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HOMOMORPHISMS OF TRIANGLE GROUPS WITH LARGE INJECTIVITY RADIUS

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ABSTRACT. We prove a new upper bound on the smallest order $o(l, m, n; r)$ of a finite group that is a homomorphic image of a triangle group $T(l, m, n)$ with injectivity radius at least r.

1. Introduction

Triangle groups $T(l, m, n) = \langle a, b, c | a^l = b^m = c^n = abc = 1\rangle$ play an important role in the theory of Riemann surfaces. In the theory of maps and hypermaps [**3**], [**4**], the groups $T(l, m, n)$ appear as groups of orientation preserving automorphisms of universal hypermaps of type (l, m, n). Each such hypermap can be visualized in form of a trivalent tesselation of a simply connected surface, where a $2l$-gon, a $2m$-gon and a $2n$-gon meet at each vertex.

The groups $T(l, m, n)$ are known to be residually finite [**3**]. Closely related to residual finiteness is the concept of injectivity radius which we introduce now. For any $u \in T(l, m, n)$ we have $u = u_1 u_2 \dots u_k$ where $u_i \in \{a, b, c\}$. The smallest k with this property will be denoted by $|u|$; if u is the unit element of $T(l, m, n)$ then we set $|u| = 0$. Let $\varphi : T(l, m, n) \to H$ be a group homomorphism. We say that the homomorphism φ has *injectivity radius at least* r if $\varphi(u) \neq 1_H$ for all non-identity elements $u \in T(l, m, n)$ such that $|u| \leq r$.

The main purpose of this article is to present a bound for the smallest order $o(l, m, n; r)$ of a finite group that is a homomorphic image of a triangle group $T(l, m, n)$ with injectivity radius at least r and to improve an earlier result on $o(2, m, n; r)$ of [**6**], [**7**].

2. Auxiliary results related to polynomial rings over the integers

Let Z be the ring of integers and let $g(x_1, x_2, \dots, x_n) \in Z[x_1, x_2, \dots, x_n]$ be a polynomial in n variables over Z. To simplify the notation we will write $\vec{x} = (x_1, x_2, \dots, x_n)$, $g(\vec{x}) \in Z[\vec{x}]$ and $\vec{h} = (h_1(x_1), h_2(x_2), \dots, h_n(x_n))$. We define the *norm* $||g(\vec{x})||$ of a polynomial $g(\vec{x}) \in Z[\vec{x}]$ as the largest absolute value of all

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its coefficients. Obviously we have the inequality

$$||f(\vec{x}) + g(\vec{x})|| \le ||f(\vec{x})|| + ||g(\vec{x})|| \tag{1}$$

for all $f(\vec{x}), g(\vec{x}) \in Z(\vec{x})$. Further, for any $f(\vec{x}) \in Z[\vec{x}]$ we define $w(f)$, the *width* of $f(\vec{x})$, as the number of non-zero coefficients of $f(\vec{x})$. It is easy to see that

$$||f(\vec{x})g(\vec{x})|| \le \min\{w(f), w(g)\} \cdot ||f(\vec{x})|| \cdot ||g(\vec{x})||. \tag{2}$$

Let $g(\vec{x}) \in Z[\vec{x}]$ be a polynomial. Denote by $\deg(h(x))$ the degree of the polynomial $h(x)$ (over one variable) and by $\delta_i(g)$ the degree of the polynomial $g(\vec{x})$ with respect to x_i.We say that a polynomial $h(x) \in Z[x]$ is *monic* if its leading coefficient is 1. In the following Lemma we show a relationship between norms of remainders and norms of divisors and dividents.

Lemma 1. *Let $g(\vec{x}) \in Z[\vec{x}]$ and let $h_i(x_i) \in Z[x_i] < Z[\vec{x}]$ (that is, $Z[x_i]$ is a subring of $Z[\vec{x}]$); let $k_i = \delta_i(g), d_i = \deg(h_i(x_i))$ and let the polynomials $h_i(x_i)$ be monic. Let $g(\vec{x}) = q_1(\vec{x})h_1(x_1) + q_2(\vec{x})h_2(x_2) + \cdots + q_n(\vec{x})h_n(x_n) + r(\vec{x})$, where $\delta_i(r) < d_i$ for all $i \in \{1, 2, \ldots, n\}$. If $\lambda_i = \max\{0, k_i - d_i + 1\}$, then*

$$||r(\vec{x})|| \le ||g(\vec{x})||(1 + ||h_1(x_1)||)^{\lambda_1}(1 + ||h_2(x_2)||)^{\lambda_2} \ldots (1 + ||h_n(x_n)||)^{\lambda_n}.$$

Proof. We present an argument for two variables; the proof for n variables is similar. Let $g(x_1, x_2) = q_1(x_1, x_2)h_1(x_1) + r_1(x_1, x_2)$, where $\delta_1(r_1) < d_1$ and let $r_1(x_1, x_2) = q_2(x_1, x_2)h_2(x_2) + r(x_1, x_2)$, where $\delta_2(r) < d_2$ (and, obviously, $\delta_1(r) < d_1$). If $k_1 < d_1$, then $q_1(x_1, x_2) = 0, \lambda_1 = 0$ and $r_1(x_1, x_2) = g(x_1, x_2)$. For $k_1 \ge d_1$ we continue by induction on k_1. Let us write $g(x_1, x_2)$ in the form $g(x_1, x_2) = {x_1}^{k_1} b_0(x_2) + {x_1}^{k_1 - 1} b_1(x_2) + \cdots + b_{k_1}(x_2)$; in this notation $b_0(x_2)$ will be called the leading term of $g(x_1, x_2)$ with respect to x_1. Let $g'(x_1, x_2) = g(x_1, x_2) - b_0(x_2){x_1}^{k_1 - d_1} h_1(x_1)$. It is easy to check that $||g'(x_1, x_2)|| \le ||g(x_1, x_2)|| \cdot (1 + ||h_1(x_1)||)$. By the division algorithm (recall that $h_1(x_1)$ is monic) we have $g'(x_1, x_2) = {q_1}'(x_1, x_2)h_1(x_1) + r_1(x_1, x_2)$ for some ${q_1}'(x_1, x_2) \in Z[x_1, x_2]$. Now, let ${k_1}' = \delta_1(g')$. If ${k_1}' < d_1$, then ${q_1}'(x_1, x_2) = 0$ and

$$\begin{aligned} ||r_1(x_1, x_2)|| = ||g'(x_1, x_2)|| &\le ||g(x_1, x_2)||(1 + ||h_1(x_1)||) \\ &\le ||g(x_1, x_2)||(1 + ||h_1(x_1)||)^{k_1 - d_1 + 1}. \end{aligned}$$

Using the inductive hypothesis (for $d_1 \le {k_1}' \le k_1 - 1$), we have $||r_1(x_1, x_2)|| \le ||g'(x_1, x_2)||(1 + ||h_1(x_1)||)^{{k_1}' - d_1 + 1}$. Furthermore, with the approximation for the norm $||g'(x_1, x_2)||$ we obtain

$$\begin{aligned} ||r_1(x_1, x_2)|| &\le ||g(x_1, x_2)||(1 + ||h_1(x_1)||)^{{k_1}' - d_1 + 2} \\ &\le ||g(x_1, x_2)||(1 + ||h_1(x_1)||)^{k_1 - d_1 + 1}. \end{aligned}$$

In the same way we can see that if $r_1(x_1, x_2) = q_2(x_1, x_2)h_2(x_2) + r(x_1, x_2)$, then $||r(x_1, x_2)|| \le ||r_1(x_1, x_2)||(1 + ||h_2(x_2)||)^{\lambda_2}$ and consequently

$$||r(x_1, x_2)|| \le ||g(x_1, x_2)||(1 + ||h_1(x_1)||)^{\lambda_1}(1 + ||h_2(x_2)||)^{\lambda_2}.$$

□

Let the *quotient ring* $Z[\xi]$ be a ring obtained by adding a root ξ of a polynomial $h(x) \in Z[x]$ to Z; so $Z[\xi] = Z[x]/h(x)$, where $h(x)$ is a monic polynomial over Z with positive degree. Inductively, we define the quotient ring $Z[\vec{\xi}] = Z[\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_n]$ as a ring obtained by adding a root ξ_n of a polynomial $h_n(x_n)$ to $Z[\xi_1, \xi_2, \ldots, \xi_{n-1}]$. In all our applications the monic polynomial $h_n(x_n)$ will belong to $Z[x_n]$ and hence our quotient rings $Z[\vec{\xi}]$ will be more restrictive than usual. We will simply write $Z[\vec{\xi}] = Z[\vec{x}]/\vec{h}$, where $\vec{h} = (h_1(x_1), h_2(x_2), \ldots, h_n(x_n))$. Now we give a bound on the norm $||f(\vec{\xi})g(\vec{\xi})||$ of the product $f(\vec{\xi})g(\vec{\xi})$.

Lemma 2. *Let $h_i(x_i) \in Z[x_i]$ be monic polynomials of degrees $d_i > 0$ and let $s_i = 1 + ||h_i(x_i)||$. Furthermore, let $Z[\vec{x}]/\vec{h} = Z[\vec{\xi}]$, let $f(\vec{\xi}), g(\vec{\xi}) \in Z[\vec{\xi}]$ be arbitrary polynomials and let $w = \min\{w(f), w(g)\}$. Then*

$$||f(\vec{\xi})g(\vec{\xi})|| \leq w \left(\prod_{i=1}^{n} {s_i}^{d_i - 1} \right) ||f(\vec{\xi})|| \cdot ||g(\vec{\xi})||.$$

Proof. Again, we present the proof for two variables. Let $f(\xi_1, \xi_2)$ and $g(\xi_1, \xi_2)$ be members of $Z[\xi_1, \xi_2]$. We want to find the unique $r(x_1, x_2)$ with $\delta_1(r) < d_1$ and $\delta_2(r) < d_2$ such that

$$f(x_1, x_2)g(x_1, x_2) = q_1(x_1, x_2)h_1(x_1) + q_2(x_1, x_2)h_2(x_2) + r(x_1, x_2);$$

then $r(\xi_1, \xi_2) = f(\xi_1, \xi_2)g(\xi_1, \xi_2)$. Since $h_1(x_1)$ has degree d_1, the polynomials $f(x_1, x_2)$ and $g(x_1, x_2)$ have degrees at most $d_1 - 1$; therefore $\delta_1(fg) \leq 2d_1 - 2$ and similarly $\delta_2(fg) \leq 2d_2 - 2$. From the definitions of norms, Lemma 1 and equation (2) we have

$$\begin{aligned}
&||f(\xi_1, \xi_2)g(\xi_1, \xi_2)|| = ||r(\xi_1, \xi_2)|| = ||r(x_1, x_2)|| \leq \\
&\leq ||f(x_1, x_2)g(x_1, x_2)||(1 + ||h_1(x_1)||)^{(2d_1 - 2) - d_1 + 1}(1 + ||h_2(x_2)||)^{(2d_2 - 2) - d_2 + 1} \leq \\
&\leq w{s_1}^{d_1 - 1}{s_2}^{d_2 - 1}||f(x_1, x_2)|| \cdot ||g(x_1, x_2)|| = \\
&= w{s_1}^{d_1 - 1}{s_2}^{d_2 - 1}||f(\xi_1, \xi_2)|| \cdot ||g(\xi_1, \xi_2)||.
\end{aligned}$$

□

For the rest of this article we need to extend the concepts of norm and width over to matrices in $SL_q(Z[\vec{\xi}])$. If $A \in SL_q(Z[\vec{\xi}])$ is a matrix with entries $A_{jk} = A_{jk}(\vec{\xi}) \in Z[\vec{\xi}]$, we define the norm $||A||$ and the width $w(A)$ as the maximum of $||A_{jk}(\vec{\xi})||$ and $w(A_{jk}(\vec{\xi}))$, respectively. In the next Lemma we present a result for products in $SL_q(Z[\vec{\xi}])$.

Lemma 3. *Let $Z[\vec{\xi}] = Z[\vec{x}]/\vec{h}$, where $h_i(x_i)$ are monic polynomials of degrees d_i, and let $s_i = 1 + ||h_i(x_i)||$. Let $A, B \in SL_q(Z[\vec{\xi}])$ be any matrices and let $w = \min\{w(A), w(B)\}$. Then*

$$||AB|| \leq qw(\prod_{i=1}^{n} {s_i}^{d_i - 1})||A|| \cdot ||B||.$$

Proof. Let j, k be arbitrary indices and let C be the product $C = AB$. For $C_{jk}(\vec{\xi}) \in Z[\vec{\xi}]$) we have $C_{jk}(\vec{\xi}) = \sum_{l=1}^{q} A_{jl}B_{lk}$. Further, let w_l be the minimum of $\{w(A_{jl}(\vec{\xi})), w(B_{lk}(\vec{\xi}))\}$. If we repeatly apply the inequality (1) and Lemma 2, we obtain the following result:

$$||C_{jk}(\vec{\xi})|| = ||\sum_{l=1}^{q} A_{jl}(\vec{\xi})B_{lk}(\vec{\xi})|| \leq \sum_{l=1}^{q} ||A_{jl}(\vec{\xi})B_{lk}(\vec{\xi})|| \leq$$
$$\leq \sum_{l=1}^{q} w_l(\prod_{i=1}^{n} {s_i}^{d_i-1})||A_{jl}(\vec{\xi})||||B_{lk}(\vec{\xi})|| \leq qw(\prod_{i=1}^{n} {s_i}^{d_i-1})||A|| \cdot ||B||.$$

Therefore for the norm of the product $||AB||$ it holds that

$$||AB|| \leq qw(\prod_{i=1}^{n} {s_i}^{d_i-1})||A|| \cdot ||B||.$$

□

We generalize the above result to arbitrary products in the next Lemma.

Lemma 4. *Let $Z[\vec{\xi}] = Z[\vec{x}]/\vec{h}$ where $h_i(x_i)$ are monic polynomials with positive degrees d_i, and let $s_i = 1 + ||h_i(x_i)||$. Let $A_1, A_2, \ldots, A_r$ be any set of r matrices in $SL_q(Z[\vec{\xi}])$, all of width $\leq w$. Then*

$$||A_1 \cdot A_2 \cdot \ldots \cdot A_r|| \leq \left(qw\prod_{i=1}^{n} {s_i}^{d_i-1}\right)^{r-1} \cdot ||A_1|| \cdot ||A_2|| \cdot \ldots \cdot ||A_r||.$$

Proof. Using induction on $r \geq 3$ we apply Lemma 3 to A and B (here $A = A_1 \cdot A_2 \cdot \ldots \cdot A_{r-1}$ and $B = A_r$). From the assumptions of Lemma 4 it follows that $\min\{w(A), w(B)\} \leq w$. For the norm of product AB we have $||AB|| \leq qw(\prod_{i=1}^{n} {s_i}^{d_i-1})||A|| \cdot ||B||$. Furthermore, we obtain

$$||A|| = ||A_1 \cdot A_2 \cdot \ldots \cdot A_{r-1}|| \leq \left(qw\prod_{i=1}^{n} {s_i}^{d_i-1}\right)^{r-2} ||A_1|| \cdot ||A_2|| \cdot \ldots \cdot ||A_{r-1}||.$$

The required result is obtained by combining the last two inequalities. □

The next theorem gives a construction of homomorphisms of hyperbolic triangle groups into finite groups of order less than C^r and with injectivity radius at least r. Here, the number C depends only on the hyperbolic type. The statement of Theorem 5 also makes use of the fact that there exist faithful representations of triangle groups in special linear groups; an explicit example will be given later in Section 3.

Theorem 5. *Let $Z[\eta, \zeta, \xi] = Z[x, y, z]/(h_1(x), h_2(y), h_3(z))$, where $h_1(x) \in Z[x]$, $h_2(y) \in Z[y]$, $h_3(z) \in Z[z]$, are monic polynomials of degree $d_1, d_2, d_3 > 0$; let $s_1 = 1 + ||h_1(x)||, s_2 = 1 + ||h_2(y)||$ and let $s_3 = 1 + ||h_3(z)||$. Assume that we have a faithful representation $\theta : T(l, m, n) \to SL_q(Z[\eta, \zeta, \xi])$ of the triangle group $T(l, m, n) = \langle a, b, c | a^l = b^m = c^n = abc = 1\rangle$. Further, let all the matrices*

$\theta(a), \theta(b)$ and $\theta(c)$ have norm $\leq t$ and width $\leq w$. Then, for any positive integer r there exists a faithful representation of $T(l,m,n)$ with injectivity radius at least r in a finite group H such, that

$$|H| < \left(qwts_1{}^{d_1-1}s_2{}^{d_2-1}s_3{}^{d_3-1}\right)^{q^2 d_1 d_2 d_3 r}. \tag{3}$$

Proof. Let $\theta : T(l,m,n) \to SL_q(Z[\eta,\zeta,\xi])$ be a faithful representation of the triangle group $T(l,m,n) = \langle a,b,c | a^l = b^m = c^n = abc = 1\rangle$ and let r be a natural number. Further, let S_r be the set of all non-identity elements u of $T(l,m,n)$ which can be written as words of lenghts at most r in symbols a, b and c. For every u in S_r, we can express the matrix $\theta(u)$ as a product of at most r matrices from $SL_q(Z[\eta,\zeta,\xi])$ and every such matrix has norm $\leq t$ and width $\leq w$. From Lemma 4 (for $n = 3$) it follows that for each $u \in S_r$ we have:

$$||\theta(u)|| \leq \left(qws_1{}^{d_1-1}s_2{}^{d_2-1}s_3{}^{d_3-1}\right)^{r-1}t^r. \tag{4}$$

Let $\phi_p : Z \to Z_p$ be the standard ring epimorphism. This epimorphism extends in an obvious way to the ring epimorphism $\phi_{p,x} : Z[x,y,z] \to Z_p[x,y,z]$, and also to the ring epimorphism

$$\phi_{p,h} : Z[x,y,z]/(h_1(x),h_2(y),h_3(z)) \to Z_p[x,y,z]/(\phi_{p,x}(h_1(x),h_2(y),h_3(z))).$$

As $h_1(x), h_2(y)$ and $h_3(z)$ are monic, we can construct the epimorphism $\phi_{p,h}$ in the following way. Let $Z[\eta,\zeta,\xi]$ and $Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}]$ be the rings obtained by adjoining roots η,ζ,ξ of the polynomials $h_1(x), h_2(y)$ and $h_2(z)$, respectively, to Z, and roots $\tilde{\eta},\tilde{\zeta},\tilde{\xi}$ of the polynomials $\phi_{p,x}(h_1(x)), \phi_{p,x}(h_2(y))$ and $\phi_{p,x}(h_3(z))$, respectively, to Z_p. The mapping $\phi_{p,h} : Z[\eta,\zeta,\xi] \to Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}]$ is the unique epimorphism which sends η to $\tilde{\eta}$, ζ to $\tilde{\zeta}$ and ξ to $\tilde{\xi}$, and whose restriction to Z is ϕ_p. It is easy to see that $\phi_{p,h}$ extends to a group homomorphism

$$\Phi_{p,h} : SL_q(Z[\eta,\zeta,\xi]) \to SL_q(Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}]). \tag{5}$$

We need to specify the value of p in (5). Let p be a prime number such that $\left(qws_1{}^{d_1-1}s_2{}^{d_2-1}s_3{}^{d_3-1}\right)^{r-1}t^r < p < \left(qwts_1{}^{d_1-1}s_2{}^{d_2-1}s_3{}^{d_3-1}\right)^r$. (It was proved by Chebyshev that if $n > 1$, then there exists a prime number p between n and $2n$; this theorem is known as Bertrand's postulate.) We claim that the group $H = SL_q(Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}])$ together with the representation $\vartheta = \Phi_{p,h}\theta : T(l,m,n) \to SL_q(Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}])$ has the required properties. To prove this, it is sufficient to show that for any non-identity element $u \in S_r$, the matrix $\vartheta(u)$ is never the identity matrix in $SL_q(Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}])$. Since θ is faithful and $u \in S_r$, the image $\theta(u)$ of an element u is not the identity matrix in $SL_q(Z[\eta,\zeta,\xi])$. Further, by the inequality (4) and by our choice of the number p, the image $\Phi_{p,h}(\theta(u))$ of u cannot be the identity matrix in $SL_q(Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}])$ (because $\theta(u)$ is not the identity matrix in $SL_q(Z[\eta,\zeta,\xi])$ and the coefficients of $\theta(u)$ are all in absolute value smaller than p). Therefore, the representation ϑ is faithful and it has injectivity radius at least r. In each matrix H in $SL_q(Z_p[\tilde{\eta},\tilde{\zeta},\tilde{\xi}])$ we have q^2 elements. Each such element is a polynomial in $\tilde{\eta},\tilde{\zeta},\tilde{\xi}$ with width less than $d_1d_2d_3$, and for each of the $d_1d_2d_3$

coefficients we have p choices. Thus, for the order of the group H we have

$$|H| = |SL_q(Z_p[\tilde{\eta}, \tilde{\zeta}, \tilde{\xi}])| < p^{q^2 d_1 d_2 d_3} < (qwts_1{}^{d_1-1} s_2{}^{d_2-1} s_3{}^{d_3-1})^{q^2 d_1 d_2 d_3 r}.$$

□

3. Quotients of triangle groups and injectivity radius

In this section we present a result for homomorphisms of triangle groups in finite groups with injectivity radius at least r with help of Chebyshev polynomials. Let $l, m, n \geq 2$ be natural numbers such that $1/l + 1/m + 1/n < 1$. Further, let $\eta = 2\cos(\pi/l)$, $\zeta = 2\cos(\pi/m)$ and $\xi = 2\cos(\pi/n)$. Next, let the matrices $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be given by

$$\mathbf{A} = \begin{pmatrix} 1 & \xi & \eta\xi + \zeta \\ 0 & -1 & -\eta \\ 0 & \eta & \eta^2 - 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} \zeta^2 - 1 & 0 & \zeta \\ \eta\zeta + \xi & 1 & \eta \\ -\zeta & 0 & -1 \end{pmatrix} \quad \text{and}$$

$$\mathbf{C} = \begin{pmatrix} -1 & -\xi & 0 \\ \xi & \xi^2 - 1 & 0 \\ \zeta & \zeta\xi + \eta & 1 \end{pmatrix}.$$

Then, the assignment $a \mapsto \mathbf{A}, b \mapsto \mathbf{B}$ and $c \mapsto \mathbf{C}$ extends to a faithful representation of the hyperbolic triangle group $T(l, m, n) = \langle a, b, c | a^l = b^m = c^n = abc = 1\rangle$ in $SL_3(Z[\eta, \zeta, \xi])$ [**5**]. If $T_k(x) = \cos(k \arccos x)$ is the $k-$th Chebyshev polynomial and if $P_k(x) = 2T_k(x/2)$, then the polynomial $P_k(x)$ is monic and has integer coefficients. It is easy to verify that η, ζ and ξ are roots of the polynomials $P_l(x) + 2$, $P_m(x) + 2$ and $P_n(x) + 2$, respectively. Let $h_1(x) = P_l(x) + 2$, $h_2(x) = P_m(x) + 2$ and $h_3(y) = P_n(y) + 2$. We see that the norm and width of the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ are $t = 1$ and $w = 2$. For the norm of the polynomial $P_k(x)$ we have the inequality $||P_k(x)|| \leq 2^{k-1}$. It follows that $s_1 = 1 + ||h_1(x)|| \leq 1 + 2^{l-1} < 2^l$, $s_2 = 1 + ||h_2(x)|| \leq 1 + 2^{m-1} < 2^m$ and $s_3 = 1 + ||h_2(y)|| \leq 1 + 2^{n-1} < 2^n$. Theorem 5 (for $q = 3$, $d_1 = l$, $d_2 = m$, and $d_3 = n$) then implies the following upper bound on the smallest order $o(l, m, m; r)$ of a homomorphic image of the triangle group $T(l, m, n)$ in a finite group with injectivity radius at least r.

Theorem 6. *Let $1/l + 1/m + 1/n < 1$. Then for every $r \geq 1$ there exists a homomorphic image of the triangle group $T(l, m, n)$ in a finite group of order at most C^r with injectivity radius at least r, where*

$$C = C(l, m, n) < 2^{[(l^2+m^2+n^2)-(l+m+n)+3]9lmn}.$$

In [**6**], the author derived an upper bound on the order $o(2, m, n; r)$ of a homomorphic image of the triangle groups $T(2, m, n)$ with injectivity radius at least r. We note that in [**6**], the problem is studied in the terminology of r-locally faithful representations. Using extensions of the ring Z by a single algebraic element, it was shown in [**6**], that $o(2, m, n; r) < \tilde{C}^r$ where $\tilde{C} = \tilde{C}(m, n) < 2^{72(mn)^3}$.

Multiple extensions of the ring Z used for constructions of homomorphic images of $T(2, m, n)$ clearly yield better results. In particular, setting $l = 2$, $||h_1(x)|| =$

$||P_2(x) + 1|| = 1$, $s_1 = 1$ and $d_1 = 2$ in the discussion preceding Theorem 6 we obtain:

Corollary 1. *Let $1/m + 1/n < 1/2$. Then for every $r \geq 1$ the triangle group $T(2, m, n)$ has an $r-$locally faithful representation in a finite group of order at most C^r where*

$$C = C(m, n) < 2^{[(m^2+n^2)-(m+n)+5]18mn}.$$

For example, if $n = m$, then the ratio of the estimates from [**6**] and Corollary 1 is $\frac{C}{\bar{C}} = 2^{18m^2(-4m^4+2m^2-2m+5)}$ which is asymptotically 2^{-72m^6} for large m.

We conclude with an application of our results to regular hypermaps. For basic concepts regarding hypermaps and representations of their embeddings on surfaces we refer to [**1**]. In particular, we will use the description of surface embeddings of hypermaps via associated cubic maps as introduced in [**2**].

We say that a hypermap $\mathcal{M}$ on a surface $\mathcal{S}$ has *planar width larger than* r if every non-contractible simple closed curve on $\mathcal{S}$ intersects the associated cubic map of $\mathcal{M}$ in more than r points.

The interesting problem of constructing regular hypermaps of arbitrarily large planar width turns out to be equivalent to constructing quotients of triangle groups by homomorphisms of arbitrarily large injectivity radius. On the group theory level this is equivalent to the fact that triangle groups $T(l, m, n)$ are residually finite, which means that the intersection of all normal subgroups of $T(l, m, n)$ of finite index is trivial. While residual finiteness as such does not provide any upper bounds, the result of Theorem 6 can be used for estimating the number of elements (hypervertices, hyperedges or hyperfaces) of a regular hypermap with planar width larger than r.

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