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BOUNDARY BEHAVIOR IN STRONGLY DEGENERATE PARABOLIC EQUATIONS

M. WINKLER

Abstract. The paper deals with the initial value problem with zero Dirichlet boundary data for

$$u_t = u^p \Delta u \quad \text{in } \Omega \times (0, \infty)$$

with $p \geq 1$. The behavior of positive solutions near the boundary is discussed and significant differences from the case of the heat equation ($p = 0$) and the porous medium equation ($p \in (0, 1)$) are found. In particular, for $p \geq 1$ there is a large class of initial data for which the corresponding solution will never enter the cone $\{v : \Omega \to \mathbb{R} \mid \exists\, c > 0 : \; v(x) \geq c\mathrm{dist}(x, \partial\Omega)\}$.
Finally, for $p > 2$ a solution u with $u(t) \in C_0^\infty(\Omega)\ \forall\, t \geq 0$ is constructed.

Introduction

This paper is concerned with nonnegative solutions of

$$\begin{aligned} u_t &= u^p \Delta u \quad \text{in } \Omega \times (0, \infty), \\ u|_{\partial\Omega} &= 0, \\ u|_{t=0} &= u_0, \end{aligned} \tag{0.1}$$

where $p \geq 1$ and Ω is a bounded domain in $\mathbb{R}^n$ with C^2-boundary. Here $0 \not\equiv u_0 \in C^0(\bar{\Omega})$ is assumed to be nonnegative with $u_0|_{\partial\Omega} = 0$.

Due to the degeneracy in (0.1), we expect that diffusive effects are weakened in regions where u is small which should primarily affect the behavior of u near the boundary of its support.

To explain this, let us recall the well-known fact that in case of the heat equation ($p = 0$) all nontrivial nonnegative solutions of (0.1) become positive in all of Ω instantaneously; in fact, the strong maximum principle even states that

$$u(t) \in K \quad \forall\, t > 0, \tag{0.2}$$

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where the cone K is defined by

$$K := \{v : \Omega \to \mathbb{R} \mid \exists\, c > 0 : \; v(x) \geq c\,\mathrm{dist}(x, \partial\Omega)\,\forall\, x \in \Omega\}.$$

This is no longer true in the *weakly degenerate* case $p \in (0, 1)$, where the PDE in (0.1) transforms into the porous medium equation $v_t = \Delta v^m$ via the substitution $u = av^m$ with $m = \frac{1}{1-p} > 1$ and $a = m^{\frac{1}{p}}$. Then, (0.2) is to be replaced with

$$\exists\, t_0 \geq 0 : \; u(t) \in K \quad \forall\, t > t_0, \tag{0.3}$$

and it depends on the behavior of u_0 near $\partial\Omega$ whether or not t_0 can be chosen equal to zero ([**BP**], [**Fr**], [**Ar**]). As to the *strongly degenerate* case $p \geq 1$, however, it has been shown in [**Win2**] that $\mathrm{supp}\, u(t) \equiv$ *const*. for all $t \geq 0$ (cf. also [**LDalP**] and [**BU**]), so that $u(t)$ will never enter K if $\mathrm{supp}\, u_0$ is a compact subset of Ω.

The properties (0.2) and (0.3) have been widely used as a powerful tool in the description of the qualititive properties of solutions to (0.1) as well as to a large class of related semilinear and quasilinear problems with additonal source or sink terms, including various topics such as stability, convergence rates or localization of blow-up points (see [**Li**], [**AP**] or [**FMcL1**], for instance).

In [**Win4**], the reader may find an example of how the *absence* of (0.3) may influence the asymptotics of solutions to $u_t = u^p \Delta u + u^{p+1}$, $p \in [1, 3)$ (in domains with a special size): Namely, there it is shown that whenever u_0 is such that u enters K at some time then $u(t)$ approaches a *positive* equilibrium as $t \to \infty$, while there are other initial data for which $u(t)$ remains outside K and for which $u(t) \to 0$ as $t \to \infty$.

The main objective of the present work will be to find conditions on positive initial data which either enforce or rule out (0.3). To illustrate our results as transparently as possible, let us assume that

$$u_0(x) \sim (\mathrm{dist}(x, \partial\Omega))^\alpha \qquad \text{near } \partial\Omega$$

for some $\alpha > 1$. (Note that the statements in the following sections are in part actually much sharper.)

- If $p \in [1, 2)$ and
  - $\alpha < \frac{1}{p-1}$ (∞ if $p = 1$) then there is $t_0 > 0$ such that $u(t) \in K \quad \forall\, t \geq t_0$ (Corollary 2.3);
  - $\alpha \geq \frac{1}{p-1}$ then $u(t) \notin K$ for all $t \geq 0$ (Lemma 2.1).
- If $p > 2$ then $u(t) \sim (\mathrm{dist}(x, \partial\Omega))^\alpha$ continues to hold for all $t \geq 0$, so that $u(t) \notin K$ for all $t \geq 0$ (Corollary 4.2).

Actually, we shall see that for $p > 2$ even superpolynomial boundary decay of u_0 can be inherited by the solution. As a consequence (and as the second topic of this work), we will present in Theorem 4.4 a somewhat 'strange' solution of (0.1) which has a property that seems to be fairly uncommon in the context of quasilinear parabolic equations:

- If $p > 2$ then (0.1) has a classical solution $u \not\equiv 0$ with

$$u(t) \in C_0^\infty(\Omega) \quad \forall\, t \geq 0.$$

1. Some preliminaries

Unless otherwise stated (and this will be the case only in Theorem 4.4), we will assume

$$u_0 \in C^0(\bar{\Omega}), \quad u_0|_{\partial\Omega} = 0 \quad \text{and} \quad u_0 > 0 \quad \text{in } \Omega.$$

For such initial data, we obtain a unique classical solution to (0.1). For a proof of this fact, we refer to Theorem 1.2.2 in [**Win1**]; a similar reasoning can be found in [**Wie2**] or in [**FMcL2**].

Lemma 1.1. *Problem (0.1) admits a unique positive classical solution u which can be obtained as the $C^0(\bar{\Omega} \times [0,\infty))) \cap C^{2,1}(\Omega \times (0,\infty))$-limit of a decreasing sequence of solutions u_ε, $\varepsilon = \varepsilon_j \searrow 0$, of*

$$\begin{aligned} \partial_t u_\varepsilon &= u_\varepsilon^p \Delta u_\varepsilon \quad \textit{in } \Omega \times (0,\infty), \\ u_\varepsilon|_{\partial\Omega} &= \varepsilon, \\ u_\varepsilon|_{t=0} &= u_{0,\varepsilon}, \end{aligned} \tag{1.1}$$

where $(u_{0,\varepsilon})_{\varepsilon=\varepsilon_j\searrow 0} \subset C^1(\bar{\Omega})$ is any decreasing sequence of functions with $u_{0,\varepsilon}|_{\partial\Omega} = \varepsilon$ and $\max\{u_0 + \frac{\varepsilon}{2}, \varepsilon\} \leq u_{0,\varepsilon} \leq u_{0,\varepsilon} + 2\varepsilon$.

As a consequence of uniqueness, it follows that if $u_0 \leq v_0$ in Ω then the corresponding solutions u and v of (0.1) satisfy $u \leq v$ in $\Omega \times (0,\infty)$. For a version of the parabolic comparison principle appropriate for degenerate problems of the above type, we refer to [**Wie2**]. The following useful semi-convexity estimate is also well-known (cf. [**Ga**], [**Win2**] or also [**Ar**]).

Lemma 1.2. i) *We have*

$$\frac{u_t}{u} \geq -\frac{1}{pt} \quad \textit{in } \Omega \times (0,\infty).$$

ii) *Suppose that, additionally, $u_0 \in C^2(\bar{\Omega})$. Then there is $C > 0$ such that*

$$\frac{u_t}{u} \geq -C \quad \textit{in } \Omega \times (0,\infty). \tag{1.2}$$

Proof. We only prove ii), since the proof of i) can be accomplished by a simplified version of this (see [**Win2**] for details). First, we mollify $v_\varepsilon := (u_0 - \frac{\varepsilon}{4})_+$ in $\mathbb{R}^n$ to a function w_ε with compact support in Ω satisfying $\max\{u_0 - \frac{\varepsilon}{2}, \varepsilon\} \leq w_\varepsilon \leq u_0$. As $\Delta v_\varepsilon \geq \inf_\Omega \Delta u_0 \geq -c$ for sufficiently small $\varepsilon > 0$ in the sense of distributions on $\mathbb{R}^n$, we also have $\Delta w_\varepsilon \geq -c$, so that $u_{0,\varepsilon} := w_\varepsilon + \varepsilon$ is in $C^\infty(\bar{\Omega})$ and fulfils $u_{0,\varepsilon}|_{\partial\Omega} = \varepsilon$ as well as $u_0 + \frac{\varepsilon}{2} \leq u_{0,\varepsilon} \leq u_0 + 2\varepsilon$. Since $u_{0,\varepsilon}$ is constant near $\partial\Omega$, the compatibility condition of first order for (1.1) is valid (that is, $u_{0\varepsilon}\Delta u_{0,\varepsilon}|_{\partial\Omega} = 0$) so that $z := \frac{\partial_t u_\varepsilon}{u_\varepsilon} \equiv u_\varepsilon^{p-1}\Delta u_\varepsilon$ is in $C^0(\bar\Omega\times[0,\infty))\cap C^{2,1}(\Omega\times(0,\infty))$. By differentiation of (1.1),

$$z_t = pz^2 + u_\varepsilon^{p-1}(2\nabla u_\varepsilon\cdot\nabla z + u_\varepsilon\Delta z) \quad \text{in } \Omega\times(0,\infty);$$

as $z \geq 0$ on $\partial\Omega$ and $z \geq -c\|u_\varepsilon\|_{L^\infty(\Omega)}^{p-1} \geq -C$ at $t=0$, we obtain from parabolic comparison that $z \geq -C$ in $\Omega\times(0,\infty)$. $\square$

As a simple consequence of Lemma 1.2 i), we note that

$$u(t_0)\in K \text{ for some } t_0>0 \quad \text{implies} \quad u(t)\in K\ \forall\, t\geq t_0.$$

2. The case $1 \leq p < 2$

Lemma 2.1. *Suppose $p\in[1,2)$ and*

$$\int_\Omega u_0^{1-p} = \infty \qquad \text{if } p>1,$$
$$\int_\Omega \ln u_0 = -\infty \qquad \text{if } p=1.$$

Then

$$u(t)\neq K \quad \text{for all } t>0.$$

Proof. Assume $u(t_0)\in K$ for some $t_0>0$. Then, if $p\in(1,2)$, $\int_\Omega u^{1-p}(t_0)<\infty$. Dividing (1.1) by u^p and integrating, we obtain

$$\int_\Omega u_\varepsilon^{1-p}(t_0) - \int_\Omega u_{0\varepsilon}^{1-p} = -(p-1)\int_0^{t_0}\int_{\partial\Omega}\partial_N u_\varepsilon,$$

where the right hand side is nonnegative since $u_\varepsilon\geq\varepsilon$ in $\Omega\times(0,\infty)$ by comparison. But the monotone convergence theorem implies that the left hand side tends to $-\infty$ as $\varepsilon\to 0$, a contradiction. The proof in the case $p=1$ is similar. $\square$

In both the radial and the one-dimensional case the previous lemma is complemented by

Lemma 2.2. *Suppose*

$$\int_\Omega u_0^{1-p} < \infty \qquad \text{if } p>1,$$
$$\int_\Omega \ln u_0 > -\infty \qquad \text{if } p=1,$$

and assume that either Ω is a ball and u_0 is radially symmetric in Ω, or that $n=1$. Then there exists $t_0>0$ such that

$$u(t)\in K \qquad \text{for all } t\geq t_0.$$

Proof. We only prove the case $1 < p < 2$, since the proof for $p = 1$ runs along the same lines. Let us start with the radial case and hence we may assume $\Omega = B_R(0)$ for some $R > 0$. We first briefly outline a proof of the well-known fact that $u(t) \to 0$ uniformly as $t \to \infty$ (cf. [**Win3**]): Let $e_1 \in C^2(\bar{B}_R(0))$ solve $-\Delta e_1 = 1$ in $B_R(0)$, $e_1|_{\partial B_R(0)} = 1$, and let $y(t)$ denote the solution of $y' = -y^{p+1}$ in $(0,\infty)$ with $y(0) = \|u_0\|_{L^\infty(B_R(0))}$. Then, as $e_1 \ge 1$ in $B_R(0)$, $v(x,t) := y(t)e_1(x)$ satisfies

$$\begin{aligned} v_t - v^p \Delta v &= y' e_1 + y^{p+1} e_1^p \\ &\ge (y' + y^{p+1}) e_1 \quad = 0 \qquad \text{in } B_R(0) \times (0,\infty), \end{aligned}$$

so that comparison yields $u \le v$ in $B_R(0) \times (0,\infty)$, whence indeed $u(t) \to 0$ uniformly in $B_R(0)$ as $t \to \infty$.

In particular, this together with the hypothesis implies the existence of $t_0 > 0$ such that

$$\int_{B_r(0)} u^{1-p}(t_0) \ge \int_{B_R(0)} u_0^{1-p} + 1 \quad \forall\, r \in \Big(\frac{R}{2}, R\Big).$$

Dividing (0.1) by u^p and integrating, we see that $z(r) := \int_0^{t_0} \int_{\partial B_R(0)} u$ fulfils

$$\begin{aligned} z'(r) &= \int_0^{t_0} \int_{\partial B_r(0)} \partial_N u + \frac{n-1}{r} \int_0^{t_0} \int_{\partial B_r(0)} u \\ &= -\frac{1}{p-1} \int_{B_r(0)} u^{1-p}(t_0) + \frac{1}{p-1} \int_{B_r(0)} u_0^{1-p} + \frac{n-1}{r} z(r) \\ &\le -\frac{1}{p-1} + \frac{2(n-1)}{R} z(r) \qquad \forall\, r \in \Big(\frac{R}{2}, R\Big), \end{aligned}$$

from which it follows, since $z(R) = 0$, that

$$z(r) \ge c_0 (R - r) \qquad \forall\, r \in \Big(\frac{R}{2}, R\Big)$$

for some $c_0 > 0$. Consequently, for any $r \in (\frac{R}{2}, R)$ there exists $t_r \in (\frac{t}{2}, t_0)$ such that

$$u(r, t_r) \equiv \frac{1}{r^{n-1}\omega_n} \int_{\partial B_r(0)} u(t_r) \ge c_2 (R - r)$$

with $c_2 = \frac{2c_0}{R^{n-1}\omega_n t_0}$, where ω_n denotes the area of the unit sphere in $\mathbb{R}^n$. Now Lemma 1.1 i) shows that

$$\begin{aligned} u(r, t_0) &\ge \Big(\frac{t_r}{t_0}\Big)^{\frac{1}{p}} u(r, t_r) \\ &\ge 2^{-\frac{1}{p}} c_2 (R - r) \qquad \forall\, r \in \Big(\frac{R}{2}, R\Big), \end{aligned}$$

which implies the claim.

In the one-dimensional case, we make use of the result just proved and take advantage of the fact that $\partial\Omega$ contains only two points. We may assume $\Omega = (-2a, a)$ for some $a > 0$. Let $\tilde{v}_0(x) := u_0(x)$ for $x \in [0, a]$ and $\tilde{v}_0(x) := u_0(-x)$ for $x \in [-a, 0)$. Then $\tilde{v}_0$ is continuous and symmetric in $[-a, a]$, $\tilde{v}_0(\pm a) = 0$ and $\tilde{v}_0 > 0$ in $(-a, a)$. From $u_0 > 0$ in $[-a, 0]$ it is clear that $v_0 := \eta\tilde{v}_0 \le u_0$ in $[-a, a]$ for some small $\eta > 0$. Consequently, the solution v of $v_t = v^p v_{xx}$ in $(-a, a) \times (0, \infty)$, $v(\pm a, t) = 0$, $v|_{t=0} = v_0$, lies below u. But since $\int_{-a}^{a} v_0^{1-p} = 2\eta^{1-p} \int_0^a u_0^{1-p} < \infty$, it follows from what we have shown before that $v(x, t_0) \ge c(a - x)$ for some $t_0 > 0$ and all $x \in (0, a)$. A similar argument near $x = -a$ and Lemma 1.2 i) complete the proof. $\square$

Corollary 2.3. *Suppose that*

$$u_0(x) \ge c_0(\mathrm{dist}(x, \partial\Omega))^\alpha \quad \textit{in } \Omega \quad \textit{for some } \alpha \in \Big(1, \frac{1}{p-1}\Big) \quad (\textit{resp. } \alpha \in (1, \infty)$$

if $p = 1$) *and some* $c_0 > 0$. *Then there is* $t_0 > 0$ *such that*

$$u(t) \in K \qquad \textit{for all } t \ge t_0.$$

Proof. Due to the smoothness of $\partial\Omega$ there is $R > 0$ with the property that for all $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) < R$ there exists $x_0 = x_0(x) \in \Omega$ such that $\mathrm{dist}(x, \partial\Omega) = \mathrm{dist}(x, \partial B_R(x_0))$. (Indeed, let R be small enough such that to each x with $\mathrm{dist}(x, \partial\Omega) < R$ there corresponds exactly one $y = y(x) \in \partial\Omega$ with $|x - y| = \mathrm{dist}(x, \partial\Omega)$. Then for any such x, the point $x_0(x) := y(x) + R\frac{x-y(x)}{|x-y(x)|}$ satisfies the above requirements.)
Let $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) < R$ be given and let $x_0 := x_0(x)$. Then $v_0^{(x)}(z) := c_0(R - |z - x_0|)^\alpha$ is positive in $B_R(x_0)$, vanishes on $\partial B_R(x_0)$ and is symmetric with respect to x_0. Since evidently $v_0^{(x)} \le u_0$ in $B_R(x_0)$, Lemma 2.2 together with the comparison principle yields $t_0 > 0$ and $c_1 > 0$ such that $u(z, t_0) \ge c_1 \mathrm{dist}(z, \partial B_R(x_0))$ holds for all $z \in B_R(x_0)$. In particular, $u(x, t_0) \ge c_1 \mathrm{dist}(x, \partial B_R(x_0)) = c_1 \mathrm{dist}(x, \partial\Omega)$. But t_0 and c_1 are the same for all x due to the fact that for different x, the functions $v_0^{(x)}$ are transferred into each other by a spatial shift. Therefore the proof is complete. $\square$

3. The case $p > 2$

The crucial step for the proof of 'conservation of boundary decay' in the case $p > 2$ is done in

Lemma 3.1. *Let* $d_0 := \max_{x\in\Omega} \mathrm{dist}(x, \partial\Omega)$ *and suppose* $\varphi \in C^1([0, d_0]) \cap C^2((0, d_0))$ *is an increasing function with* $\varphi(0) = 0$ *and such that*

$$\varphi^{p-1}\varphi'' \textit{ is nondecreasing,} \tag{3.1}$$

$$\varphi' \le c\varphi'' \quad \textit{in } (0, d_0) \quad \textit{for some } c > 0 \textit{ and} \tag{3.2}$$

$$\lim_{d \searrow 0} \frac{\varphi(d)}{d^p \varphi''(d)} = +\infty. \tag{3.3}$$

Then for all $c_1 > 0$ *and* $T > 0$ *there is* $c_1' > 0$ *such that under the assumption*

$$u_0 \leq c_1 \varphi(dist(x, \partial\Omega)) \quad in\ \Omega, \tag{3.4}$$

the solution u *of* (0.1) *satisfies*

$$u \leq c_1' \varphi(dist(x, \partial\Omega)) \quad in\ \Omega \times (0, T). \tag{3.5}$$

Before proving this lemma, let us give an example which particularly shows that even very fast boundary decay of u_0 can be inherited by the solution.

Corollary 3.2. i) *For any* $\alpha > 1$, *from*

$$u_0(x) \leq c_1 (\mathrm{dist}(x, \partial\Omega))^\alpha$$

it follows that

$$u(x,t) \leq c_1'(c_1, T)(\mathrm{dist}(x, \partial\Omega))^\alpha \quad in\ \Omega \times (0, T).$$

ii) *For any* $\alpha \in (0, \frac{p-2}{2})$ *there is* $A(\alpha, \Omega) > 0$ *such that for all* $A > A(\alpha, \Omega)$,

$$u_0(x) \leq c_1 e^{-A(\mathrm{dist}(x, \partial\Omega))^{-\alpha}}$$

implies

$$u(x,t) \leq c_1'(c_1, T) e^{-A(\mathrm{dist}(x, \partial\Omega))^{-\alpha}} \quad in\ \Omega \times (0, T).$$

Proof. It is easily verified that $\varphi(d) := d^\alpha$ fulfils the assumptions of Lemma 3.1, which proves i). To check the same for $\varphi(d) := e^{-Ad^{-\alpha}}$, we compute

$$\begin{aligned} \varphi^{p-1}(d)\varphi''(d) &= \alpha A[\alpha A - (\alpha+1)d^\alpha] d^{-2\alpha-2} e^{-pAd^{-\alpha}}, \\ \frac{\varphi''(d)}{\varphi(d)} &= \frac{\alpha A - (\alpha+1)d^\alpha}{d^{\alpha+1}}, \end{aligned}$$

whence (3.1) and (3.2) hold with $A > A(\alpha, \Omega)$ and $A(\alpha, \Omega)$ large enough. Furthermore,

$$\frac{\varphi(d)}{d^p \varphi''(d)} = \frac{d^{2\alpha+2-p}}{\alpha A[\alpha A - (\alpha+1)d^\alpha]} \to +\infty \quad \text{as } d \to 0,$$

since $p > 2\alpha + 2$. □

Proof. (of the lemma). We first observe that as $\varphi \in C^1$, (3.4) implies $u_0(x) \leq$ $\leq c\mathrm{dist}(x, \partial\Omega)$ and hence

$$u(x,t) \leq c_2 \mathrm{dist}(x, \partial\Omega) \quad \text{in } \Omega \times (0, T), \tag{3.6}$$

which easily follows from comparison of u with the stationary supersolution e of (0.1), where $-\Delta e = 1$ in Ω and $e|_{\partial\Omega} = 0$.

We now follow a barrier-type technique as demonstrated in a slightly different setting in [**FMcL2**] and in [**Wie1**] for $\varphi(s) = s$. On $\Omega' := B_{R+d}(x_0) \cap \Omega$, $d > 0$ to be specified later, with x_0 the center of a ball $B_R(x_0)$ touching $\partial\Omega$ from outside at $y \in \partial\Omega$, introduce the function $w(x) := \varphi(\xi)$, $\xi := |x - x_0| - R$. Due to (3.1) and (3.2),

$$\begin{aligned} w^{p-1}\Delta w &= \varphi^{p-1}(\xi)\Big(\varphi''(\xi) + \frac{n-1}{|x-x_0|}\varphi'(\xi)\Big) \\ &\leq c\varphi^{p-1}(d)\varphi''(d) =: \varrho(d). \end{aligned} \tag{3.7}$$

Letting $y(t)$ satisfy $y' = \varrho(d)y^{p+1}$, that is, $y(t) = (y_0^{-p} - p\varrho(d)t)^{-\frac{1}{p}}$ with $y_0 := y(0)$, we see that y exists on $(0, T_y)$ with $T_y = (p\varrho(d)y_0^p)^{-1}$. In order to compare u in $\Omega' \times (0,T)$ with $v(x,t) := y(t)w(x)$, we observe that by (3.7),

$$v_t - v^p\Delta v = w \cdot (y' + w^{p-1}\Delta w \cdot y^{p+1}) \leq 0.$$

At $t = 0$, we have

$$u_0(x) \leq c_1\varphi(\operatorname{dist}(x,\partial\Omega)) \leq c_1\varphi(|x-x_0| - R) = c_1 w(x),$$

while if $|x - x_0| = R + d$, (3.6) implies that for d small enough, $u(x,t) \leq c_2 d$. Hence, $u \leq v$ on the parabolic boundary if $y_0 := \max\{c_1, \frac{c_2 d}{\varphi(d)}\}$, so that $y_0 \leq c\frac{d}{\varphi(d)}$. Consequently, using (3.7), we estimate

$$T_y \geq c\frac{\varphi^p(d)}{\delta^p\varphi^{p-1}(d)\varphi''(d)} = c\frac{\varphi(d)}{d^p\varphi''(d)}.$$

By assumption (3.3), we can now fix $d > 0$ small enough such that $T_y > T$, so that the comparison principle yields $u(x,t) \leq c\varphi(\xi)$ on $\Omega' \times (0,T)$ and thus the claim follows. $\square$

4. A C_0^∞-solution

We start with a simple consequence of Lemmata 1.2 and 3.1 that provides a *two-sided* estimate for u near the boundary. This will be necessary in Lemma 4.3, where, roughly speaking, for a suitably rescaled equation the lower bound will be used to control the ellipticity constant, while the upper bound ensures that the rescaled function is a bounded solution.

Corollary 4.1. *Let φ meet the conditions of Lemma 3.1 and suppose that*

$$c_0\varphi(\operatorname{dist}(x,\partial\Omega)) \leq u_0 \leq c_1\varphi(\operatorname{dist}(x,\partial\Omega)) \quad \text{in } \Omega$$

for positive constants c_0, c_1. Then for all $T > 0$ there are $c_0', c_1' > 0$ such that

$$c_0' u_0(x) \leq u(x,t) \leq c_1' u_0(x) \quad \text{in } \Omega \times (0,T).$$

Proof. Integrating (1.2) and using Lemma 3.1, we immediately obtain $e^{-CT}u_0(x) \leq u(x,t) \leq c\varphi(\operatorname{dist}(x,\partial\Omega)) \leq \frac{c}{c_0}u_0$. $\square$

Without further comment, we state the following immediate consequence of Corollaries 4.1 and 3.2.

Corollary 4.2. i) *From*

$$c_0(\mathrm{dist}(x,\partial\Omega))^\alpha \le u_0(x) \le c_1(\mathrm{dist}(x,\partial\Omega))^\alpha, \qquad \alpha > 1,$$

it follows that

$$c_0'(\mathrm{dist}(x,\partial\Omega))^\alpha \le u(x,t) \le c_1'(\mathrm{dist}(x,\partial\Omega))^\alpha \quad \text{in } \Omega\times(0,T).$$

ii) *For* $\alpha \in (0, \frac{p-2}{2})$ *and* $A > A(\alpha,\Omega) > 0$,

$$c_0 e^{-A(\mathrm{dist}(x,\partial\Omega))^{-\alpha}} \le u_0(x) \le c_1 e^{-A(\mathrm{dist}(x,\partial\Omega))^{-\alpha}}, \qquad 0<\alpha<\frac{p-2}{2}$$

implies

$$c_0' e^{-A(\mathrm{dist}(x,\partial\Omega))^{-\alpha}} \le u(x,t) \le c_1' e^{-A(\mathrm{dist}(x,\partial\Omega))^{-\alpha}} \quad \text{in } \Omega\times(0,T).$$

In order to establish a connection between the boundary decay and regularity up to $\partial\Omega$, we introduce a positive function $\delta:\ \Omega\to\mathbb{R}^+$ such that for some $\kappa>1$

$$\frac{1}{\kappa}\sup_{|x-z|<\delta(x)} u_0(z) \le u_0(x) \le \kappa \inf_{|x-z|<\delta(x)} u_0(z); \tag{4.1}$$

note that these inequalities are satisfied if we set for instance

$$\delta(x) := \sup\Big\{\eta>0 \mid \frac{1}{\kappa}\sup_{|x-z|<\eta} u_0(z) \le u_0(x) \le \kappa \inf_{|x-z|<\eta} u_0(z)\Big\}, \quad x\in\Omega.$$

For certain types of boundary behavior, however, we can choose δ much more conveniently:
i) If $c_0(\mathrm{dist}(x,\partial\Omega))^\alpha \le u_0(x) \le c_1(\mathrm{dist}(x,\partial\Omega))^\alpha$ holds for some $\alpha>0$, then it is easily verified that we may choose $\delta(x) := cd(x)$ with suitably small $c>0$ and $\kappa > \frac{c_1}{c_0}$.
ii) In view of Theorem 4.4 we also consider the case

$$c_0\varphi(\mathrm{dist}(x,\partial\Omega)) \le u_0(x) \le c_1\varphi(\mathrm{dist}(x,\partial\Omega)) \quad \text{with } \varphi(d) = e^{-Ad^{-\alpha}}, \quad A,\alpha>0.$$

We claim that we may use

$$\delta(x) = c(\mathrm{dist}(x,\partial\Omega))^{1+\alpha} \quad \text{for some small } c>0.$$

Indeed, observe that for $d>0$ the equations $\frac{1}{e^A}\varphi(d+\eta_-(d)) = \varphi(d) = e^A\varphi(d-\eta_+(d))$ are solved by $\eta_-(d) = [(1-d^\alpha)^{-\frac{1}{\alpha}}-1]d$ and $\eta_+(d) = [1-(1+d^\alpha)^{-\frac{1}{\alpha}}]d$, respectively. Both expressions equal $\frac{1}{\alpha}d^{1+\alpha}+O(d^{1+2\alpha})$ as $d\to 0$, hence $\eta_\pm(d) \ge cd^{1+\alpha}$ for $d\le d_1$, $d_1>0$ small.

Now if $d(x)\le d_1$ and $|z-x|<\eta_-(d(x))$ (where we have abbreviated $d(x) := \mathrm{dist}(x,\partial\Omega)$) then $u_0(z) \le c_1\varphi(d(x)) \le c_1\varphi(d(x)+\eta_-(d(x))) \le c_1e^A\varphi(d(x)) \le \frac{c_1e^A}{c_0}u_0(x)$; similarly we obtain for $|z-x|<\eta_+(d(x))$ that $u_0(z) \ge c_0\varphi(d(x)) \ge c_0\varphi(d(x)-\eta_+(d(x))) \ge \frac{c_0}{e^A}\varphi(d(x)) \ge \frac{c_0}{c_1e^A}u_0(x)$. Thus, it follows that in fact an admissible choice is $\delta(x) = cd^{1+\alpha}(x)$.

Lemma 4.3. *Let δ be a function satisfying (4.1). Suppose that the solution u of (0.1) obeys a two-sided estimate*

$$c_0 u_0(x) \le u(x,t) \le c_1 u_0(x) \quad \text{in } \Omega \times (0,T) \tag{4.2}$$

with constants $0 < c_0 < c_1$. Assume furthermore that $u_0 \in C^{2m+\beta}(\bar{\Omega})$ and $\partial\Omega \in C^{2m+\beta}$ for some $m \in \mathbb{N}$ and some $\beta > 0$. Then for all $|\sigma| + 2k \le 2m$, we have

$$|D_x^\sigma \partial_t^k u(x,t)| \le c\delta^{-|\sigma|-2k}(x) u_0^{1+kp}(x) \quad \text{in } \Omega \times (0,T). \tag{4.3}$$

Consequently, if in addition $\delta^{-2m}(x)u_0(x) \to 0$ as $\operatorname{dist}(x,\partial\Omega) \to 0$ then $u \in C^{2m,m}(\mathbb{R}^n \times [0,T])$, where u has been extended by zero outside Ω.

Proof. Fix $x_0 \in \Omega$ and let

$$v(y,s) := \frac{1}{u_0(x_0)} \cdot u\Big(x_0 + \delta(x_0)y,\ \delta^2(x_0)u_0^{-p}(x_0)s\Big), \quad (y,s) \in B_1(0) \times (0, T_{x_0}),$$

where $T_{x_0} := \delta^{-2}(x_0)u^p(x_0)T$. Clearly,

$$D_y^\sigma \partial_t^k v(y,s) = \delta^{|\sigma|+2k}(x_0)u^{-1-kp}(x_0) D_x^\sigma \partial_t^k u(x,t) \quad \text{for } \sigma \in \mathbb{N}_0^n \text{ and } k \in \mathbb{N}_0,$$

so that v again satisfies $v_s = v^p \Delta v \equiv \nabla \cdot (v^p \nabla v) - pv^{p-1}|\nabla v|^2$. As

$$\frac{1}{\kappa}c_0 \le \frac{c_0 u_0(x_0 + \delta(x_0)y)}{u_0(x_0)} \le v(y,s) \le \frac{c_1 u_0(x_0 + \delta(x_0)y)}{u_0(x_0)} \le \kappa c_1,$$

Theorems V.1.1 and III.12.1 in [**LSU**] provide a uniform interior estimate

$$\|v\|_{C^{2m+\theta, m+\frac{\theta}{2}}(\bar{B}_{1/2}(0)\times[0,T_{x_0}])} \le c$$

for some $\theta > 0$, which in the original coordinates in particular means that the quantities

$$\delta^{|\sigma|+2k}(x_0)u_0^{-1-kp}(x_0)\left|D_x^\sigma \partial_t^k u(x,t)\right|, \quad |\sigma| + 2k \le 2m,$$

are all bounded in $B_{\delta(x_0)/2}(x_0) \times (0,T)$, uniformly with respect to the choice of x_0. We may now set $x = x_0$ to obtain (4.3). $\square$

Theorem 4.4. *Suppose $p > 2$ and $B_R(0) \subset \Omega$ for some $R > 0$. Then there exists a nontrivial classical solution u of (0.1) with the property*

$$u(t) \in C_0^\infty(\Omega) \text{ with } \operatorname{supp} u(t) \equiv B_R(0) \quad \forall t \in (0,T).$$

Proof. Choosing $\alpha \in (0, \frac{p-2}{2})$ and $A > A(\alpha, \Omega)$ (cf. Corollary 3.2), we define u to be the positive solution in $B_R(0) \times (0,T)$ evolving from $u_0(x) := := e^{-A(r-|x|)^{-\alpha}}$, $x \in B_R(0)$, extended by zero to all of Ω. Then $u_0 \in C_0^\infty(\Omega)$ and $c_0\varphi(\operatorname{dist}(x,\partial B_R(0))) \le u_0(x) \le c_1\varphi(\operatorname{dist}(x,\partial B_R(0)))$ holds in $B_R(0)$ for $c_0 = c_1 = 1$ and $\varphi(d) := e^{-Ad^{-\alpha}}$. By Corollary 4.2, $c_0' u_0(x) \le u(x,t) \le c_1' u_0(x)$. Now the assertion follows, because due to our above considerations we may choose $\delta(x) = c(\operatorname{dist}(x,\partial B_R(0)))^{1+\alpha}$ in Lemma 4.3 for some $c > 0$. $\square$

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References

[**Ar**] Aronson D. G., *The porous medium equation.* Nonlinear diffusion problems, Lect. 2nd 1985 Sess. C.I.M.E. Montecatini Terme/Italy 1985, Lect. Notes Math. **1224** (1986), 1–46.

[**ACP**] Aronson D. G., Crandall M. G. and Peletier L. A., *Stabilization of solutions of a degenerate nonlinear diffusion problem.* Nonlin. Anal. **6** (10)(1982), 1001–1022.

[**AP**] Aronson D. G. and Peletier L. A., *Large Time Behaviour of Solutions of the Porous Medium Equation in Bounded Domains.* J. Diff. Eqns. **39** (1981), 378–412.

[**BP**] Bertsch M. and Peletier L. A., *A positivity property of Solutions of Nonlinear Diffusion Equations.* J. Diff. Eqns. **53** (1984), 30–47.

[**BU**] Bertsch, M. and Ughi M., *Positivity properties of viscosity solutions of a degenerate parabolic equation.* Nonlin. Anal. **14** (7) (1990), 571–592.

[**Fr**] Friedman, A.: *Variational priciples and free-boundary problems.* Wiley (1982).

[**FMcL1**] Friedman, A. and McLeod, B., *Blow-up of solutions of semilinear heat equations.* Indiana Univ. Math. J. **34** (1985), 425–447.

[**FMcL2**] ———, *Blow-up of Solutions of Nonlinear Degenerate Parabolic Equations.* Arch. Rat. Mech. Anal. **96** (1987), 55–80.

[**Ga**] Gage M. E., *On the size of the blow-up set for a quasilinear parabolic equation.* Contemp. Math. **127** (1992), 41–58.

[**LSU**] Ladyzenskaja O. A., Solonnikov V. A. and Ural'ceva N. N., *Linear and Quasi-linear Equations of Parabolic Type.* AMS, Providence (1968).

[**Li**] Lions P. L., *Structure of the Set of Steady-State Solutions and Asymptotic Behaviour of Semilinear Heat Equations.* J. Diff. Eqns. **53** (1982), 362–386.

[**LDalP**] Luckhaus S. and Dal Passo R., *A Degenerate Diffusion Problem Not in Divergence Form.* J. Diff. Eqns. **69** (1987), 1–14.

[**Wie1**] Wiegner M., *Blow-up for solutions of some degenerate parabolic equations.* Diff. Int. Eqns. **7** (5-6) (1994), 1641–1647.

[**Wie2**] ———,*A Degenerate Diffusion Equation with a Nonlinear Source Term.* Nonlin. Anal. TMA **28** (1997), 1977–1995.

[**Win1**] Winkler M., *Some results on degenerate parabolic equations not in divergence form.* PhD Thesis, `www.math1.rwth-aachen.de/Forschung-Research/d_emath1.html` (2000).

[**Win2**] ———, *On the Cauchy Problem for a Degenerate Parabolic Equation.* J. Anal. Appl. **20** (3) (2001), 677–690.

[**Win3**] ———, *Propagation versus constancy of support in the degenerate parabolic equation* $u_t = f(u)\Delta u$. Preprint (2002).

[**Win4**] ———, *A doubly critical degenerate parabolic problem.* Preprint (2003).

M. Winkler, Department of Mathematics I, RWTH Aachen, Wüllnerstr. 5-7, 52056 Aachen, Germany, *current address*: Institute of Applied Mathematics, Comenius University, Mlynská dolina, 84248 Bratislava, Slovakia, *e-mail*: `winkler@math1.rwth-aachen.de`