \Sigma_2$, $\Phi(x) = \tilde{x} = \tilde{x}_1\tilde{x}_2\tilde{x}_3\dots$, where $\tilde{x}_k = x_s$ if $k \in N_s$, i.e. $\Phi(x) = x_1x_2x_1x_3x_1x_2x_1x_4x_1x_2x_1\dots$. Then $\Phi(x)$ is not only uniformly recurrent but the blocks in $\Phi(x)$ are even periodic. This follows from the next lemma whose proof is obvious.

**Lemma 3.2.** *Let $B = \tilde{x}_i\tilde{x}_{i+1}\dots\tilde{x}_j$ be a block of $\Phi(x)$, and let $n$ be the maximal positive integer such that $i \leq 2^n \leq j$. Then the block $B$ is periodic in the sequence $\Phi(x)$, with period $2^{n+1}$.*

Let $\{a^i\}_{i=0}^\infty$ be a set of all sequences in $\Sigma_2$ such that

$$a^i = 0^{(2^i-1)}\ 1\ 0^{(2^{i+1}-1)}\ 1\ 0^{(2^{i+1}-1)}\ 1\dots$$

for each $i$ in $\mathbb{N}_0$, where the upper right index of $0^{(2^{i+1}-1)}$ means that a zero repeats $(2^{i+1}-1)$ times (i.e., $0^{(3)} = 000$), and hence for any distinct indices $i$ and $j$ the sequence $a^i$ differes from $a^j$ on infinitely many positions. It is worth noticing that each sequence $a^i$ contains infinitely many zeros and infinitely many ones. Thus, keeping our notation, we have $\Phi(a^i) = \tilde{a}^i = \tilde{a}^i_1\tilde{a}^i_2\tilde{a}^i_3\dots$, for each $i$. The sets $\omega_\sigma(\tilde{a}^i)$ are minimal and uncountable, since the sequences $\tilde{a}^i$ are uniformly recurent but not periodic.

For $x = x_1x_2x_3\dots$ and $y = y_1y_2y_3\dots$ in $\Sigma_2$, put

$$x \quad \diamond^i \quad y = X^{(2^i-1)}\ Y\ X^{(2^{i+1}-1)}\ Y\ X^{(2^{i+1}-1)}\ Y\dots,$$

where $X$ is a block $x_1x_2\dots x_k$, $k \in \mathbb{N}$, of the sequence $x$, the upper right index of $X^{(2^i-1)}$ means that the block $X$ repeats $(2^i-1)$ times and the length of each block $X$ is equal to the position on which it is (i.e., $x \quad \diamond^1 \quad y = XYXXXY\dots =$

$= \underline{x_1}\ \underline{y_1y_2}\ \underline{x_1x_2x_3}\ \underline{x_1x_2x_3x_4}\ \underline{x_1x_2x_3x_4x_5}\ \underline{y_1y_2y_3y_4y_5y_6}\ldots$, the blocks are underlined to be well-arranged). Finally, let

$$\alpha^i = \tilde{a}^{i+1} \quad \diamond^i \quad \tilde{a}^0,$$

and let

$$\mathcal{X} = \bigcup_{i=0}^{\infty} \overline{\mathrm{Orb}(\alpha_\mathrm{i})}.$$

Let us note that for the constructions of the sequences $a^i$ and $\alpha^i$ was used similar principle as for the construction of the map $\Phi$ defined above.

The proof of the second condition of the following Lemma was motivated by Lemma 1.2 from [**11**].

**Lemma 3.3.** *Let $\alpha^i$ and $\tilde{a}^i$ be as above, for each $i$. Then*

(i) $\overline{\mathrm{Orb}(\alpha^i)} = \mathrm{Orb}(\alpha^i) \cup \omega_\sigma(\alpha^i)$.

(ii) $\overline{\mathrm{Orb}(\tilde{a}^{i+1})} \cup \overline{\mathrm{Orb}(\tilde{a}^0)} \subset \omega_\sigma(\alpha^i) = \overline{\mathrm{Orb}(\tilde{a}^{i+1})} \cup \overline{\mathrm{Orb}(\tilde{a}^0)} \cup C_{a^{i+1}} \cup C_{a^0}$

*where $C_{a^{i+1}}$ is a subset of the set $\mathrm{Orb}^{-1}(\tilde{a}^{i+1})$ of all $\sigma$-preimages of the points from $\mathrm{Orb}(\tilde{a}^{i+1})$, and similarly for $C_{a^0}$. Consequently, each $C_{a^i}$ is countable.*

*Proof.* (i) This equality is true, since $\omega_\sigma(\alpha^i)$ is the set of accumulation points of $\mathrm{Orb}(\alpha^i)$.

(ii) Obviously, $\tilde{a}^{i+1}$ and $\tilde{a}^0$ belong to $\omega_\sigma(\alpha^i)$, and $\omega_\sigma(\alpha^i)$ is closed and invariant. Therefore, it contains $\overline{\mathrm{Orb}(\tilde{a}^{i+1})}$ and $\overline{\mathrm{Orb}(\tilde{a}^0)}$. This proves the first inclusion.

To prove the second equality, let $u \in \omega_\sigma(\alpha^i)$. There is a sequence $\{p_k\} \in \mathcal{A}$ such that $\sigma^{p_k}(\alpha^i) \to u$. Consider the four possible cases:

1. Infinitely many terms in the sequence $\sigma^{p_k}(\alpha^i)$ begin with a block of $\tilde{a}^{i+1}$ of the same length $\lambda \geq 0$ (i.e., block of the form $a_{m_s-\lambda+1}\ldots a_{m_s}$), followed by a block $\tilde{a}^0_1\tilde{a}^0_2\ldots,\tilde{a}^0_{n_k}$ of $\tilde{a}^0$. Since the length of blocks $\tilde{a}^0$ tends to infinity, $u = \tilde{a}^{i+1}_i\ldots\tilde{a}^{i+1}_j\tilde{a}^0$, where $j \geq i$. Thus, $u \in \mathrm{Orb}^{-1}(\tilde{a}^0)$.

2. Similarly, if infinitely many terms in the sequence $\sigma^{p_k}(\alpha^i)$ begin with a block of $\tilde{a}^0$ of the same length, followed by a block $\tilde{a}^{i+1}_1\tilde{a}^{i+1}_2\ldots,\tilde{a}^{i+1}_{m_l}$ of $\tilde{a}$, $u \in \mathrm{Orb}^{-1}(\tilde{a}^{i+1})$.

3. If infinitely many terms in the sequence $\sigma^{p_k}(\alpha^i)$ begin with a block of $\tilde{a}^{i+1}$ whose length is unbounded as $k$ tends to infinity then $u \in \overline{\mathrm{Orb}(\tilde{a}^{i+1})}$.

4. If infinitely many terms in the sequence $\sigma^{p_k}(\alpha^i)$ begin with a block of $\tilde{a}^0$ whose length is unbounded as $k$ tends to infinity then $u \in \overline{\mathrm{Orb}(\tilde{a}^0)}$. □

**Lemma 3.4.** *The map $\sigma$ restricted to $\mathcal{X}$ is $\omega^u_\infty C$.*

*Proof.* We show that $\{\alpha^i\}_{i=0}^{\infty}$ is an $\omega^u$-scrambled set. By Lemma 3.3., $\omega_\sigma(\alpha^i) \setminus \omega_\sigma(\alpha^j) \supset \omega_\sigma(\tilde{a}^{i+1})$ is uncountable and $\omega_\sigma(\alpha^i) \cap \omega_\sigma(\alpha^j) \supset \omega_\sigma(\tilde{a}^0) \neq \emptyset$ for each $i \neq j$, and since $\omega_\sigma(\tilde{a}^0)$ is infinite and minimal, it contains no periodic point. □

**Lemma 3.5.** *Let $i > j$, then $\rho(\sigma^n(\tilde{a}^i), \sigma^m(\tilde{a}^j)) > 1/(2^{j+2})$, for any $m, n \in \mathbb{N}_0$.*

*Proof.* By a direct computation, each block of length $2^{i+2}$ contains at most three ones and at least $2^{i+2}-3$ zeros of the sequence $\tilde{a}^i$. Hence, for each $m, n$ in $\mathbb{N}_0$ and each $i > j$ is $\rho(\sigma^n(\tilde{a}^i), \sigma^m(\tilde{a}^j)) > 1/(2^{j+2})$. $\square$

For simplicity put $P_x = \overline{\mathrm{Orb}(\tilde{x})} \cup \mathrm{Orb}^{-1}(\tilde{x})$, for $x \in \Sigma_2$.

**Lemma 3.6.** *The sets $P_{a^i}$ are distal (and hence, disjoint). Thus, for $u \in P_{a^i}$ and $v \in P_{a^j}$ where $i \neq j$, $\liminf_{k\to\infty} \rho(\sigma^k(u), \sigma^k(v)) > 0$.*

*Proof.* Apply Lemma 3.5. $\square$

Our next aim is to show that $\sigma$ is LYC on $P_x$ for no $x \in \Sigma_2$. For simplicity, we will consider only sequences $x$ which contain infinitely many zeros and infinitely many ones, if it contains finitely many zeros or ones, then the assertion is obvious. In this case, it is possible to reconstruct the original sequence $x$ from $\tilde{y} = \sigma^k(\tilde{x})$ without knowing $k$. In fact, majority of the digits in $\tilde{y}$ must be equal to $x_1$: either the digits on odd places in $\tilde{y}$ are the same and equal to $x_1$, or the digits on the even places in $\tilde{y}$ are equal to $x_1$. Next, having fixed $x_1$ among the digits $\tilde{x}_j$, we remove from $\tilde{y}$ the digits corresponding to $x_1$ (i.e., either all digits on the odd places, or all digits on the even places), and we obtain a new sequence, to which the previous procedure is applicable.

However, in a similar way we can reconstruct the first $n$ digits in $x$ from any block of a sequence in $\omega_\sigma(\tilde{x})$, with a sufficient length $\delta_n$. Indeed, let $d = d_1 d_2 d_3 \cdots \in \omega_\sigma(\tilde{x})$. Then for any $m = 1, 2, \ldots$ there is $\{n_k\} \in \mathcal{A}$ such that $d_1 d_2 \ldots d_m = \tilde{x}_{n_k+1}\tilde{x}_{n_k+2}\ldots\tilde{x}_{n_k+m}$, for any $k$. Let $r = \min\{i : x_i \neq x_1\}$. Then it suffices to take $m = 2^r$ to see, which members of $d_1 d_2 \ldots d_m$ are equal to $x_1$. (Thus, in our case, $\delta_1 = 2^r$.) Define a map $\mu : \omega_\sigma(\tilde{x}) \to \Sigma_2$ such that, for $d = d_1 d_2 d_3 \cdots \in \omega_\sigma(\tilde{x})$, $\mu(d) = s = s_1 s_2 s_3 \ldots$, where $s_n$ is given inductively in the following way:

*Stage 1*: Let

$$s_1 = \begin{cases} 0, & \text{if } x_1 = d_1 = d_3 = d_5 \ldots, \\ 1, & \text{if } x_1 = d_2 = d_4 = d_6 \ldots \end{cases}$$

Let $d^1 = d_1^1 d_2^1 d_3^1 \ldots$ be a subsequence of $d$ obtained by removing $d_1, d_3, d_5 \ldots$ from $d$ if $s_1 = 0$, and by removing $d_2, d_4, d_6 \ldots$ otherwise.

*Stage n*: Sequence $d^{n-1} = \{d_i^{n-1}\}_{i=1}^{\infty}$ is available from the *stage n-1*. Let

$$s_n = \begin{cases} 0, & \text{if } x_n = d_1^{n-1} = d_3^{n-1} = d_5^{n-1} \ldots, \\ 1, & \text{if } x_n = d_2^{n-1} = d_4^{n-1} = d_6^{n-1} \ldots, \end{cases}$$

and let $d^n$ be obtained from $d^{n-1}$ by removing the odd or even members, if $s_n = 0$ or $s_n = 1$, respectively. Obviously, we have the following

**Lemma 3.7.** *Let $x$ be a sequence in $\Sigma_2$ having infinitely many zeros and infinitely many ones. Then the map $\mu$ is a bijection from $\omega_\sigma(\tilde{x})$ to $\Sigma_2$.*

**Lemma 3.8.** *Let $x$ be a sequence in $\Sigma_2$ having infinitely many zeros and infinitely many ones. Then, for any distinct $d, h$ in $P_x$,*

$$\liminf_{n\to\infty} \rho(\sigma^n(d), \sigma^n(h)) > 0.$$

*Proof.* Since any point in $P_x$ is eventually in $\omega_\sigma(\tilde{x})$ we may assume, without loss of generality, that $d, h \in \omega_\sigma(\tilde{x})$. By Lemma 3.7., $\mu$ is bijective, hence $\mu(d) \neq \mu(h)$. Then, for some $m \in \mathbb{N}$, the sequences $\mu(d), \mu(h)$ differ on the $m$-th coordinate, $\mu(d)_m \neq \mu(h)_m$. But then $\liminf_{n\to\infty} \rho(\sigma^n(d), \sigma^n(h)) \geq 1/2^m$ (cf. the construction of $\mu$). □

Now we can return to our special sequences $\alpha^i$.

**Lemma 3.9.** *The map $\sigma$ restricted to* $\mathrm{Orb}(\alpha^i)$ *is not $LY_2C$ for each $i$. In particular* $\liminf_{n\to\infty} \rho(\sigma^n(\alpha^i), \sigma^{n+k}(\alpha^i)) > 0$ *whenever $k \in \mathbb{N}$.*

*Proof.* Let us suppose that $\liminf_{n\to\infty} \rho(\sigma^n(\alpha^i), \sigma^{n+k}(\alpha^i)) = 0$. Then there is $\{n_l\}_{l=1}^\infty \in \mathcal{A}$ such that $\lim_{n\to\infty} \rho(\sigma^{n_l}(\alpha^i), \sigma^{n_l+k}(\alpha^i)) = 0$. Then $\sigma^{n_l}(\alpha^i) \to d$ and $\sigma^{n_l+k}(\alpha^i) \to d$ so $\sigma^k(d) = d$ and $d \in X$ is a periodic point. But $\sigma$ restricted to $\mathcal{X}$ has no periodic point (cf. Lemma 3.3.) – a contradiction. □

**Lemma 3.10.** *The map $\sigma$ restricted to $\bigcup_{i=0}^\infty$* $\mathrm{Orb}(\alpha^i)$ *is not $LY_2C$.*

*Proof.* Because of the symmetry it suffices to show that, for any $l \in \mathbb{N}_0$, $\liminf_{n\to\infty} \rho(\sigma^{n+l}(\alpha^i), \sigma^n(\alpha^j)) > 0$, where $i \neq j$. Assume, contrary to what we wish to show, that this is not true. Then, by Lemma 3.5., in both sequences $\sigma^l(\alpha^i)$, $\alpha^j$, there must be arbitrarily large blocks of $\tilde{a}^0$ at the same positions. However, if $l = 0$, then any blocks of $\tilde{a}^0$ in $\alpha^i$ and $\alpha^j$ are at complementary positions (cf. the definition of $\alpha^i$ and $\alpha^j$). If $l$ is positive, then the blocks are shifted, and there is some overlapping of the blocks of $\tilde{a}^0$. But since the lenght of the blocks $\tilde{a}^0$, $\tilde{a}^{i+1}$ and $\tilde{a}^{j+1}$ tends to infinity, the parts of the blocks of $\tilde{a}^0$ in $\sigma^l(\alpha^i)$, $\alpha^j$, respectively, that are overlapping, are small — their length is $l$. Consequently, by Lemma 3.5., we get $\liminf_{n\to\infty} \rho(\sigma^{n+l}(\alpha^i), \sigma^n(\alpha^j)) \geq 1/(2^j + 2 + l) > 0$. □

Now we are able to prove our main result.

**Theorem 3.11.** *There is a compactum $X \subset \Sigma_2$ such that $\sigma(X) \subset X$, $\sigma$ has no periodic points in $X$, $\sigma$ restricted to $X$ is $\omega_\infty^u C$ and any LY-scrambled set has only two points.*

*Proof.* Let $X = \mathcal{X}$. By Lemma 3.4., $\sigma$ is $\omega_\infty^u$C on $\mathcal{X}$. On the other hand, by Lemma 3.3.,

$$\mathcal{X} \subset \left( \bigcup_{i=0}^{\infty} \mathrm{Orb}(\alpha^i) \right) \cup \left( \bigcup_{i=0}^{\infty} P_{a^i} \right).$$

By Lemmas 3.6. and 3.8. – 3.10. any two points set in $\bigcup_{i=0}^\infty \mathrm{Orb}(\alpha^i)$ or in $\bigcup_{i=0}^\infty P_{a^i}$ is not LY-scrambled. Hence, $\{u, v\}$ is LY-scambled, if $u \in \bigcup_{i=0}^\infty \mathrm{Orb}(\alpha^i)$ and $v \in \bigcup_{i=0}^\infty P_{a^i}$ are suitable points. □

**Concluding remarks**. (i) R. Pikula [**11**] recently proved, that there is an $\omega$C map $f$ of a compact metric space with the property that any LY-scrambled set has not more than 8 points. He considers uncountable $\omega^u$-scrambled sets. Our Theorem 3.11. gives a stronger result. On the other hand, our $\omega^u$-scrambled set is infinity.

(ii) The systems obtained in Theorems 2.3. – 3.11. can be inserted to the real line so that there is a continuous map $f$ of the unit interval $I$ which has as factors the systems from Theorems 2.3. – 3.11.

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